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Logarithmic lepton-mass singularities in baryon β decay

A. Garcia*
Departamento de Física, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Apartado Postal 14-740, 07000-México, D.F., México

S. Rebeca Juárez W.
Escuela Superior de Física y Matemáticas del Instituto Politécnico Nacional, Unidad Profesional Zacatenco, Edifico 6, 07300-México, D.F., México

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We study the logarithmic lepton-mass singularities contributions to the semileptonic hyperon β-decay probability. We find that such contributions are independent of strong interactions. This result is consistent with and a useful extension of a theorem by Marciano and Sirlin on π_{l2} decay.

I. INTRODUCTION

Leptonic and semileptonic weak decay processes have been discussed with great detail in the past.[1,2] Two remarkable conclusions arise from this research. The first one is that the knowledge of radiative corrections is crucial to make meaningful comparisons between theoretical results and experimental measurements. The second is that terms that depend logarithmically on the mass of the lepton (electron) are very important, because their contribution to the decay probability amounts to a very large correction.[3-5] The logarithmic lepton-mass singularity (LMS) arises when one considers the hypothetical problem in which the electron mass vanishes. This hypothesis is not too far from reality.

There are two model dependences in the semileptonic weak decay processes. They come from the strong- and weak-interaction effects. The development of unified models of weak and electromagnetic interactions, in particular the renormalizable gauge-theory model[6] by Weinberg—Salam—'t Hooft, solves part of the difficulties, the well known problem of the ultraviolet divergence of the radiative corrections. Still, the radiative corrections can not yet be rigorously computed in many cases of physical interest because of the complications arising from the strong interactions. Using the Weinberg—Salam—'t Hooft model and elementary considerations of analyticity, we analyze the LMS radiative corrections to the hyperon β decay, looking for the relevance of the role that strong interactions play in the LMS terms.

Marciano and Sirlin[3] pointed out recently that the coefficient of the LMS in the radiative corrections to the total decay rate of $\pi\to l+\nu_e$ (π_{l2}), where l is an electron or a muon, is independent of strong interactions. Motivated by this conclusion, we have investigated if this result is a property peculiar to π_{l2} decay or if it is a property that is also found for other decay processes.

In semileptonic hyperon decay, the type of kinematics is totally different. The hadrons are spin-$\frac{1}{2}$ particles, and strong interactions introduce many more structure-dependent terms. The most general expression allowed by relativistic covariance and gauge invariance allows for more than a 100 form factors to start with. The result that emerges from this paper agrees with and extends the applicability of the theorem by Marciano and Sirlin. Our conclusion may be stated as a theorem:

The coefficient of the logarithmic lepton-mass singularity in the radiative corrections of order α to the total semileptonic decay probability of hyperons is independent of strong interactions, and can therefore be rigorously computed.

This result about the effect of the strong interactions is nevertheless unexpected because there is less model dependence in the total decay probability when more model-dependent amplitudes are included, namely the inner-bremsstrahlung amplitudes.

The sequence of this paper is as follows. In Sec. II, we describe briefly the theoretical framework of the general radiative corrections for the semileptonic β decay of a baryon. We classify the amplitudes according to their model dependence. We propose a general expression that contains the structure dependence, without neglecting the four-momentum transfer q. For definiteness we choose $\Sigma^-\to ne\nu$, but our results apply equally well to the other semileptonic hyperon decays, where the lepton is either an electron or a muon. In Sec. III, we evaluate for both the virtual and bremsstrahlung cases the model-dependent contributions to the LMS. We establish a parallelism between the corresponding partial results. In Sec. IV, we summarize and combine our results establishing the theorem.

II. TRANSITION AMPLITUDES

The LMS contributions arise as a consequence of the vanishing of the $[(k-l)^2-m_e{}^2]^{-1}$ term of the electron propagator in the limit of zero lepton mass, where k and l are the intermediate boson or photon and lepton four-momenta, respectively.

In searching for the LMS contribution, within the framework of the $V-A$ theory and the Weinberg-Salam theory, we have to distinguish between the massless photon and the heavy intermediate bosons and Higgs scalars, because the masses act like infrared cutoffs that prevent the emergence of the LMS. The LMS appears only for the photon, when the invariants k^2 and $l\cdot k$ vanish for quanta of nonzero frequency in the limit of zero lepton mass m_e.

Many authors have extensively studied hadron-structure effects arising from strong interactions.[4,7] Sirlin[4] developed a strategy for evaluating the finite part of the electromagnetic corrections to the neutron β decay in a model-independent fashion. This approach has been extended[5] to evaluate the general form of the β energy spectrum in charged-hyperon β decay.

We shall use the fact that the total gauge-invariant amplitude can be split according to their dependence on the details of the strong interactions. In order to have a good estimate of the corrections we are looking for, to first order in $\alpha=e^2/4\pi$ and G, we do not neglect the four-momentum transfer, which is appreciable, because of the hadronic mass difference. The relation between the g coupling constant in Weinberg-Salam theory and the vector weak coupling constant G (which we use) is $g^2/8m_W{}^2=G/\sqrt{2}$. Let us now look at the virtual emission and reabsortion of photons.

The rate Γ for the semileptonic decay process containing virtual radiative corrections is

$$\Gamma^v=\frac{1}{(2\pi)^5}\frac{1}{2m_1}\int\frac{d^3p_2}{2E_2}\frac{d^3p_\nu}{2E_\nu}\frac{d^3l}{2E}\delta^4(p_1-p_2-l-p_\nu)\times\tfrac{1}{2}\sum_{\text{pol}}|M_v|^2 \tag{1}$$

in the Σ rest frame. In this equation, the p_1, p_2, l, k, and p_ν stand for the Σ, neutron, electron, photon, and antineutrino four-momentum, respectively. Our conventions for the metric and γ matrices are the same ones as in Ref. 8.

In order to analyze the model-dependent contribution, we will consider as in Ref. 4 that the total gauge-invariant amplitude M is divided into three parts:

$$M_v=M_0+M_v^I+M_v^D\ , \tag{2}$$

where M_0 is the uncorrected transition amplitude, M_v^I is the model-independent amplitude, and M_v^D is the model-dependent one.

M_0 is given in terms of the lepton current L_μ and hadron current H_μ as

$$M_0=\frac{G}{\sqrt{2}}\sin\theta_C L_\mu H^\mu\ , \tag{3}$$

where

$$L_\mu=\bar{u}_e O_\mu v_\nu,\quad O_\mu=\gamma_\mu(1+\gamma_5) \tag{4}$$

and

$$H_\mu=\langle n\,|\,J_\mu\,|\,\Sigma^-\rangle,\quad J_\mu=V_\mu-A_\mu\ . \tag{5}$$

The parameter θ_C is the Cabibbo angle.[9]

The explicit form of the hadronic current matrix element[10] is

$$H_\mu=u_n W_\mu(p_1,p_2)u_\Sigma=\bar{u}_n\left\{f_1(q^2)\gamma_\mu+\frac{f_2(q^2)}{m_1}\sigma_{\mu\nu}q_\nu+\frac{f_3(q^2)}{m_1}q_\mu+\left[g_1(q^2)\gamma_\mu+\frac{g_2(q^2)}{m_1}\sigma_{\mu\nu}q_\nu+\frac{g_3(q^2)}{m_1}q_\mu\right]\gamma_5\right\}u_\Sigma\ , \tag{6}$$

f_i and g_i are the Dirac form factors, and $q=p_1-p_2$. The decaying-particle mass is m_1, and the final-baryon mass is m_2. The model-independent amplitude M_v^I of Eq. (2) contains three elements:

$$M_v^I=A_1^I+A_2^I+A_3^I\ , \tag{7}$$

where A_1^I is the "vertex form" contribution, A_2^I is the electron wave-function renormalization after mass renormalization, and A_3^I is the convection-convection contribution[11] that comes from the virtual-photon emission and reabsorption by hadronic lines.

The M_v^I amplitude contains the large terms of logarithmic order in the electron energy and mass, is infrared divergent, finite in the ultraviolet region, and gauge-invariant under gauge transformations of the photon propagator. It has already been computed. There is no need to reproduce here the steps involved in the evaluation of the model-independent contribution. We shall limit ourselves to referring the reader to Eqs. (7) and (8) of Ref. 5.

In fact, the amplitudes that specifically contribute to the LMS are A_1^I and A_2^I. All the unknown virtual structure-dependent contributions to the LMS are contained in M_v^D, the model-dependent amplitude. Its explicit form in the 't Hooft–Feynman gauge is

$$M_v^D=\frac{\alpha}{4\pi^3 i}\frac{G}{\sqrt{2}}\sin\theta_C\int\frac{d^4k}{k^2}\frac{(2l_\mu L_\lambda-k_\alpha L_{\mu\alpha\lambda})}{(k^2-2l\cdot k+i\epsilon)}H_{\mu\lambda}^D\ , \tag{8}$$

where

$$L_{\mu\alpha\lambda}=\bar{u}_e\gamma_\mu\gamma_\alpha O_\lambda v_\nu\ . \tag{9}$$

The model dependence is contained in

$$H_{\mu\lambda}^D=\bar{u}_2\frac{T_{\mu\lambda}}{(1-s/m_W{}^2)}u_1\ , \tag{10}$$

where s is defined as $s=(q+k)^2$ and m_W is the mass of the W meson. Let us now proceed to study the model-dependent tensor $T_{\mu\lambda}$. It is regular as $k\to 0$ and transverse as $k_\mu T_{\mu\lambda}=0$. By construction,

$$T_{\mu\lambda}=\frac{1}{(k^2-2p_1\cdot k+i\epsilon)}\{k_\alpha G_{\alpha\lambda}(2p_{1\mu}-k_\mu)-\tfrac{1}{2}W_\lambda(p_2,p_1-k)[\not k,\gamma_\mu]\}+G_{\mu\lambda}(p_1,p_2,k)+X_{\mu\lambda}\ . \tag{11}$$

Our $X_{\mu\lambda}$ contains the gauge-invariant part of the off-mass-shell electromagnetic form factors of the hadrons and the residual model-dependent part of the renormalized hadronic line A_3^D.

$W_\lambda(p_1,p_2,k)$ is the weak vertex with a photon interaction and $G_{\mu\lambda}(p_1,p_2,k)$ is the matrix element for the emission of a photon of four-momentum k and polarization index μ from any internal line of the proper vertex of the weak-interaction current. It is required that[4]

$$k_\mu G_{\mu\lambda}=W_\lambda(p_2,p_1,k)-W_\lambda(p_2,p_1)\ . \tag{12}$$

The most general form of $W_\lambda(p_1,p_2,k)$ allowed by Lorentz invariance is

$$W_\lambda(p_1,p_2,k)=\hat J_1\gamma_\lambda+\frac{\hat J_2}{m_1}p_{1\lambda}+\frac{\hat J_3}{m_1}p_{2\lambda}+\frac{\hat J_4}{m_1}k_\lambda+\frac{\hat J_5}{m_1{}^3}\epsilon_{\lambda\nu\rho\alpha}p_{1\nu}p_{2\rho}k_\alpha$$

$$+\left[\frac{\hat J_6}{m_1}\gamma_\lambda+\frac{\hat J_7}{m_1{}^2}p_{1\lambda}+\frac{\hat J_8}{m_1{}^2}p_{2\lambda}+\frac{\hat J_9}{m_1{}^2}k_\lambda+\frac{\hat J_{10}}{m_1{}^4}\epsilon_{\lambda\nu\rho\alpha}p_{1\nu}p_{2\rho}k_\alpha\right]\not k\ . \tag{13}$$

Each $\hat J$ of this equation contains at the same time two form factors in the following way:

$$\hat J_i=J_i(1+\rho_{Ji}\gamma_5),\quad i=1,\ldots,10\ . \tag{14}$$

The products $\rho_{Ji}J_i$ correspond to the axial-vector form factors. The relation between these form factors and the Dirac form factors of $W_\lambda(p_1,p_2)$ is illustrated in the Appendix. It is easy to check that when $k\to 0$, $W_\lambda(p_1,p_2,k)\to W_\lambda(p_1,p_2)$. The form of Eq. (13) can be changed to other equivalent forms. Our choice makes calculations simpler.

By using Eq. (12) $G_{\mu\lambda}$ becomes

$$G_{\mu\lambda}=\frac{\hat J_4}{m_1}g_{\mu\lambda}+\frac{\hat J_5}{m_1{}^3}\epsilon_{\lambda\nu\rho\alpha}p_{1\nu}p_{2\rho}g_{\alpha\mu}+\left[\frac{\hat J_6}{m_1}\gamma_\lambda+\frac{\hat J_7}{m_1{}^2}p_{1\lambda}+\frac{\hat J_8}{m_1{}^2}p_{2\lambda}+\frac{\hat J_9}{m_1{}^2}k_\lambda+\frac{\hat J_{10}}{m_1{}^4}\epsilon_{\lambda\nu\rho\alpha}p_{1\nu}p_{2\rho}k_\alpha\right]\gamma_\mu\ . \tag{15}$$

Let us now define the most general relativistic covariant expression for $X_{\mu\lambda}$ to represent the effects of the strong interactions. This expression contains 116 structure factors. In order to preserve the transversality of $T_{\mu\lambda}$ we have to impose the gauge-invariant condition on it. This leaves us with 96 independent structure factors. There are alternative forms for the $X_{\mu\lambda}$, but using the Dirac equation and the γ matrix identities, they can be reduced to a form that is convenient for calculations. According to its gauge-invariance characteristics we bring together the several structure factors in three terms

$$X_{\mu\lambda}=X_{\mu\lambda}^E+X_{\mu\lambda}^N+X_{\mu\lambda}^{NC} \tag{16}$$

in such a way that, since

$$k_\mu X_{\mu\lambda}=0 \tag{17}$$

and since $X_{\mu\lambda}^E$ is explicitly gauge-invariant, then

$$k_\mu(X_{\mu\lambda}^N+X_{\mu\lambda}^{NC})=0\ , \tag{18}$$

but

$$k_\mu X_{\mu\lambda}^N\neq 0\ . \tag{19}$$

The general form for the explicit gauge-invariant part $X_{\mu\lambda}^E$ is

$$
\begin{aligned}
X^{E}_{\mu\lambda}=\gamma_\lambda &\left[\hat{B}'_1\Omega^{(1)}_\mu+\hat{B}'_2\Omega^{(2)}_\mu+\hat{B}'_3\Omega^{(3)}_\mu+\left[\hat{B}_5+\hat{B}'_5\frac{k\!\!\!/}{m_1}\right]\Omega^{(4)}_\mu\right]+\frac{p_{2\lambda}}{m_1}\left[\hat{C}'_1\Omega^{(1)}_\mu+\hat{C}'_2\Omega^{(2)}_\mu+\hat{C}'_3\Omega^{(3)}_\mu+\left[\hat{C}_5+\hat{C}'_5\frac{k\!\!\!/}{m_1}\right]\Omega^{(4)}_\mu\right]\\
&+\frac{p_{1\lambda}}{m_1}\left[\hat{D}'_1\Omega^{(1)}_\mu+\hat{D}'_2\Omega^{(2)}_\mu+\hat{D}'_3\Omega^{(3)}_\mu+\left[\hat{D}_5+\hat{D}'_5\frac{k\!\!\!/}{m_1}\right]\Omega^{(4)}_\mu\right]\\
&+\frac{k_\lambda}{m_1}\left\{\hat{E}'_1\Omega^{(1)}_\mu+\hat{E}'_2\Omega^{(2)}_\mu+\hat{E}'_3\Omega^{(3)}_\mu+\left[\hat{E}_5+\hat{E}'_5\frac{k\!\!\!/}{m_1}\right]\Omega^{(4)}_\mu+\frac{1}{m_1}\left[\hat{E}_2\frac{p_{2\mu}}{m_1}+\hat{E}_3\frac{p_{1\mu}}{m_1}+\left[\hat{E}_1+\hat{E}'_2\frac{p_2\cdot k}{m_1{}^2}+\hat{E}'_3\frac{p_1\cdot k}{m_1{}^2}\right]\gamma_\mu\right]\right\}\\
&-\frac{g_{\mu\lambda}}{m_1}\left[\hat{E}_2\frac{p_2\cdot k}{m_1{}^2}+\hat{E}_3\frac{p_1\cdot k}{m_1{}^2}+\left[\hat{E}_1+\hat{E}'_2\frac{p_2\cdot k}{m_1{}^2}+E'_3\frac{p_1\cdot k}{m_1{}^2}\right]\frac{k\!\!\!/}{m_1}\right]\\
&+i\epsilon_{\lambda\nu\rho\alpha}\frac{p_{1\nu}p_{2\rho}k_\alpha}{m_1^3}\left\{\hat{F}'_1\Omega^{(1)}_\mu+\hat{F}'_2\Omega^{(2)}_\mu+\hat{F}'_3\Omega^{(3)}_\mu+\left[\hat{F}_5+\hat{F}'_5\frac{k\!\!\!/}{m_1}\right]\Omega^{(4)}_\mu\right.\\
&\qquad\qquad\left.+\frac{1}{m_1}\left[\hat{F}_2\frac{p_{2\mu}}{m_1}+\hat{F}_3\frac{p_{1\mu}}{m_1}+\left[\hat{F}_1+\hat{F}'_2\frac{p_2\cdot k}{m_1{}^2}+F'_3\frac{p_1\cdot k}{m_1{}^2}\right]\gamma_\mu\right]\right\}\\
&-i\epsilon_{\lambda\mu\nu\alpha}\frac{p_{1\nu}p_{2\alpha}}{m_1{}^3}\left[\hat{F}_2\frac{p_2\cdot k}{m_1{}^2}+\hat{F}_3\frac{p_1\cdot k}{m_1{}^2}+\left[\hat{F}_1+\hat{F}'_2\frac{k\cdot p_2}{m_1{}^2}+\hat{F}'_3\frac{p_1\cdot k}{m_1{}^2}\right]\frac{k\!\!\!/}{m_1}\right]\\
&+i\epsilon_{\lambda\mu\nu\alpha}\frac{k_\nu p_{1\alpha}}{m_1{}^3}\left[\hat{H}_1+\hat{H}'_1\frac{k\!\!\!/}{m_1}\right]+i\epsilon_{\lambda\mu\nu\alpha}\frac{k_\nu p_{2\alpha}}{m_1{}^3}\left[\hat{I}_1+\hat{I}'_1\frac{k\!\!\!/}{m_1}\right],
\end{aligned}
\tag{20}
$$

where

$$
\Omega^{(1)}_\mu=\frac{\gamma_\mu k\!\!\!/}{m_1{}^2},\quad \Omega^{(2)}_\mu=\frac{1}{m_1{}^3}(p_{2\mu}k\!\!\!/-p_2\cdot k\gamma_\mu)\ ,
$$

$$
\Omega^{(3)}_\mu=\frac{1}{m_1{}^3}(p_{1\mu}k\!\!\!/-p_1\cdot k\gamma_\mu)\ ,
\tag{21}
$$

and

$$
\Omega^{(4)}_\mu=\frac{i}{m_1{}^4}\epsilon_{\mu\nu\rho\alpha}p_{1\nu}p_{2\rho}k_\alpha\ .
$$

The $X^{N}_{\mu\lambda}$ is not gauge-invariant by itself, each of its terms has a non-LMS contributing companion, proportional to k_μ,[12] that belongs to $X^{NC}_{\mu\lambda}$. The 14 elements of $X^{NC}_{\mu\lambda}$ can be ignored, and from this point on the ellipses indicate non-LMS contributions. We have

$$
\begin{aligned}
X^{N}_{\mu\lambda}=\frac{1}{m_1}\gamma_\lambda&\left[\hat{B}_2\frac{p_{2\mu}}{m_1}+\hat{B}_3\frac{p_{1\mu}}{m_1}\right]\\
&+\frac{p_{2\lambda}}{m_1{}^2}\left[\hat{C}_2\frac{p_{2\mu}}{m_1}+\hat{C}_3\frac{p_{1\mu}}{m_1}\right]\\
&+\frac{p_{1\mu}}{m_1{}^2}\left[\hat{D}_2\frac{p_{2\mu}}{m_1}+\hat{D}_3\frac{p_{1\mu}}{m_1}\right]\ .
\end{aligned}
\tag{22}
$$

As in the former cases, each coefficient is composed of two structure factors. We also assume that the strong interactions are well enough behaved to make all the form factors of the same order of magnitude. Their dimensions are already matched by dividing them by m_1. For convenience in the following detailed calculations we propose a more suitable form for the $T_{\mu\lambda}$ of Eq. (11) by dividing it into five components,

$$
T_{\mu\lambda}=\sum_{n=1}^{5}T_{\mu\lambda}(n)\ ,
\tag{23}
$$

where

$$
\begin{aligned}
&T_{\mu\lambda}(1)=X^{E}_{\mu\lambda},\quad T_{\mu\lambda}(2)=G_{\mu\lambda}(p_1,p_2,k)\ ,\\
&T_{\mu\lambda}(3)=\frac{k_\alpha G_{\alpha\lambda}(2p_{1\mu}-k_\mu)}{k^2-2p_1\cdot k+i\epsilon}\ ,\\
&T_{\mu\lambda}(4)=-\frac{1}{2}\frac{W_\lambda(p_1,p_2,k)}{k^2-2p_1\cdot k+i\epsilon}[k\!\!\!/,\gamma_\mu],\quad T_{\mu\lambda}(5)=X^{N}_{\mu\lambda}\ .
\end{aligned}
\tag{24}
$$

From now on, the integer n will be related to the nth component of $T_{\mu\lambda}$, i.e., $T_{\mu\lambda}(n)$.

Notice in Eq. (10) that the hadronic matrix element is accompanied by the factor $(1-s/m_W{}^2)^{-1}$. This fact and the compactness of notation suggests to us a redefinition of the form factors. We do this in such a way that the new structure factors S_{kn} absorb the variable s. Therefore, as functions of s they can be expanded in a power series in this variable, in the region of analyticity of the form factors. The lower-indicies k,n in the structure factors indicate the relation between these new structure factors and the form factors that appear in the $T_{\mu\lambda}(n)$.

The relation between contributing form factors is given as

$$\begin{aligned}
&S_{12}=S_{13}=J_5(1-s/m_W{}^2)^{-1} ,\\
&S_{22}=S_{23}=J_6(1-s/m_W{}^2)^{-1} ,\\
&S_{32}=S_{33}=J_7(1-s/m_W{}^2)^{-1} ,\\
&S_{42}=S_{43}=J_8(1-s/m_W{}^2)^{-1} ,\\
&S_{52}=S_{53}=J_{10}(1-s/m_W{}^2)^{-1} ,\\
&S_{15}=B_2(1-s/m_W{}^2)^{-1},\quad S_{25}=B_3(1-s/m_W{}^2)^{-1} ,\\
&S_{35}=C_2(1-s/m_W{}^2)^{-1},\quad S_{45}=C_3(1-s/m_W{}^2)^{-1} ,\\
&S_{55}=D_2(1-s/m_W{}^2)^{-1},\quad S_{65}=D_3(1-s/m_W{}^2)^{-1} .
\end{aligned} \tag{25}$$

The model-dependent LMS contribution of order α to the total virtual transition probability arises from the interference of M_0 with each of the n elements of $T_{\mu\lambda}$ in A^D.

Let us now look at the inner bremsstrahlung. The decay rate for this real-photon emission case is given by

$$\Gamma_B=\int \frac{1}{(2\pi)^8}\frac{1}{2m_1}\delta^4(p_1-p_2-k-p_\nu-l)\times\frac{d^3p_2}{2E_2}\frac{d^3p_\nu}{2E_\nu}\frac{d^3k}{2\omega}\frac{d^3l}{2E}\frac{1}{2}\sum_{\text{pol}}|M_B|^2 . \tag{26}$$

Just as in the virtual case we shall proceed to perform a separation of the model-dependent and -independent parts as follows:

$$M_B=M_B^I+M_B^D . \tag{27}$$

The model-independent term contains the infrared divergence. Using the previous definitions, we obtain for the amplitude

$$M_B^I=\frac{eG}{\sqrt{2}}\sin\theta_C\epsilon_\mu\times\left[L_\lambda H_{\mu\lambda}^I+\frac{(2l_\mu L_\lambda+k_\alpha L_{\mu\alpha\lambda})}{(l+k)^2-m_e{}^2+i\epsilon}\frac{H_\lambda}{(1-s/m_W{}^2)}\right] . \tag{28}$$

The first term contains the infrared-divergent part which is extracted from the hadron-line contribution,[13] and the second term corresponds to the emission of the real photon from the line of the electron; ϵ_μ is the photon polarization four-vector. This second term is the one that contains the $(l\cdot k)^{-1}$ factor which is responsible for the LMS contribution.

For the model-dependent amplitude we have

$$M_B^D=\frac{eG}{\sqrt{2}}\sin\theta_C\epsilon_\mu L_\lambda H_{\mu\lambda}^D . \tag{29}$$

The former $T_{\mu\lambda}(p_1,p_2,k)$ holds true if the virtual photon is adequately substituted by the real photon ($k^2=0$, $\epsilon\cdot k=0$). The infrared-convergence and gauge-invariance properties are preserved.

The squared model-independent amplitude contributions to the total transition probability have already been computed for soft photons [see Eqs. (14) and (15) Ref. 5]. As is well known, the infrared divergence of the virtual part is canceled by the infrared divergence of the inner bremsstrahlung. The interference of the M_B^D and M_B^I gives the additional LMS contributions. The squared model-dependent amplitude does not contain any lepton propagator, therefore, we do not expect any LMS contribution from this term.

III. A PARALLELISM BETWEEN MODEL-DEPENDENT AMPLITUDES

In order to proceed with the detailed calculation we have to square the amplitudes and sum over all polarizations. The result after a fairly long but straightforward calculation[14] of both traces (virtual and bremsstrahlung) shows invariance under each of the following transformations:

$$\begin{aligned}
&J_2^\dagger\leftrightarrow J_3^\dagger,\quad \rho_{J_2^\dagger}J_2^\dagger\leftrightarrow\rho_{J_3^\dagger}J_3^\dagger ,\\
&J_7\leftrightarrow J_8,\quad \rho_{J7}J_7\leftrightarrow\rho_{J8}J_8 ,\\
&C_2\leftrightarrow D_2,\quad \rho_{C2}C_2\leftrightarrow D_2\rho_{D2} ,\\
&C_3\leftrightarrow D_3,\quad \rho_{C3}C_3\leftrightarrow D_3\rho_{D3} .
\end{aligned} \tag{30}$$

And $m_2\rightarrow m_2$ if

$$J_i^\dagger\leftrightarrow J_i^\dagger\rho_{J_i^\dagger} \text{ combined with } \begin{cases} J_k\leftrightarrow J_k\rho_{Jk} ,\\ B_j\leftrightarrow -B_j\rho_{Bj} ,\\ C_j\leftrightarrow C_j\rho_{Cj} ,\\ D_j\leftrightarrow D_j\rho_{Dj} .\end{cases} \tag{31}$$

The values for the lower indices are $i=1,2,3$, $k=5,6,7,8,10$, and $j=2,3$. The traces should be integrated over the kinematical variables. For the virtual case, the indicated order is d^3k, d^3p_2, d^3p_ν, and d^3l, for the bremsstrahlung case we integrate successively over d^3p_2, d^3p_ν, photon-lepton angle $d\Omega_k$, d^3l, and the energy of the photons in this order.

A. Virtual case

The integration over the virtual-photon four-momentum is performed, using Feynman[15] technique and a Wick rotation in order to go over into Euclidean four-dimensional space. The approach we use closely parallels the one explained in Ref. 3; we divide the domain of integration into two parts, characterized by (a) $k''^2=k_0''^2+\vec{k}''^2\leq m_1{}^2$ and (b) $k''^2>m_1{}^2$.

In the domain (a) the LMS appears and the restriction implies that we are in the region of analyticity of the new form factors, with respect to the variable s. The domain (b) cannot contribute to the LMS because the lower limit of k''^2 acts as an infrared cutoff, so that the corresponding integral is finite as $m_e\rightarrow 0$. After the integration over

d^3p_2 which fixes the value of E_ν,[16] we find the differential decay rate, which is divided into five elements for convenience,

$$d\Gamma_v = \omega^v(E,\Omega_l,\Omega_\nu)dE\, d\Omega_l d\Omega_\nu \tag{32}$$

and

$$\omega^v(E,\Omega_l,\Omega_\nu) = \sum_{n=1}^{5} \omega_n^v(E,\Omega_l,\Omega_\nu) . \tag{33}$$

Each of these elements is associated with the components $T_{\mu\lambda}(n)$ of the model-dependent tensor.

Let us recall that in this case we deal with the interference of M_0 and $T_{\mu\lambda}(n)$.

Specifically, the contribution of the nth term of the decay rate is

$$\omega_n^v(E,\Omega_l,\Omega_\nu)dE\, d\Omega_l d\Omega_\nu = -\frac{\alpha}{\pi}\frac{G^2}{2\pi^3}\sin^2\theta_C \ln\left[\frac{m_e}{m_1}\right]\int_{x'=0}^{1}\frac{E(1-x')l\, dE(Em-E)^2}{[1-E/m_1+(\hat{p}_\nu\cdot\vec{l}\,)/m_1]^3}\frac{d\Omega_\nu}{4\pi}\frac{d\Omega_l}{4\pi}C_n^v(l,p_\nu)dx' + \cdots . \tag{34}$$

The $C_n^v(l,p_\nu)$ is compactly written as a linear combination of bilinear products of structure factors where the coefficients are denoted by $a(n,S_{kn}J^\dagger)$. Each coefficient is characterized by n, and the product of the form factors to which it belongs. We express $C_n^v(l,p_\nu)$ as follows:

$$C_n^v(l,p_\nu) = \sum_{i=1,k}^{i=3} \mathrm{Re}(a(n,S_{kn}J_i^\dagger)S_{kn}J_i^\dagger + a(n,S_{kn}J_i^\dagger\rho_{J_i^\dagger})S_{kn}J_i^\dagger\rho_{J_i^\dagger}) + \text{symmetric} . \tag{35}$$

The " + symmetric" at the end of this expression means that one should add to the given expression all the additional terms obtained from it by making the substitutions indicated in Eqs. (30) and (31).

In the following we shall only mention the nonvanishing coefficients. The resulting 42 coefficients for $n=2$ and an equivalent number for $n=3$, may be obtained by using Eqs. (30), (31), and the following list:

$$\begin{aligned}
a(2,J_5J_1^\dagger) &= \left[1+\frac{m_2}{m_1}\right]a_5 ,\\
a(2,J_5J_2^\dagger) &= \left[1+\frac{m_2}{m_1}-\left[\frac{E}{m_1}+\frac{E_\nu}{m_1}\right]\right]a_5 ,\\
a(2,J_6J_1^\dagger) &= -\left[\left[1+\frac{m_2}{m_1}\right]\left[\frac{l\cdot p_\nu}{m_1{}^2}\right]\right.\\
&\quad\left.-\left[1-\frac{m_2}{m_1}\right]\frac{l\cdot(2p_1-p_\nu)}{m_1{}^2}\right]a_6 ,\\
a(2,J_6J_1^\dagger\rho_{J_1^\dagger}) &= a(2,J_6J_1^\dagger) ,\\
a(2,J_6J_2^\dagger) &= \frac{EE_\nu}{m_1{}^2}\left[2-\frac{l\cdot p_\nu}{EE_\nu}\right]a_6 ,\\
a(2,J_6J_2^\dagger\rho_{J_2^\dagger}) &= a(2,J_6J_2^\dagger) ,\\
a(2,J_7J_1^\dagger) &= \frac{l\cdot(2p_1-p_\nu)}{m_1{}^2}a_7 ,\\
a(2,J_7J_2^\dagger) &= \frac{1}{2}\left[\left[1+\frac{m_2}{m_1}\right]\frac{l\cdot(2p_1-p_\nu)}{m_1{}^2}\right.\\
&\quad\left.-\left[1-\frac{m_2}{m_1}\right]\frac{l\cdot p_\nu}{m_1{}^2}\right]a_7 ,\\
a(2,J_7J_1^\dagger\rho_{J_1^\dagger}) &= \frac{l\cdot p_\nu}{m_1{}^2}a_7 ,\\
a(2,J_{10}J_1^\dagger) &= \frac{l\cdot(2p_1-p_\nu)}{m_1{}^2}a_{10} ,\\
a(2,J_{10}J_2^\dagger) &= \frac{1}{2}\left[\left[1+\frac{m_2}{m_1}\right]\frac{l\cdot(2p_1-p_\nu)}{m_1{}^2}\right.\\
&\quad\left.-\left[1-\frac{m_2}{m_1}\right]\frac{l\cdot p_\nu}{m_1{}^2}\right]a_{10} ,\\
a(2,J_{10}J_1^\dagger\rho_{J_1^\dagger}) &= \frac{l\cdot p_\nu}{m_1{}^2}a_{10} ,
\end{aligned} \tag{36}$$

where we define

$$\begin{aligned}
a_5 &= \left[2-\frac{l\cdot p_\nu}{EE_\nu}\right]\left[-\frac{l\cdot p_\nu}{m_1{}^2}\right] ,\\
a_6 &= \frac{l\cdot p_\nu}{EE_\nu} ,\\
a_7 &= \left[2-\frac{l\cdot p_\nu}{EE_\nu}\right] ,\\
a_{10} &= x'a_5 ,
\end{aligned} \tag{37}$$

while for $n=3$,

$$\begin{aligned}
a(2,J_i^\dagger S_{k2}) &= -\beta a(3,J_i^\dagger S_{k3}) ,\\
a(2,J_i^\dagger\rho_{J_i^\dagger}S_{k2}) &= -\beta a(3,J_i^\dagger\rho_{J_i^\dagger}S_{k3}) .
\end{aligned} \tag{38}$$

We recall that $\beta = |\,\vec{l}\,|/E$.

The evaluation of the d^3p_ν integration yields, for the $n=5$ case,

$$\omega_5^{\nu}(E,\Omega_l)dE\,d\Omega_l=-\frac{2\alpha}{\pi}\frac{G^2}{2\pi^3}\sin^2\theta_C\ln\left[\frac{m_e}{m_1}\right]\frac{E}{m_1}\int_0^1\frac{E(1-x')l\,dE(E_m-E)^2}{(1-2E/m_1)^2}\frac{d\Omega_l}{4\pi}C_5^{\nu}(E)dx'+\cdots\ . \tag{39}$$

The $C_5^{\nu}(l,p_\nu)$ of Eq. (34) was integrated taking into account the phase space, becoming $C_5^{\nu}(E)$:

$$C_5^{\nu}(E)=\int\frac{d\Omega_\nu}{4\pi}\frac{C_5^{\nu}(l,p_\nu)}{[1-E/m_1+(\hat{p}_\nu\cdot\vec{l}\,)/m_1]^3}\ . \tag{40}$$

The 40 contributing coefficients that are contained in $C_5^{\nu}(E)$ become

$$\begin{aligned}
a(5,B_2J_1^\dagger)={}&2\left[1-\frac{E}{m_1}\right]-\left[\frac{E}{m_1}+\frac{m_2}{m_1}\right]\left[1-\frac{E}{m_1}+\beta^2\frac{E}{m_1}\right]-\left[3-\frac{E}{m_1}-\frac{m_2}{m_1}\right]H+\beta\left[3-\frac{2E}{m_1}-\frac{2m_2}{m_1}\right]J\\
&+\beta^2\left[\frac{E}{m_1}+\frac{m_2}{m_1}\right]K+L-2\beta M+\beta^2N\ ,\\
a(5,B_2J_1^\dagger\rho_{J_1^\dagger})={}&-\frac{E}{m_1}\left[1-\frac{E}{m_1}+\beta^2\frac{E}{m_1}\right]+\left[1+\frac{E}{m_1}\right]H-\beta\left[1+\frac{2E}{m_1}\right]J+\beta^2\frac{E}{m_1}K-L+2\beta M-\beta^2N\ ,\\
a(5,B_2J_2^\dagger)={}&\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{E}{m_1}-\beta^2\frac{E}{m_1}-H+\beta^2K\right]\ ,\\
a(5,B_3J_1^\dagger)={}&2\left[1-\frac{E}{m_1}\right]-\left[\frac{E}{m_1}+\frac{m_2}{m_1}\right]\left[1-\frac{E}{m_1}+\frac{E\beta^2}{m_1}\right]-H+\beta J\ ,\\
a(5,\beta_3J_1^\dagger\rho_{J_1^\dagger})={}&-\frac{E}{m_1}+\frac{E^2}{m_1{}^2}(1-\beta^2)+H-\beta J\ ,\\
a(5,B_3J_2^\dagger)={}&\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{E}{m_1}-\frac{E\beta^2}{m_1}\right]\ ,\\
a(5,C_2J_1^\dagger)={}&\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{E}{m_1}-\beta^2\frac{E}{m_1}-H+\beta^2K\right]\ ,\\
a(5,C_2J_2^\dagger)={}&\frac{1}{2}\left[\left[1+\frac{m_2}{m_1}-\frac{E}{m_1}\right]\left[1-\frac{E}{m_1}-\beta^2\frac{E}{m_1}-H+\beta^2K\right]-H-\beta J+L-\beta^2N\right].,\\
a(5,C_3J_1^\dagger)={}&\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{E}{m_1}-\frac{E\beta^2}{m_1}\right]\ ,\\
a(5,C_3J_2^\dagger)={}&\frac{1}{2}\left[\left[1+\frac{m_2}{m_1}-\frac{E}{m_1}\right]-\frac{E}{m_1}(1+\beta^2)\left[1+\frac{m_2}{m_1}-\frac{E}{m_1}\right]-H-\beta J\right]\ ,
\end{aligned} \tag{41}$$

with the complementary definitions,

$$H(E)=H=\frac{(E_m-E)}{m_1}\frac{1}{(1-2E/m_1)}\frac{1}{3}\left[3\left[1-\frac{E}{m_1}\right]^2+\frac{E^2\beta^2}{m_1{}^2}\right],$$

$$J(E)=J=\frac{(E_m-E)}{m_1}\frac{1}{(1-2E/m_1)}\frac{E\beta}{m_1}(-\tfrac{4}{3})\left[1-\frac{E}{m_1}\right],$$

$$K(E)=K=\frac{(E_m-E)}{m_1}\frac{1}{(1-2E/m_1)}\frac{1}{3}\left[\frac{3E^2\beta^2}{m_1}+\left[1-\frac{E}{m_1}\right]^2\right],\tag{42}$$

$$L(E)=L=\frac{(E_m-E)^2}{m_1{}^2}\frac{1}{(1-2E/m_1)^2}\left[1-\frac{E}{m_1}\right]\left[1-\frac{2E}{m_1}+\frac{2E^2}{m_1{}^2}\right].$$

$$M(E)=M=\frac{(E_m-E)^2}{m_1{}^2}\frac{1}{(1-2E/m_1)^2}\left[-\frac{E\beta}{3m_1}\right]\left[5-\frac{10E}{m_1}+\frac{6E^2}{m_1{}^2}\right],$$

$$N(E)=N=\frac{(E_m-E)^2}{m_1{}^2}\frac{1}{(1-2E/m_1)^2}\frac{1}{3}\left[1-\frac{E}{m_1}\right]\left[1-\frac{2E}{m_1}+\frac{6E^2}{m_1{}^2}\right].$$

At this stage we realize that the factor $E(1-x')$ appears in all our results, and that a suitable variable transformation is the key that will allow us to compare easily the final virtual result with the final bremsstrahlung result. Then let us define a variable s, such that $s=E(1-x')$.

B. Inner-bremsstrahlung case

For the differential decay rate, after the d^3p_2 integration which fixes the value of E_ν,[17] we have the expression

$$d\Gamma_B=\sum_{n=1}^{5}\omega_n^B(E,\Omega_l,\omega_0,\Omega_k,\Omega_\nu)dE\,d\omega_0 d\Omega_l d\Omega_k d\Omega_\nu \tag{43}$$

and we obtain

$$\omega_n^B(E,\Omega_l,\omega_0,\Omega_k,\Omega_\nu)dE\,d\omega_0 d\Omega_l d\Omega_k d\Omega_\nu$$
$$=-\frac{\alpha}{\pi}\frac{G^2}{2\pi^3}\sin^2\theta_C\beta\,dE\,dz\frac{d\Omega_l}{4\pi}dx\left[\frac{d\phi_k}{4\pi}\frac{d\Omega_\nu}{4\pi}\frac{E(E_m-z)^2C_n^B(l,p_\nu,k)}{(1-\beta x)(1-z+\hat{p}_\nu\cdot(\vec{l}+\vec{k})^3}\right]+\cdots . \tag{44}$$

We introduced the definitions

$$z=E+\omega_0 \text{ and } l\cdot k=\omega_0 E(1-\beta x)\ , \tag{45}$$

ω_0 is the energy of the real photon and x is related to the photon lepton angle, $x=\cos\theta_{pl}$.

$C_n^B(l,p_\nu,k)$ is also compactly written as a linear combination of bilinear products of structure factors. The bremmstrahlung coefficients are $b(n,S_{kn}J^\dagger)$ characterized in the same way as the virtual ones. The expression for the $C_n^B(l,p_\nu,k)$ is

$$C_n^B(l,k,p_\nu)=\sum_{i=1,k}^{i=3}\mathrm{Re}[b(n,S_{kn}J_i^\dagger)S_{kn}J_i^\dagger+b(n,S_{kn}J_i^\dagger\rho_{J_i^\dagger})S_{kn}J_i^\dagger\rho_{ji^\dagger}]+\text{symmetric}\ . \tag{46}$$

The " + symmetric" has the same meaning as before. The nonvanishing coefficients are

$$
\begin{aligned}
&b(2,J_5J_1^\dagger)=\left[1+\frac{m_2}{m_1}\right]b_5,\quad b(2,J_5J_2^\dagger)=\left[1+\frac{m_2}{m_1}-\frac{E}{m_1}-\frac{\omega_0}{m_1}-\frac{E_\nu}{m_1}\right]b_5\,,\\
&b(2,J_6J_1^\dagger)=-\left[\left[1+\frac{m_2}{m_1}\right]\frac{l\cdot p_\nu}{Em_1}-\left[1-\frac{m_2}{m_1}\right]\frac{l\cdot(2p_1-p_\nu)}{Em_1}\right]b_6,\quad b(2,J_6J_1^\dagger\rho_{J1^\dagger})=b(2,J_6J_1^\dagger)\,,\\
&b(2,J_6J_2^\dagger)=\left[\frac{E_\nu}{m_1}\left[2-\frac{l\cdot p_\nu}{EE_\nu}\right]\right]b_6,\quad b(2,J_6J_2^\dagger\rho_{J2^\dagger})=b(2,J_6J_2^\dagger)\,,\\
&b(2,J_7J_1^\dagger)=\frac{(l+k)\cdot(2p_1-p_\nu)}{m_1{}^2}b_7\,,\\
&b(2,J_7J_2^\dagger)=\left[\frac{1}{2}\left[1+\frac{m_2}{m_1}\right](l+k)\cdot(2p_1-p_\nu)-\tfrac{1}{2}(l+k)\cdot p_\nu\left[1-\frac{m_2}{m_1}\right]\right]\frac{b_7}{m_1{}^2}\,,\\
&b(2,J_7J_1^\dagger\rho_{J1^\dagger})=(l+k)\cdot p_\nu\frac{b_7}{m_1{}^2},\quad b(2,J_{10}J_1^\dagger)=(l+k)\cdot(2p_1-p_\nu)\frac{b_{10}}{m_1{}^2}\,,\\
&b(2,J_{10}J_2^\dagger)=\left[\frac{1}{2}\left[1+\frac{m_2}{m_1}\right](l+k)\cdot(2p_1-p_\nu)-\frac{1}{2}\left[1-\frac{m_2}{m_1}\right](l+k)\cdot p_\nu\right]\frac{b_{10}}{m_1{}^2}\,,\\
&b(2,J_{10}J_1^\dagger\rho_{J1^\dagger})=(l+k)\cdot p_\nu\frac{b_{10}}{m_1{}^2}\,.
\end{aligned}
\tag{47}
$$

For brevity we introduced b_5, b_6, b_7, and b_{10}, which are defined as

$$b_5=\left[2-\frac{l\cdot p_\nu}{EE_\nu}\right]\left[-\frac{p_\nu\cdot(l+k)}{m_1{}^2}\right],\quad b_6=\frac{p_\nu\cdot(l+k)}{E_\nu m_1},\quad b_7=2-\frac{l\cdot p_\nu}{EE_\nu},\quad b_{10}=\left[2-\frac{l\cdot p_\nu}{EE_\nu}\right]\frac{k\cdot p_\nu}{m_1{}^2}\,. \tag{48}$$

For $n=3$ the surviving coefficients are such that

$$b(3,J_i^\dagger S_{k3})=-b(2,J_i^\dagger S_{k2}),\quad b(3,J_i^\dagger\rho_{J_i^\dagger}S_{k3})=-b(2,J_i^\dagger\rho_{J_i^\dagger}S_{k2})\,. \tag{49}$$

It is obvious now that in Eq. (24) the contribution of $T_{\mu\lambda}(3)$ cancels the contribution of $T_{\mu\lambda}(2)$. This result does not come as a surprise. None of the explicit gauge-invariant terms contribute to the LMS.

Let us recall that

$$k_\mu T_{\mu\lambda}(1)=0,\quad k_\mu[T_{\mu\lambda}(2)+T_{\mu\lambda}(3)]=0,\quad k_\mu T_{\mu\lambda}(4)=0 \tag{50}$$

and the coefficients for $n=1$ and $n=4$ vanish.

After integrating over $d\Omega_\nu$ we obtain for the $n=5$ case

$$\omega_5^B(E,\Omega_l,\omega_0,\Omega_k)dE\,d\omega_0 d\Omega_l d\Omega_k=-\frac{2\alpha}{\pi}\frac{G^2}{2\pi^3}\sin^2\theta_C\frac{E}{m_1}z\frac{(E_m-z)^2}{(1-\beta x)}\frac{\beta\,dE\,dz\,C_5^B(z,l\cdot k)}{[1-2z/m_1+(2l\cdot k)/m_1{}^2]^2}\frac{d\Omega_l}{4\pi}\frac{d\Omega_k}{4\pi}+\cdots\,. \tag{51}$$

The contributing coefficients contained in $C_5^B(z,l\cdot k)$ are

$$
\begin{aligned}
b(5,B_2J_1^\dagger)&=2\left[1-\frac{z}{m_1}\right]-\left[\frac{z}{m_1}+\frac{m_2}{m_1}\right]-\left[3-\frac{z}{m_1}-\frac{m_2}{m_1}\right]H_B+\left[3-\frac{2z}{m_1}-\frac{2m_2}{m_1}\right]J_B\\
&\quad+\left[\frac{z}{m_1}+\frac{m_2}{m_1}\right]K_B+L_B-2M_B+N_B\ ,\\
b(5,B_2J_1^\dagger\rho_{J_1^\dagger})&=-\frac{z}{m_1}+\left[1+\frac{z}{m_1}\right]H_B-\left[1+\frac{2z}{m_1}\right]J_B+\frac{z}{m_1}K_B-L_B+2M_B-N_B\ ,\\
b(5,B_2J_2^\dagger)&=\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{2z}{m_1}-H_B+K_B\right],\quad b(5,B_3J_1^\dagger)=2\left[1-\frac{z}{m_1}\right]-\left[\frac{z}{m_1}+\frac{m_2}{m_1}\right]-H_B+J_B\ ,\\
b(5,B_3J_1^\dagger\rho_{J_1^\dagger})&=-\frac{z}{m_1}+H_B-J_B,\quad b(5,B_3J_2^\dagger)=\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{2z}{m_1}\right],\\
b(5,C_2J_1^\dagger)&=\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{2z}{m_1}-H_B+K_B\right],\\
b(5,C_2J_2^\dagger)&=\frac{1}{2}\left[\left[1+\frac{m_2}{m_1}-\frac{z}{m_1}\right]\left[1-\frac{2z}{m_1}-H_B+K_B\right]-H_B-J_B+L_B-N_B\right],\\
b(5,C_3J_1^\dagger)&=\frac{1}{2}\left[1+\frac{m_2}{m_1}\right]\left[1-\frac{2z}{m_1}\right],\quad b(5,C_3J_2^\dagger)=\frac{1}{2}\left[\left[1+\frac{m_2}{m_1}-\frac{z}{m_1}\right]-2\left[1+\frac{m_2}{m_1}-\frac{z}{m_1}\right]\frac{z}{m_1}-H_B-J_B\right].
\end{aligned}
\tag{52}
$$

We have checked that, neglecting[14] terms of order m_e^2, the relation between $H_B, J_B, \ldots, M_B, N_B$ and the virtual $H, J, \ldots, M, N$, respectively, is

$$
\{H_B,J_B,K_B\}=\{H,J,K\}_z\left[1-\frac{2z}{m_1}\right]\left[1-\frac{2z}{m_1}+\frac{2l\cdot k}{m_1^2}\right]^{-1}+\cdots
$$

and (53)

$$
\{L_B,M_B,N_B\}=\{L,M,N\}_z\left[1-\frac{2z}{m_1}\right]^2\left[1-\frac{2z}{m_1}+\frac{2l\cdot k}{m_1^2}\right]^{-2}+\cdots\ .
$$

The integration over the photon-lepton angle gives the relation that exists between H_B, J_B, K_B, etc., and the virtual H, J, K, etc., respectively.

$$
\int_{-1}\frac{dx}{(1-\beta x)(c+dx)^2}\{1,H_B,J_B,\ldots,N_B\}=-\frac{2\ln(m_e)}{\beta(1-2z/m_1)^2}\{1,H,J,\ldots,N\}_z+\cdots\ ,
\tag{54}
$$

where

$$
c=1-\frac{2z}{m_1}+\frac{2E\omega_0}{m_1^2},\quad d=-\frac{2E\omega_0\beta}{m_1^2}
$$

are kinematical coefficients. It is worthwhile to mention that all the bremsstrahlung results turn out to be in exact correspondence with the results of the virtual case. The previous virtual factor $l(1-x')$ plays the role of $(l+k)$ in the bremsstrahlung corrections. This establishes the parallelism between both corrections, which is very useful, because in this way one can compare terms with the same form factors, one by one.

IV. CONCLUSION

We are already in a position to summarize our results. It is interesting to point out that the only surviving partial contribution in each case comes from the $n=5$ term.[18]

The discussion of the bremsstrahlung case leads us, according to Eqs. (43), (51), and (54), to

$$
\begin{aligned}
\Gamma^B&=\frac{2\alpha}{\pi}\frac{G^2}{2\pi^3}\sin^2\theta_C\\
&\quad\times\int_0^{E_m}\frac{z\,dz}{m_1}\int_0^{E}E\,dE\,\ln(m_e)\frac{(E_m-z)^2}{(1-2z/m_1)^2}C_5^B(z)\\
&\quad+\cdots\ .
\end{aligned}
\tag{55}
$$

Here

$$
C_5^B(z)=\int\frac{C_5^B(z,x)dx}{(1-\beta x)(c+dx)^2}\ .
\tag{56}
$$

Similarly, from Eqs. (32), (33), and (39), the result for the virtual correction to the rate is

$$\Gamma^{v}=-\frac{2\alpha}{\pi}\frac{G^2}{2\pi^3}\sin^2\theta_C\ln(m_e)\times\int_0^{E_m}\frac{E\,dE}{m_1}\int_0^{E}s\,ds C_5^v(E)\frac{(E_m-E)^2}{(1-2E/m_1)^2}+\cdots . \tag{57}$$

In searching for a suitable correlation between variables we find that the E of Eq. (57) has all the features of the z of Eq. (55). On the other hand, we identify the E of Eq. (55) with the variable s of Eq. (57). Relying on the former identification we propose the substitutions in Eq. (55), $z\rightarrow E$ and $E\rightarrow s$.

We notice by inspection of Eqs. (40), (41), (56), and (52) that

$$C_5^B(z\rightarrow E)=C_5^v(E)+\cdots . \tag{58}$$

Now the theorem becomes obvious by adding these two partial decays. Clearly Eq. (55) cancels Eq. (57). Let us finally state the theorem:

The coefficient of the logarithmic lepton-mass-singularity in the radiative corrections to order α to the total semileptonic baryon β-decay probability is not affected by the strong interactions and can therefore be rigorously computed.

This theorem holds true when the lepton is an electron or a muon. The strong interactions may be present in terms proportional to m_e^2, terms which were neglected in our calculation, because they do not show LMS.

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APPENDIX

We give the relations between the Dirac form factors[10] and the J_i form factors:

$$J_1=F_1,\quad J_2=F_2+F_3,\quad J_3=-F_3 ,$$

$$J_1\rho_{J1}=-F_1\rho_{F1},\quad J_2\rho_{J2}=F_2\rho_{F2}+\rho_{F3}F_3 ,$$

$$J_3\rho_{J3}=-F_3\rho_{F3} ,$$

$$J_1\rho_{J1}=-G_1,\quad J_2\rho_{J2}=G_2+G_3 ,$$

$$J_3\rho_{J3}=-G_3$$

and

$$J_1=f_1+\left[1+\frac{m_2}{m_1}\right]f_2 ,$$

$$\rho_{J1}J_1=-\left[g_1-\left[1-\frac{m_2}{m_1}\right]g_2\right] ,$$

$$J_2=f_3-f_2,\quad J_2\rho_{J2}=g_3-g_2 ,$$

$$J_3=-(f_2+f_3),\quad J_3\rho_{J3}=-(g_2+g_3) .$$

*Also at Escuela Superior de Física y Matemáticas del Instituto Politécnico Nacional, Mexico.

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[11]D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961); N. Meister and D. R. Yennie, Phys. Rev. 130, 1210 (1963).

[12]k_μ prevents the appearance of the LMS because

$$\bar{u}_e \not{k}\frac{1}{\not{l}+\not{k}-m_e}\cdots=\bar{u}_e\cdots .$$

[13]The explicit form of $H^{I}_{\mu\lambda}$ is

$$H^{I}_{\mu\lambda}=(2p_{1\mu}-k_\mu)H_\lambda/(\lambda^2-2p_1\cdot k+i\epsilon) ;$$

see [1] and [2] of Eq. (13) in Ref. 5.

[14]Terms proportional to m_e^2, $l\cdot k$, and k^2 are neglected since they do not contribute to the LMS.

[15]R. P. Feynman, Phys. Rev. 76, 769 (1949).

[16]From four-momentum conservation we find for the antineutrino energy that

$$E_\nu^v=(E_m-E)(1-E/m_1+E\beta\hat{l}\cdot\hat{p}_\nu/m_1)^{-1} ,$$

where $E_m=(m_1^2+m_e^2-m_2^2)/2m_1$, $\hat{l}$ and $\hat{p}_\nu$ are unit vectors along the directions of the electron and antineutrino three-momentum, respectively.

[17]When the real photon is included as an outgoing particle, four-momentum conservation implies that

$$E_\nu^B=\left[E_m-z+\frac{k\cdot l}{m_1}\right]\left[1-\frac{z}{m_1}+\hat{p}_\nu\cdot\frac{(\vec{l}+\vec{k})}{m_1}\right]^{-1} ,$$

where $z=E+\omega_0$ and ω_0 is the real-photon energy.

[18]The sum of the virtual $T_{\mu\lambda}(2)$ plus $T_{\mu\lambda}(3)$ coefficients is proportional to m_e^2, which is dropped.