Analytical approach to the overshoot phenomenon for a coasting beam in particle accelerators

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A nonlinear perturbation theory is developed for longitudinal instabilities of a coasting beam in particle accelerators. In contrast to the linear theory, the present perturbation approach to the Vlasov equation demonstrates that the part of the particle distribution function averaged over the azimuthal angle around the ring has the time dependence of second order in perturbed quantities. A set of differential equations in time is derived for the average distribution (or the energy spread) and the perturbation amplitude. A mapping of the time derivative of the energy spread onto the complex impedance plane is obtained and it shows that the region where the energy spread increases exists even in the stable area. The energy spread for an initially unstable beam is shown to increase beyond the threshold of the stability and eventually to converge to a final value determined by the initial energy spread and the threshold value. Quantitative agreements are obtained between the experimental results at the CERN ISR and the theoretical overshoot formula in the case where the initial energy spread is close to the threshold.

I. INTRODUCTION

The interaction of a high-density coasting beam with its environments in circular accelerators or storage rings leads to various collective phenomena of the beam. Electromagnetic fields created by the beam in vacuum chambers, resistive walls, etc., act back on the particles in the beam and change their distribution. If the initial disturbance is enhanced by this beam-environment interaction, the beam is called unstable. Such collective phenomena are described by the Vlasov equation for the particle distribution function in the phase space, and beam dynamics is examined by solving it in a self-consistent manner. Since the self-excited field is a function of the particle distribution function f, the Vlasov equation is nonlinear in f; therefore it is very difficult to solve the complete Vlasov equation exactly. Instead, the linearized Vlasov equation for the perturbed distribution function is generally used to calculate its initial growth rate or stability limit.[1–3] Keil and Messerschmid[4] have demonstrated that the thresholds of the instability predicted by the linear theory agree well with those obtained from their computer simulation which includes nonlinear beam dynamics.

Once an instability occurs, after a sufficient time, the growing amplitude of the perturbation might reach the region where nonlinear terms in the Vlasov equation become important and the linearization is no longer valid. The evolving disturbance will diffuse the particle distribution due to these nonlinear effects. Resulting increase in the energy spread of the beam in turn alters linear properties of the instability, e.g., its growth rate. However the linear theory cannot predict whether the energy spread will stop blowing up or not when the energy spread exceeds the threshold value of the stability. Experiments with the experimental cavity at the CERN ISR[5] and computer simulation by Keil and Messerschmid[4] have shown that the energy spread continues to increase even after it passes the threshold. In general, final energy spreads are larger for a beam with smaller initial spread. This phenomenon is called an overshoot phenomenon and first predicted by Dory[6] using computer simulations. The following empirical formula (Dory's law) for the final energy spread σ_f is commonly used in the case where the initial energy spread σ_i is not too small compared with the threshold σ_{th}:

$$\sigma_f \sigma_i \approx \sigma_{\text{th}}^2 \ .$$

An explanation of the overshoot phenomenon for a bunched beam has been proposed by Ruggiero[7] with a model in which several resonating unstable modes are overlapped in the longitudinal phase space. His method is, however, not based on the nonlinearity of the Vlasov equation and the energy spread of a bunched beam is determined only by the contribution of the perturbed distribution. Bonch-Osmolovskii *et al.*[8] have investigated the time evolution of the particle distribution in terms of solving the Vlasov equation numerically.

When the amplitude of the initial disturbance is small and the instability does not develop too rapidly, we can assume that the nonlinearity modifies the particle distribution at a rate much smaller than the linear response of the system. Under this adiabaticity assumption, the perturbation method developed for the linear theory can be employed even in the nonlinear regime. In this paper, we shall develop a nonlinear perturbation theory for instabilities of a coasting beam and present an explanation of the overshoot phenomenon in the blowup of the energy spread. The mathematical technique used is familiar in the quasilinear theory[9–11] of plasma waves.

The plan of the paper is as follows. In Sec. II the nonlinear Vlasov equation is formulated in parallel with the conventional linear theory. The dispersion relation is derived in the usual manner. In Sec. III a differential equation in time for the energy spread is given together with a subsidiary equation for the perturbation amplitude. In

Sec. IV we solve the set of equations and discuss the overshoot phenomenon together with the experimental results at the ISR. Our conclusions are presented in Sec. V.

II. NONLINEAR VLASOV EQUATION

The distribution function f of circulating particles in the longitudinal phase space is expressed by using the azimuthal angle θ around the machine circumference and its canonical conjugate variable defined by

$$w=2\pi\int_{E_c}^{E}\frac{dE}{\omega}\ , \tag{2.1}$$

where E and ω is the energy and the revolution frequency of a particle, respectively, and E_c is the energy of the center of the distribution. Since a relative energy spread is rather small and the correction from the revolution frequency to w is still smaller, we replace ω by its value ω_0 of the synchronous particles and use the energy error $\epsilon=E-E_c$ as variable. In a linear approximation we can write[12]

$$\omega=\omega_0+k_0\epsilon\ , \tag{2.2}$$

where

$$k_0=-\eta\frac{\omega_0}{\beta^2E_c}\ , \tag{2.3}$$

β is the velocity of the synchronous particles in the unit of light velocity, $\eta=\alpha-\overline{\gamma}^{\,2}$ is the dispersion of the revolution frequency (defined such that $\eta>0$ above the transition energy), α is the momentum compaction factor, and γ is the Lorentz factor. The Vlasov equation is

$$\frac{\partial f(\epsilon,\theta,t)}{\partial t}+\frac{\partial f(\epsilon,\theta,t)}{\partial\theta}\omega+\frac{\partial f(\epsilon,\theta,t)}{\partial\epsilon}\dot{\epsilon}=0\ . \tag{2.4}$$

By a perturbation approach, the particle distribution function f is divided into two portions f_0 and f_1:

$$f=f_0+f_1\ . \tag{2.5}$$

In the linear theory[2] f_0 is the unperturbed time-dependent distribution function and f_1 is the perturbed distribution function. The function f_1 can be decomposed into a Fourier series without zero harmonics:

$$f_1(\epsilon,\theta,t)=\mathrm{Re}\left[\sum_{n\neq0}h_n(\epsilon,t)e^{in\theta}\right] \tag{2.6}$$

and each component vanishes when averaged over the angle θ from 0 to 2π. This fact suggests that we can redefine the distribution function f_0 as the average of the total distribution function f over the azimuthal angle around the ring:

$$f_0(\epsilon,t)=\frac{1}{2\pi}\int_0^{2\pi}f(\epsilon,\theta,t)d\theta\equiv\langle f\rangle_\theta\ . \tag{2.7}$$

The definition (2.7) uniquely separates the distribution function f into f_0 and f_1 which are slowly varying and rapidly oscillating in time, respectively. The time derivative $\dot{\epsilon}$ of the particle energy is expressed as

$$\dot{\epsilon}=-\frac{(e\omega_0)^2}{2\pi}\mathrm{Re}\left[\sum_{n\neq0}Z_n\int_{-\infty}^{\infty}h_n(\epsilon,t)d\epsilon e^{in\theta}\right]$$
$$=-e\omega_0\mathrm{Re}\left[\sum_{n\neq0}Z_n\phi_n(t)e^{in\theta}\right]\ , \tag{2.8}$$

where

$$\phi_n(t)=\frac{e\omega_0}{2\pi}\int_{-\infty}^{\infty}h_n(\epsilon,t)d\epsilon \tag{2.9}$$

and Z_n is the coupling impedance at the frequency $\omega=n\omega_0$. We assume that each Fourier component can be treated individually and that the nonlinear coupling between different components can be neglected. Under these assumptions, we shall consider only a certain Fourier harmonic n and then drop the suffix n in $\phi_n(t)$, $h_n(\epsilon,t)$, and Z_n. Averaging the Vlasov equation (2.4) over θ, we obtain a differential equation in time for f_0:

$$\frac{\partial f_0}{\partial t}+\left\langle\frac{\partial f_1(\epsilon,\theta,t)}{\partial\epsilon}\dot{\epsilon}\right\rangle_\theta=0\ . \tag{2.10}$$

Inserting Eqs. (2.6) and (2.8) into Eq. (2.10) and performing the average gives

$$\frac{\partial f_0}{\partial t}-\tfrac{1}{2}\mathrm{Re}\left[\frac{\partial h(\epsilon,t)}{\partial\epsilon}e\omega_0Z^*\phi^*(t)\right]=0\ . \tag{2.11}$$

The equation for f_1 is obtained by substituting $f=f_0+f_1$ into Eq. (2.4) and by using Eq. (2.10) in a form

$$\frac{\partial f_1}{\partial t}+\frac{\partial f_1}{\partial\theta}\omega+\frac{\partial f_0}{\partial\epsilon}\dot{\epsilon}=-\frac{\partial f_1}{\partial\epsilon}\dot{\epsilon}+\left\langle\frac{\partial f_1}{\partial\epsilon}\dot{\epsilon}\right\rangle_\theta\ . \tag{2.12}$$

For f_1, we use a linear solution to Eq. (2.12) and then neglect the second-order terms on the right-hand side to obtain the linearized Vlasov equation:

$$\frac{\partial f_1(\epsilon,\theta,t)}{\partial t}+\frac{\partial f_1(\epsilon,\theta,t)}{\partial\theta}\omega+\frac{\partial f_0(\epsilon,t)}{\partial\epsilon}\dot{\epsilon}=0\ . \tag{2.13}$$

The different point of Eq. (2.13) from the Vlasov equation in the conventional linear theory is that f_0 has the time dependence. However since we assume that the time change in f_0 is much slower than in f_1, it is reasonable in a relatively short time interval from t_1 to put

$$f_1(\epsilon,\theta,t)=\mathrm{Re}[h(\epsilon,t_1)e^{in\theta-i\Omega(t-t_1)}]\ , \tag{2.14}$$

$$\dot{\epsilon}=-e\omega_0\mathrm{Re}[Z\phi(t_1)e^{in\theta-i\Omega(t-t_1)}]\ , \tag{2.15}$$

$$\frac{\partial f_0(\epsilon,t)}{\partial\epsilon}=\frac{\partial f_0(\epsilon,t_1)}{\partial\epsilon}\ . \tag{2.16}$$

Substitution of these forms into Eq. (2.13) yields the expression for h

$$h(\epsilon,t_1)=i\frac{e\omega_0Z\phi(t_1)[\partial f_0(\epsilon,t_1)/\partial\epsilon]}{\Omega-n(\omega_0+k_0\epsilon)}\ , \tag{2.17}$$

where we replace ω by $n(\omega_0+k_0\epsilon)$ using the relation (2.2). Inserting the above equation into Eq. (2.9), we get

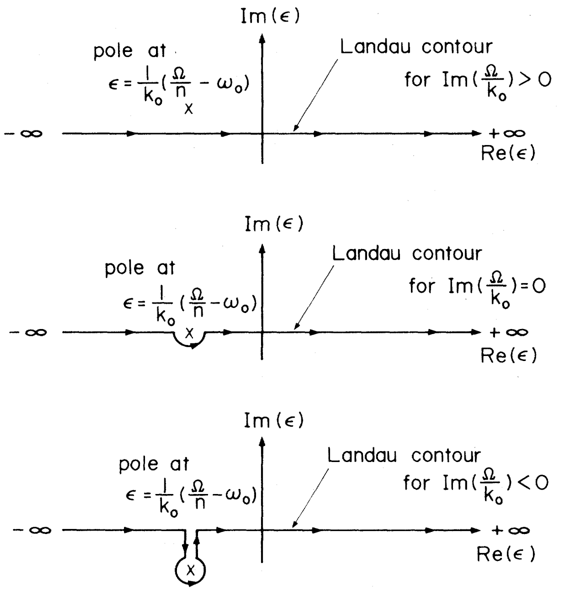

FIG. 1. Landau contour below the transition energy to evaluate the integration of the dispersion relation for three different values of $\mathrm{Im}(\Omega/k_0)$.

$$\phi(t_1)\left[1-\frac{(e\omega_0)^2}{2\pi}iZ\int\frac{(\partial f_0/\partial\epsilon)_{t_1}d\epsilon}{\Omega-n(\omega_0+k_0\epsilon)}\right]=0 . \tag{2.18}$$

In order for $\phi(t_1)$ to have a nontrivial solution, the quantity in the bracket has to vanish:

$$1-\frac{(e\omega_0)^2}{2\pi}iZ\int_{-\infty}^{\infty}\frac{(\partial f_0/\partial\epsilon)_{t_1}d\epsilon}{\Omega-n(\omega_0+k_0\epsilon)}=0 . \tag{2.19}$$

The frequency $\Omega=\Omega(t_1)$ of an oscillatory solution is given by solving this dispersion relation[2] and it varies slowly in time together with f_0. A remaining problem is a proper choice of the contour of the integration over ϵ, in particular, around the pole

$$\epsilon=\epsilon_P=\frac{1}{k_0}\left[\frac{\Omega}{n}-\omega_0\right] .$$

The above simple method provides no clue to this problem. However, Landau's method[13] for solving the Vlasov equation as an initial-value problem predicts that the contour of the integration always has to pass below (above) the pole $\epsilon=\epsilon_P$ below (above) transition, as shown in Fig. 1. This set of contours guarantees that the integral in Eq. (2.19) is continuous across the $\mathrm{Im}(\Omega)=0$. We employ the Landau contour to calculate the frequency of coherent oscillations after the beam is stabilized. A proper treatment of the linearized Vlasov equation based on the Landau prescription is intelligibly explained in Chap. 10 of Ref. 6. The reader should also refer to Ref. 14 for further details.

The equation for f_0 can be written by substituting Eq. (2.17) into Eq. (2.11) as

$$\frac{\partial f_0}{\partial t}-\frac{(e\omega_0)^2}{2}|Z|^2|\phi(t_1)|^2 \,\mathrm{Re}\left[\frac{\partial}{\partial\epsilon}\frac{i(\partial f_0/\partial\epsilon)_{t_1}}{\Delta\Omega_r-nk_0\epsilon+i\Delta\Omega_i}\right]=0 , \tag{2.20}$$

where we denote $\Omega-n\omega_0=\Delta\Omega_r+i\Delta\Omega_i$. In this equation all time dependent quantities must be evaluated at $t=t_1$. But since t_1 is arbitrarily chosen, this equation can be understood as a differential equation for f_0. We introduce a dimensionless quantity $A(t)$ normalized by the square of the dc current I_0 of a beam as

$$A(t)=\frac{|\phi(t)|^2}{I_0^2}>0 . \tag{2.21}$$

This quantity obeys the equation

$$\frac{dA(t)}{dt}=2\Delta\Omega_i A(t) \tag{2.22}$$

because

$$\phi(t)=\phi(t_1)e^{-i\Omega(t-t_1)} .$$

If we use $A(t)$ and take the real part of the quantity in square brackets in Eq. (2.20), we have the final form:

$$\frac{\partial f_0}{\partial t}-\frac{(e\omega_0)^2}{2}I_0^2|Z|^2A(t_1)\frac{\partial}{\partial\epsilon}\frac{\Delta\Omega_i(\partial f_0/\partial\epsilon)_{t_1}}{(\Delta\Omega_r-nk_0\epsilon)^2+\Delta\Omega_i^2}=0 . \tag{2.23}$$

The time development of the particle distribution can be calculated numerically in principle from the set of differential equations (2.22) and (2.23) with $\Omega=\Omega(t)$ given by the dispersion relation (2.19).

For later use, we introduce some normalizations[15] and rewrite the dispersion relation (2.19) in a more practical form. We first introduce a normalized energy error $x=\epsilon/\sigma_\epsilon$ with the rms of the energy error σ_ϵ. The rms spread S of revolution frequency is related to σ_ϵ by

$$S=\sigma_\epsilon k_0=-\eta\omega_0\frac{\sigma_\epsilon}{\beta^2E_c} . \tag{2.24}$$

We change the integral variable from ϵ to x and get the normalized distribution function $g_0(x,t_1)$ by putting $g_0(x,t_1)=f_0(\epsilon,t_1)2\pi\sigma_\epsilon/N$ so that

$$\int_{-\infty}^{\infty}g_0(x,t_1)dx=1 , \tag{2.25}$$

where N is the number of particles in a coasting beam. The dispersion relation can be rewritten as

$$1=\mathrm{sgn}(\eta)\frac{iI_0\beta^2Z}{2\pi(\sigma_\epsilon/E_c)^2|\eta|nE_c/e}\times\int_{-\infty}^{\infty}\frac{(\partial g_0/\partial x)_{t_1}dx}{x-x_1} , \tag{2.26}$$

where

$$x_1=\frac{\Omega-n\omega_0}{nS}=\frac{\Delta\Omega}{nS} \tag{2.27}$$

is the normalized frequency shift[15] and $I_0=Ne\omega_0/2\pi$.

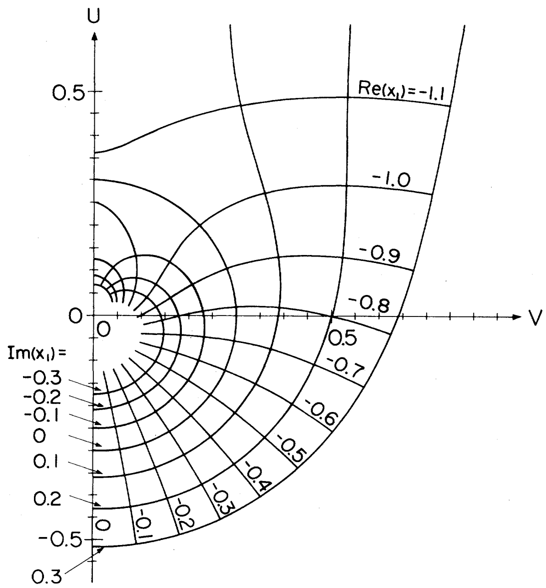

FIG. 2. The stability diagram below the transition energy for the distribution function given by Eq. (2.30).

We use the quantities U and V (Ref. 16) instead of the impedance

$$U-iV=\frac{-I_0\beta^2}{2\pi(\sigma_\epsilon/E_c)^2\,|\,\eta\,|\,nE_c/e}iZ \tag{2.28}$$

to get

$$1=-\mathrm{sgn}(\eta)\cdot(U-iV)\int\frac{(\partial g_0/\partial x)_{t_1}dx}{x-x_1}\ . \tag{2.29}$$

The mapping of the complex x_1 plane onto the U,V plane for a given distribution function is called a stability diagram.[16] Hübner and Vaccaro,[15] and Zotter[17] have presented stability diagrams for various realistic distribution functions of a coasting beam. The distribution function that gives a simple stability criterion, i.e., the one which has a circular stability limit centered at the origin, is given by[17]

$$g_0(x,t_1)=\frac{8a}{3\pi}(1-a^2x^2)^{3/2}\ \text{for}\ |\,x\,|<\frac{1}{a}\ , \tag{2.30}$$

where

$$a^2=\tfrac{5}{6}\ .$$

The stability diagram below the transition energy is shown in Fig. 2 with some curves for constant $\mathrm{Re}(x_1)$ and $\mathrm{Im}(x_1)$. The stability limit is given by the contour $\mathrm{Im}(x_1)=0$. The growth rate expressed in terms of $\mathrm{Im}(x_1)$ is found to switch sign according as the position of a U,V pair is inside or outside the stable area.

III. TIME DEVELOPMENT OF ENERGY SPREAD

We will assume as an approximation to the exact solution to the set of Eqs. (2.22) and (2.23) that the time development of the particle distribution can be characterized by the change in the energy spread only and that the shape of the average distribution remains unchanged while the energy spread is blowing up. This means that the average distribution can be factorized into the *time-dependent* energy spread and the *time-independent* normalized distribution function:

$$f_0(\epsilon,t)=\frac{N}{2\pi\sigma_\epsilon(t)}g_0(x)\ .$$

The rms energy spread is defined by

$$\sigma_\epsilon^2(t)=\frac{1}{2\pi N}\int_{-\infty}^{\infty}\int_0^{2\pi}\epsilon^2 f(\epsilon,\theta,t)d\epsilon\, d\theta$$
$$=\frac{1}{N}\int_{-\infty}^{\infty}\epsilon^2 f_0(\epsilon,t)d\epsilon\ . \tag{3.1}$$

Multiplying Eq. (2.11) by ϵ^2 and integrating over ϵ gives

$$\frac{d\sigma_\epsilon^2(t)}{dt}-\frac{1}{2N}\mathrm{Re}\left[e\omega_0 Z^*\phi^*(t)\int_{-\infty}^{\infty}\epsilon^2\frac{\partial h(\epsilon,t)}{\partial\epsilon}d\epsilon\right]=0\ . \tag{3.2}$$

If we use the expression (2.17) for $h(\epsilon,t)$ after one integration by parts, we obtain

$$\int\epsilon^2\frac{\partial h}{\partial\epsilon}d\epsilon=[2\epsilon h]_{-\infty}^{\infty}-2\int\epsilon h\, d\epsilon$$
$$=-2ie\omega_0 Z\phi(t)\int\epsilon\frac{(\partial f_0/\partial\epsilon)d\epsilon}{\Omega-n(\omega_0+k_0\epsilon)}$$
$$=-\frac{2\Delta\Omega}{k_0 n}\phi(t)ie\omega_0 Z\int\frac{(\partial f_0/\partial\epsilon)d\epsilon}{\Omega-n(\omega_0+k_0\epsilon)}\ . \tag{3.3}$$

Since the frequency Ω satisfies the dispersion relation (2.19), Eq. (3.3) can be rewritten as

$$\int\epsilon^2\frac{\partial h}{\partial\epsilon}d\epsilon=-\frac{2\Delta\Omega}{k_0 n}\phi(t)\frac{2\pi}{e\omega_0}\ . \tag{3.4}$$

If we substitute Eq. (3.4) into Eq. (3.2) and take the real part of the second term, the equation for σ_ϵ is finally given by

$$\frac{d\sigma_\epsilon^2}{dt}-\frac{\beta^2E_ce}{\eta nI_0}|\,\phi(t)\,|^2(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)=0\ , \tag{3.5}$$

where $Z=Z_r+iZ_i$ and $\Delta\Omega=\Delta\Omega_r+\Delta\Omega_i$. We introduce D, the square of the relative energy spread,

$$D=\left[\frac{\sigma_\epsilon}{E_c}\right]^2\ . \tag{3.6}$$

Using this new variable and the quantity $A(t)$ defined by Eq. (2.21), we have the set of differential equations

$$\frac{dD(t)}{dt}-\mathrm{sgn}(\eta)KA(t)(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)=0 \tag{3.7}$$

and

$$\frac{dA(t)}{dt}=2\Delta\Omega_iA(t)\ , \tag{3.8}$$

where

$$K=\frac{I_0\beta^2}{|\eta|nE_c/e} . \tag{3.9}$$

The complex frequency shift $\Delta\Omega$ is calculated from the dispersion relation (2.19) and is a function of $D(t)$ only.

The equation for $D(t)$ can readily be solved by inserting Eq. (3.7) into Eq. (3.8) and integrating over t

$$\frac{1}{\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)}\frac{dD}{dt}=2\int_{D_i}^{D}\frac{\Delta\Omega_i dD}{\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)}+A(0) . \tag{3.10}$$

Integrating again, we obtain

$$\int_{D_i}^{D}\frac{dD}{\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)\left[2\int_{D_i}^{D}\frac{\Delta\Omega_i dD}{\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)}+A(0)\right]}=t , \tag{3.11}$$

where D_i is the initial value of D and the time is chosen such that $D=D_i$ at $t=0$. To obtain the time asymptotic ($t\rightarrow\infty$) solution for D, some remarks are necessary on the sign of each term appearing in the denominator of the integrand. The result is presented in Fig. 3 in the form of a mapping of the quantity $\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)$ onto the U,V plane as well as that of the complex frequency shift $\Delta\Omega$ for particle distribution function given by Eq. (2.30). We use here the abbreviation

$$F=\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i) \tag{3.12}$$

and point out that F can be expressed in terms of U,V and x_1 instead of Z and $\Delta\Omega$ as

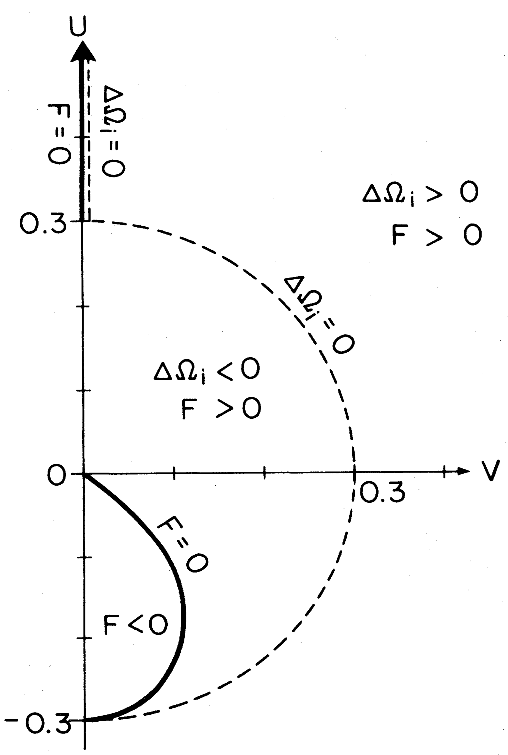

FIG. 3. Mapping of $F=\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)$ onto the U,V plane for the same distribution function as in Fig. 2.

$$F=-2\pi Dn\,|S\,|[V\text{Re}(x_1)+U\text{Im}(x_1)] \tag{3.13}$$

by use of the definitions (2.24), (2.27), and (3.9). The contours $F=0$ and $\Delta\Omega_i=0$ are denoted by the thick solid curve and dashed curve, respectively, in Fig. 3. To help explanations in what follows, we write down an alternative expression of Eq. (3.7) in terms of F:

$$\frac{dD}{dt}=F(D)A(t) . \tag{3.14}$$

The important point in Fig. 3 is that the region $F<0$ exists only inside the stable area. This fact leads to an interesting conclusion that the energy spread is always going to be larger for an unstable beam because positive $\Delta\Omega_i$ always gives positive F. Although the shape of the boundary $F=0$ depends on the particle distribution assumed, this fact can be confirmed to hold for all the realistic distribution functions as follows:

(i) Below transition ($\eta<0$), since $\text{Re}(x_1)<0$ for V (or Z_r) >0 as in Fig. 2, we find with help of Eq. (3.13)

$$\text{sgn}(\eta)KZ_r\Delta\Omega_r=-2\pi Dn\,|S\,|\text{Re}(x_1)V>0 . \tag{3.15}$$

Above transition, the graph must be reflected about the horizontal axis[15] and the sign of $\text{Re}(x_1)$ remains minus for $V>0$. We thus conclude that the first term of F, i.e., the product $\text{sgn}(\eta)KZ_r\Delta\Omega_r$ is always a positive quantity regardless of the sign of η.

(ii) On the other hand, we can state for the second term of F below transition that

$\text{sgn}(\eta)KZ_i\Delta\Omega_i>0$ (<0) outside the stable area ,

$\text{sgn}(\eta)KZ_i\Delta\Omega_i<0$ (>0) inside the stable area ,

on the lower (upper) half plane. Above transition all the directions of inequal sign are reversed.

(iii) We finally find out how the sign of F will be signed in the U,V plane. We restrict our discussion to a case below transition because the same result can be obtained above transition. From the above arguments, we see that if the regions $F<0$ exist, they are inside the stable area on the lower half plane or outside the stable area on the upper half plane. We first investigate the former case.

(a) Both the origin ($Z_r,Z_i=0$) and the intersection of the U axis ($Z_r=0$) and the curve $\Delta\Omega_i=0$ satisfy $F=\text{sgn}(\eta)K(Z_r\Delta\Omega_r+Z_i\Delta\Omega_i)=0$, while the other points

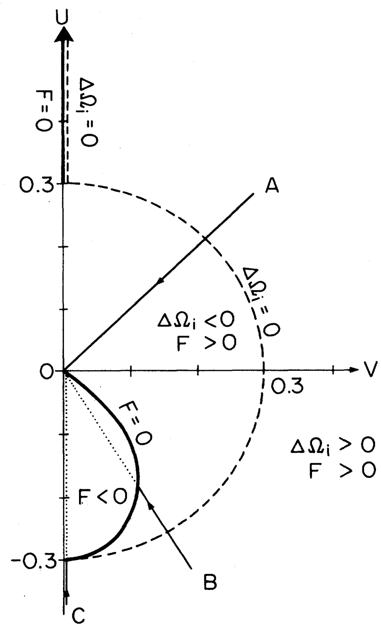

FIG. 4. Motion of U,V pairs associated with the blowup of the energy spread for three different impedances. They never can get into the region $F<0$, which is denoted by the dotted line.

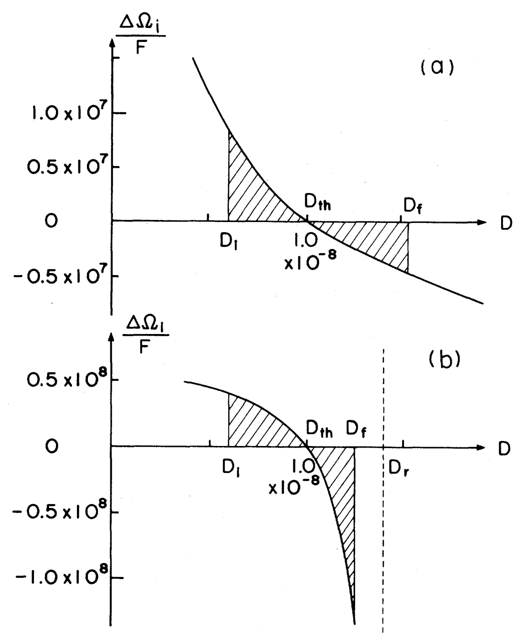

FIG. 5. The quantity $\Delta\Omega_i/F$ as a function of D for (a) the case A and (b) the case B shown in Fig. 4. The areas of the shadowed portions must be the same on both sides of D_{th}.

that lie on the U axis give negative F because the first term vanishes and the second term is negative. Noting that $F>0$ on the curve $\Delta\Omega_i=0$, we find that the region $F<0$ indeed exists and its boundary $F=0$ is confined in the stable area.

(b) Next we turn our investigation to the latter case. Since $\Delta\Omega_i \approx 0$ near the stability limit, if the region $F<0$ exists, it is expected to be in the region where $|U|$ and $|V|$ are large. In the limit of large $|x_1|$ (or $|U|$, $|V|$), for all the distribution functions, the integral of Eq. (2.29) tends to[16]

$$\int \frac{(dg_0/dx)dx}{x-x_1} = \int \frac{g_0 dx}{(x-x_1)^2} \approx \frac{1}{x_1{}^2}\int g_0 dx = \frac{1}{x_1{}^2}$$

and the quantity F can be evaluated by use of Eqs. (2.24), (2.27), (3.9), and the dispersion relation (2.29) as

$$\begin{aligned} F &= -\text{sgn}(\eta) K \text{Im}[(Z_i - iZ_r)(\Delta\Omega_r - i\Delta\Omega_i)] \\ &= 2\pi D n S \text{Im}[-\text{sgn}(\eta)(U-iV)x_1^*] \\ &= -\frac{2\pi D^{3/2} n\omega_0}{\beta^2}\eta\,|x_1|^2 \text{Im}(x_1) \geq 0 , \end{aligned} \tag{3.16}$$

where the equals sign holds on the intersection of the U axis and the curve $\Delta\Omega_i=0$. Therefore it turns out that F is always positive or zero outside the stable area.

IV. OVERSHOOT

Let us now study the overshoot phenomenon in the energy-spread blowup for an initially unstable beam, i.e., $\Delta\Omega_i>0$. The final value D_f at $t\to\infty$ is given by the value of D where the denominator of the integrand in Eq. (3.11) vanishes,

$$F(D_f)\left[2\int_{D_i}^{D_f}\frac{\Delta\Omega_i}{F(D)}dD + A(0)\right] = 0 . \tag{4.1}$$

We make an approximation that the normalized distribution function is unchanged in time. Then as the energy spread is increasing, the U,V pair for a given impedance is moving to the origin on the U,V plane along the straight line connecting the initial position and the origin as shown in Fig. 4. The energy spread increases beyond the threshold value owing to the presence of the region $F>0$ even inside the stable area. There are two cases depending on whether the contour $F=0$ intersects with the straight line (B case) or not (A case). See Fig. 4. In the latter case, since $F(D)$ is never zero, we get

$$2\int_{D_i}^{D_f}\frac{\Delta\Omega_i}{F(D)}dD + A(0) = 0. \tag{4.2}$$

The second term can be thought of as a very small positive quantity representing the initial perturbation. On inspection of Eq. (3.14), the main contribution to the increase of the energy spread is seen to come from the large $A(t)$ after the unstable perturbation grows sufficiently, so that we neglect the second term to obtain a simpler form

$$\int_{D_i}^{D_f}\frac{\Delta\Omega_i}{F(D)}dD = 0 . \tag{4.3}$$

The integrand is plotted in Fig. 5(a) as a function of D for the initial impedance $U=V=0.5$ and $D_{\text{th}}=10^{-8}$ where D_{th} is the threshold value of D which gives $\Delta\Omega_i=0$. The

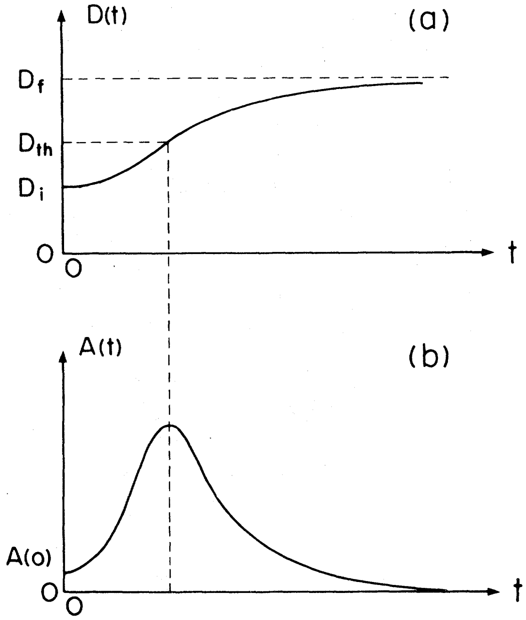

FIG. 6. Sketches of the qualitative behavior of (a) $D(t)$ and (b) $A(t)$ as a function of time t.

condition (4.3) says that the areas of the shadowed portions are the same on both sides of $D_{\rm th}$. If we take a linear approximation in the neighborhood of $D_{\rm th}$

$$\frac{\Delta\Omega_i}{F(D)} = -b(D - D_{\rm th}) \,, \tag{4.4}$$

Eq. (4.3) gives a simple relation

$$D_f + D_i \approx 2D_{\rm th} \tag{4.5}$$

which can be written in an alternative form by changing the variable from D to the rms energy spread σ_ϵ as

$$\sigma_f^{\,2} + \sigma_i^{\,2} \approx 2\sigma_{\rm th}^{\,2} \,. \tag{4.6}$$

This is an overshoot formula because the final energy spread σ_f exceeds the threshold value $\sigma_{\rm th}$ and becomes larger as the initial energy spread σ_i is smaller than the threshold. Figures 6(a) and 6(b) show the qualitative behavior of $D(t)$ and $A(t)$ as a function of time t based on Eqs. (3.7) and (3.8), respectively, from which we see that the blowup of the energy spread eventually saturates at the time asymptotic limit where the perturbation dies out. We shall compare the overshoot formula (4.6) with the experimental results at the ISR.[5] Measurements were done for the full momentum spread Δp at half maximum, but it seems to be quite all right to consider that it is proportional to the rms energy spread. The results are summarized in Table I. We see from the final column that the relation $\Delta p_i^{\,2} + \Delta p_f^{\,2} \approx 2p_{\rm th}^{\,2}$ is approximately fulfilled for most cases measured. Of course, the formula (4.6) is not exactly valid for the quantitative analysis of the overshoot phenomenon because the line $\Delta\Omega_i/F$ is not linear in general. Nevertheless the qualitative relationship between D_i, $D_{\rm th}$, and D_f is unchanged.

TABLE I. Experimental results on the overshoot in the blowup of the energy spread at the ISR. Momentum spreads were measured by Schottky scan. Each suffix is used in the same sense as for the energy spread. Data are taken from Ref. 5.

Beam No.	Current (mA)	$(\Delta p/p)_i$	$(\Delta p/p)_{\rm th}$	$(\Delta p/p)_f$	$\frac{\Delta p_i^{\,2}\Delta p_f^{\,2}}{2\Delta p_{\rm th}^{\,2}}$
1	87.5		No Schottky scan made		
2	61.0		No Schottky scan made		
3	40.8	2.48	2.71	3.73	1.36
4	41.3	2.27	2.75	5.50	2.34
5	39.4	2.30	2.68	3.35	1.15
6	22.5	2.31	2.03	stable	
7	20.6	1.20	1.93	1.99	0.72
8	30.1	2.50	2.61	stable	
9	26.8	2.26	2.22	2.81	1.32
10	89.2	3.93	4.04	5.50	1.40

A similar qualitative relationship can be obtained for the line B in Fig. 4, in which case the final value D_f cannot be larger than D_r, the zero of $F(D)$, because the integral of Eq. (4.1) goes to minus infinity as D approaches D_r as shown in Fig. 5(b). In particular, if the impedance is pure inductive below transition or pure capacitive above transition (see line C in Fig. 4), the three quantities D_f, $D_{\rm th}$, and D_r are identical with each other, so that the overshoot does not occur. This conclusion contradicts with that of Dory's computer simulation[6] which showed that the overshoot phenomenon would also take place by the negative mass instability due to the pure space-charge effect. Future subjects lie in this region and there are the following possible explanations to the paradox. (1) Our conclusion is closely related to the assumption that the particle distribution shape remains unchanged during blowing up. If we allow it to change as in computer simulations, the threshold energy spread will be floating and might take different values at the initial and the final distribution shapes. (2) The higher-order terms than the second might be dominant in such a special case. (3) The adiabaticity assumption and the perturbation method might break down. In spite of these limitations, the present method provides a simple and powerful tool to analyze nonlinear instabilities for most practical cases.

The curve $F=0$ is sensitive to the shape of the distribution function in the neighborhood of the origin. Therefore it is not always obvious whether a given impedance belongs to A type or B type. However taking into account the fact that the energy spread D becomes very large near the origin, we need not argue this problem in detail unless we treat an extremely unstable beam.

When the beam is initially stable, the initial perturbation is reduced. The first term in Eq. (4.2) has the same minus sign as $\Delta\Omega_i$ because $dD \propto F(D)$ if the range of the integration is restricted within the stable area. This term balances the small second term $A(0)$ as soon as the energy spread begins to change with time according to Eq. (3.14), with the result that the initial energy spread will vary only slightly. We remark here that $\phi(0)$ defined in Sec. II is a time asymptotic approximation determined by the normal modes of oscillation and that it is not exactly the initial perturbation itself at $t=0$. In order to estimate a precise value of the final spread, we have to know the information of the initial perturbation and have to analyze the time development of the exact solution to the Vlasov equation. From the practical point of view, we are not interested in this problem for an initially stable beam and so we do not mention it here.

V. CONCLUSIONS

The nonlinear perturbation theory has been developed for the collective phenomena of a coasting beam and, in particular, for the overshoot phenomenon in the blowup of the energy spread. The unperturbed part of the particle distribution function in the linear theory is no longer stationary in the present theory and is redefined as the average of the actual distribution function over the azimuthal angle around the ring. The nonlinearity of the Vlasov equation gives the average distribution the time dependence of second order in perturbed quantities. The linear growth rate of the instability varies slowly in time together with the average distribution through the dispersion relation.

We demonstrated that the time development of the energy spread obeys Eq. (3.7) with the subsidiary differential equation (3.8) for the perturbation amplitude. The systematic investigation of these equations leads to a mapping of F which represents the time derivative of the energy spread onto the U,V plane as shown in Fig. 3. In the process that the U,V pair for a given impedance moves to the origin, the energy spread increases beyond the threshold value of the stability owing to the presence of the region $F>0$ even inside the stable area, i.e., we find an overshoot phenomenon. The qualitative relationship may be simply described as follows: The smaller the initial energy spread is, the larger the final energy spread becomes. A quantitative agreement of the experimental results at the ISR with the theoretical overshoot formula obtained in this paper is satisfactory for beams with an initial energy spread close to the threshold value.

We see that the initial energy spread will change only a little for an initially stable beam. The precise estimation of the final energy spread requires the details of the time development of the particle distribution and so is outside the framework of the present theory.

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