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Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant

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We examine the late-time behavior of initially expanding homogeneous cosmological models satisfying Einstein's equation with a positive cosmological constant Λ. It is shown that such models of all Bianchi types except IX exponentially evolve toward the de Sitter solution, with time scale $(3/\Lambda)^{1/2}$. The behavior of Bianchi type-IX universes is similar, provided that Λ is sufficiently large compared with spatial-curvature terms. Thus, a positive cosmological constant provides an effective means of isotropizing homogeneous universes.

One of the most important reasons why cosmological models which predict an inflationary phase[1] in the early universe are considered attractive is that they hold out hope that the present state of our universe can be explained without the necessity of imposing highly special conditions on the initial state of the universe. Most investigations of inflationary cosmological models have assumed Robertson-Walker symmetry (i.e., homogeneity and isotropy) from the outset. In this case, one finds that inflation causes a large increase in the horizon size of the present universe on account of the exponential rate of expansion occurring during the "supercooled" era just prior to the phase transition, when the "false vacuum" contribution to the stress energy of the quantum field dominates, thereby effectively yielding a large positive cosmological constant Λ in the dynamical equations. This increase in the horizon size in the Robertson-Walker models suggests the possibility that interactions could have homogenized and isotropized the universe, and hence that the present state of the universe could have evolved from much more general initial conditions. However, it is not obvious that cosmological models with non-Robertson-Walker initial conditions will even enter (or "gracefully exit") an inflationary epoch, nor is it obvious that if inflation occurs, initial inhomogeneity and anisotropy will be smoothed out. (Indeed, gravitational interactions generally tend to enhance inhomogeneities rather than smooth them out.) The only way that these ideas can be tested is to examine the evolution of initially inhomogeneous and anisotropic cosmological models.

The issue of whether an inflationary epoch occurs in a class of non-Robertson-Walker models has been discussed recently by Steigman and Turner.[2] We shall not consider this issue further here. With regard to the question of whether the universe evolves to a homogeneous and isotropic state during an inflationary epoch, it has been conjectured by Gibbons and Hawking[3] and Hawking and Moss[4] that—roughly speaking—all expanding-universe models with positive cosmological constant asymptotically approach the de Sitter solution. [It is difficult to formulate a precise version of this conjecture since, firstly, it is not obvious exactly how one should define the notions of "expanding universe" and "asymptotic approach to de Sitter." Furthermore, given such definitions, it certainly should be possible for regions of the universe to collapse to black holes, so that the universe approaches a de Sitter solution with black holes rather than the de Sitter solution. In addition, other special (but presumably unstable) behavior should be possible, such as asymptotic approach to an Einstein static universe.] Some evidence for this conjecture has been discussed by Boucher and Gibbons.[5] The purpose of the present paper is to investigate this conjecture in the simple context of homogeneous cosmological models. We shall give a remarkably simple proof that all initially expanding Bianchi cosmologies except type IX evolve toward the de Sitter solution on an exponentially rapid time scale. The behavior of type-IX cosmologies is similar provided that Λ is sufficiently large. Thus our results lend support to the above "cosmic no-hair" conjecture.

We consider the dynamical evolution of Bianchi cosmologies via Einstein's equation with cosmological constant

$$G_{ab} = -\Lambda g_{ab} + 8\pi T_{ab} \quad . \tag{1}$$

(Here our sign conventions are those denoted $+++$ by Misner, Thorne, and Wheeler.[6]) We shall make no assumptions about the nature of the matter stress-energy tensor T_{ab} (which should be interpreted as the "nonvacuum" contribution to T_{ab} during an inflationary phase) except that it satisfy the dominant and strong energy conditions. The dominant energy condition states that $T_{ab}t^b$ is (past-directed) timelike or null for all (future-directed) timelike t^a. In particular, this implies that

$$T_{ab}t^a t^b \geq 0 \ . \tag{2}$$

The strong energy condition requires that

$$(T_{ab} - \tfrac{1}{2}g_{ab}T)t^a t^b \geq 0 \ . \tag{3}$$

The only equations we shall need for our analysis are the following two components of Eq. (1): the "initial-value constraint" equation

$$0 = G_{ab}n^a n^b - \Lambda - 8\pi T_{ab}n^a n^b \ , \tag{4}$$

where n^a is the unit normal to the homogeneous hypersurfaces, and the Raychaudhuri equation

$$0 = R_{ab}n^a n^b + \Lambda - 8\pi(T_{ab} - \tfrac{1}{2}g_{ab}T)n^a n^b \ . \tag{5}$$

Both $G_{ab}n^a n^b$ and $R_{ab}n^a n^b$ can be expressed in terms of the three-geometry of the homogenenous hypersurfaces and the extrinisic curvature K_{ab} of these surfaces. First, for convenience, we decompose K_{ab} into its trace K and trace-free part σ_{ab},

$$K_{ab} = \tfrac{1}{3}Kh_{ab} + \sigma_{ab} \ , \tag{6}$$

where

$$K = K_{ab}h^{ab} \ , \tag{7}$$

h_{ab} is the spatial metric,

$$h_{ab} = g_{ab} + n_a n_b \ , \tag{8}$$

and h^{ab} is its inverse. Thus σ_{ab} is the shear of the timelike geodesic congruence orthogonal to the homogeneous hypersurfaces. Note, however, that if T_{ab} is of the form of a fluid—which we do *not* assume—then σ_{ab} is *not* necessarily the shear of this fluid unless the fluid happens to be moving orthogonally to the homogeneous hypersurfaces. In terms of K, σ_{ab}, and the three-geometry, Eqs. (4) and (5) become, respectively,

$$K^2 = 3\Lambda + \tfrac{3}{2}\sigma_{ab}\sigma^{ab} - \tfrac{3}{2}{}^{(3)}R + 24\pi T_{ab}n^a n^b \tag{9}$$

and

$$\dot{K} = \Lambda - \tfrac{1}{3}K^2 - \sigma_{ab}\sigma^{ab} - 8\pi(T_{ab} - \tfrac{1}{2}g_{ab}T)n^a n^b \ , \tag{10}$$

where ${}^{(3)}R$ is the scalar curvature of the homogeneous hypersurface and $\dot{K} \equiv \mathcal{L}_n K$ is the derivative of K with respect to proper time.

The scalar curvature ${}^{(3)}R$ is given in terms of the structure-constant tensor $C^a{}_{bc}$ of the Lie algebra of the spatial symmetry group by

$${}^{(3)}R = -C^a{}_{ab}C^c{}_c{}^b + \tfrac{1}{2}C^a{}_{bc}C^c{}_a{}^b - \tfrac{1}{4}C_{abc}C^{abc} \ . \tag{11}$$

Here indices are lowered and raised with h_{ab} and h^{ab}. Equation (11) can be rewritten in a very useful form as follows: Fix a three-form $\epsilon_{abc} = \epsilon_{[abc]}$ on the Lie algebra. It follows directly from the antisymmetry property $C^c{}_{ab} = -C^c{}_{ba}$ that $C^c{}_{ab}$ can be expressed in the form[7]

$$C^c{}_{ab} = M^{cd}\epsilon_{dab} + \delta^c{}_{[a}A_{b]} \ , \tag{12}$$

where $M^{cd} = M^{dc}$. Thus we can eliminate $C^c{}_{ab}$ in favor of the tensor M^{ab} and the dual vector A_a. Substitution of Eq. (12) into the Jacobi identity

$$C^e{}_{d[a}C^d{}_{bc]} = 0 \tag{13}$$

yields

$$M^{ab}A_b = 0 \ . \tag{14}$$

By substituting Eq. (12) into Eq. (11) and using Eq. (14) we obtain the desired formula for ${}^{(3)}R$;

$${}^{(3)}R = -\tfrac{3}{2}A_b A^b - h^{-1}(M_{ab}M^{ab} - \tfrac{1}{2}M^2) \ , \tag{15}$$

where h^{-1} denotes the determinant of h^{ab}, calculated using the three-form ϵ_{abc}. From Eq. (15) it follows immediately that a necessary condition for ${}^{(3)}R$ to be positive is $M_{ab}M^{ab} < \tfrac{1}{2}M^2$. However, this can occur only if M^{ab} is positive definite (or negative definite) and, in this case, Eq. (14) implies that $A_b = 0$. This corresponds precisely to the type-IX case in the Bianchi classification. Thus we have shown that in all Bianchi models, except type IX, we have

$${}^{(3)}R \leq 0 \ . \tag{16}$$

Consider now any Bianchi model which is not type IX. Using the inequalities (2), (3), and (16) we obtain from Eqs. (9) and (10)

$$\dot{K} \leq \Lambda - \tfrac{1}{3}K^2 \leq 0 \ . \tag{17}$$

Here the first inequality follows directly from Eq. (10) and the second directly from Eq. (9). Now, since $K^2 \geq 3\Lambda$, we see immediately that K cannot pass through zero. Thus, if $K > 0$ at an (arbitrarily chosen) initial time $t = 0$ (i.e., if the universe is initially expanding), then $K > 0$ for all time (i.e., the universe expands forever) and, indeed, we have

$$K \geq (3\Lambda)^{1/2} \tag{18}$$

at all times. However, Eq. (17) also implies

$$\frac{1}{K^2 - 3\Lambda}\frac{dK}{d\tau} \leq -\frac{1}{3} \ . \tag{19}$$

Integrating the inequality, we obtain

$$K \leq \frac{(3\Lambda)^{1/2}}{\tanh(\tau/\alpha)} \ , \tag{20}$$

where

$$\alpha = (3/\Lambda)^{1/2} \ . \tag{21}$$

Thus K is "squeezed" between the lower limit $(3\Lambda)^{1/2}$ and the upper limit (20) which exponentially approaches $(3\Lambda)^{1/2}$ on the time scale α. Thus we find that the expansion rate K rapidly approaches $(3\Lambda)^{1/2}$. Returning to Eq. (9), we find from the upper limit, Eq. (20), that

$$\sigma^{ab}\sigma_{ab} \leq \tfrac{2}{3}(K^2 - 3\Lambda) \leq \frac{2\Lambda}{\sinh^2(\tau/\alpha)} \ . \tag{22}$$

Thus the shear of the homogeneous hypersurfaces rapidly approaches zero. Similarly, Eqs. (9) and (20) also imply that the matter energy density is bounded by

$$T_{ab}n^a n^b \leq \frac{\Lambda}{8\pi}\frac{1}{\sinh^2(\tau/\alpha)} \ . \tag{23}$$

Furthermore, the dominant energy condition implies that $T_{ab}n^a n^b$ is at least as large as any other orthonormal frame component of T_{ab} in a basis with n^a as the timelike vector. Thus all components of T_{ab} rapidly approach zero. Finally, the fact that $K \to (3\Lambda)^{1/2}$ while $\sigma_{ab} \to 0$ implies that at late times the time dependence of the spatial metric is approximated (to within controllable errors) by

$$h_{ab}(\tau) = e^{2(\tau-\tau_0)/\alpha} h_{ab}(\tau_0) \quad .$$

However, under such an isotropic scaling up of the spatial metric, the spatial curvature ${}^{(3)}R_{ab}$ scales away to zero. Thus we conclude that for all non-type-IX Bianchi cosmologies which are initially expanding, for $\tau \gg \alpha$, the universe will appear to be matter free, with nearly flat spatial sections which expand isotropically at the constant rate $K = (3\Lambda)^{1/2}$. Thus, for $\tau \gg \alpha$, the universe will appear locally indistinguishable from de Sitter spacetime. We emphasize, however, that the global properties of the universe need not approach those of de Sitter.

In the case of Bianchi type-IX models, the scalar curvature ${}^{(3)}R$ can become positive. However, one can show that for given ϵ_{abc} and M^{ab}, if we fix the determinant h of the spatial metric, then the largest possible positive value of ${}^{(3)}R$ is achieved by choosing h^{ab} to be proportional to M^{ab}, i.e.,

$$h^{ab} = \frac{M^{ab}}{(h \det M)^{1/3}} \quad . \tag{24}$$

(This is equivalent to saying that, at fixed proper volume, ${}^{(3)}R$ is maximized by choosing the curvature to be isotropic.) Hence, substituting this into Eq. (15) (with $A_b = 0$), we find

$${}^{(3)}R \leq \frac{3}{2} \frac{(\det M)^{2/3}}{h^{1/3}} \quad . \tag{25}$$

Suppose that initially we have $K > 0$ and, in addition, that initially we have

$$\Lambda > \frac{3}{4} \frac{(\det M)^{2/3}}{h^{1/3}} \quad , \tag{26}$$

i.e., Λ initially is greater than $\frac{1}{2}$ of what the scalar curvature would be if the universe had the same volume but were isotropic. Since we have

$$\dot{h} = 2hK \quad , \tag{27}$$

if follows that h initially increases and must continue to increase unless K passes through zero. However, by Eqs. (9) and (25), K will remain positive unless the inequality (26) is violated, which can happen only if h becomes *smaller* than its initial value. [Note that h is the only time-varying quantity in Eq. (26).] Hence the universe must expand forever. Arguments similar to those used in the non-type-IX case then can be used to show that the universe must approach de Sitter at late time.

If Eq. (26) is not satisfied initially, then it is possible that the universe may recollapse. Other types of non-de Sitter asymptotic behavior also are possible, in particular, asymptotic approach to an Einstein static universe. However, even when Eq. (26) is not satisfied, it seems reasonable to conjecture that the only types of stable asymptotic behavior are recollapse and asymptotic approach to de Sitter spacetime.

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[1]By "inflation" we mean "new inflation." For a recent discussion of various aspects of inflationary models, see *The Early Universe*, edited by G. W. Gibbons, S. W. Hawking, and S. Siklos (Cambridge Univ., Cambridge, to be published).

[2]G. Steigman and M. Turner (unpublished).

[3]G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).

[4]S. W. Hawking and I. G. Moss, Phys. Lett. 110B, 35 (1982).

[5]W. Boucher and G. W. Gibbons, in Ref. 1.

[6]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[7]See, e.g., G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. 12, 108 (1969).