Quantum noise and active feedback

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Quantum mechanics limits the noise performance of linear amplifiers, and these limits, in turn, restrict the effects of active feedback on the noise level of any device which uses feedback from such an amplifier. I present a simple quantum-mechanical characterization of feedback, and use this method to determine the limiting noise levels of active filters and oscillators. The results clarify the role of the amplifier in the measurement process and illustrate the limits of a Langevin treatment of quantum noise.

An amplifier may be described as a device which transforms some set of input modes into a set of output modes. Quantum mechanics demands that this transformation be unitary, and this implies a certain minimum noise level for any linear amplifier.[1] In a common application of linear amplifiers, active feedback, the output modes are coupled back to the input modes, and several interesting questions arise regarding noise in such coupled systems:

(1) Active feedback can be used to synthesize reactive components (e.g., an effective inductance) and thereby change the resonant frequency of a detector. To which frequency does the quantum noise (nominally $\approx \hbar\omega$) correspond?

(2) Active feedback can narrow the bandwidth of a detector and thereby reduce the effects of thermal noise; this may be described as lowering the effective temperature of the detector.[2] In the limit that the effective temperature is lowered below $\hbar\omega/k_B$, do we recover the appropriate quantum noise?

(3) If the gain of the amplifier is increased, the system becomes unstable and oscillates. What are the quantum limits on the stability of these oscillations?

In this Brief Report I show how the unitarity constraints can be applied self-consistently to the coupled modes of a system with active feedback. The approach is simple, and provides some simple answers to the questions posed above.

Consider a simple harmonic oscillator with Hamiltonian

$$H=\hbar\omega(a^\dagger a+\tfrac{1}{2})+i\Gamma(a^\dagger-a)+H_{\mathrm{hb}} \ , \tag{1}$$

where H_{hb} is the Hamiltonian of a heat bath and Γ is a coordinate of the bath which couples to the oscillator and generates a damping constant γ; some method of treating dissipation is essential for the analysis which follows,[3] and this approach is due to Senitzky.[4] Perturbation theory on the oscillator-bath coupling leads to the equations of motion (with $\hbar=1$)

$$\frac{da}{dt}=-i\omega a-\gamma a+\delta F \ , \tag{2a}$$

$$\frac{da^\dagger}{dt}=-i\omega a^\dagger-\gamma a^\dagger+\delta F^\dagger \ , \tag{2b}$$

where δF and $\delta F^\dagger$ are quantum Langevin operators[5] with properties

$$\int d\tau e^{i\Omega\tau}\langle \delta F^\dagger(t)\delta F(t-\tau)\rangle=2\gamma\bar{\nu}(\Omega) \ , \tag{3a}$$

$$\int d\tau e^{i\Omega\tau}\langle \delta F(t)\delta F^\dagger(t-\tau)\rangle=2\gamma[\bar{\nu}(\Omega)+1] \ , \tag{3b}$$

and

$$\langle \delta F(t)\delta F(t')\rangle=\langle \delta F^\dagger(t)\delta F^\dagger(t')\rangle=0 \ , \tag{3c}$$

where $\bar{\nu}(\Omega)=(e^{\hbar\Omega/k_BT}-1)^{-1}$. If the output modes of the amplifier are created by $b^\dagger$ and annihilated by b, then to simulate active feedback we must add to the Hamiltonian a term

$$H_{\mathrm{feedback}}=g(b^\dagger a+a^\dagger b) \ , \tag{4}$$

while the dynamics of the amplifier itself are described by

$$b(t)=\int dt' M(t-t')a(t')+N(t) \ , \tag{5a}$$

$$b^\dagger(t)=\int dt' M^\dagger(t-t')a^\dagger(t')+N^\dagger(t) \ , \tag{5b}$$

where N and $N^\dagger$ are the operators which express the added amplifier noise. From these equations, it may be seen that

$$\frac{da(t)}{dt}=-i\omega a(t)-\gamma a(t)-ig\int dt' M(t-t')a(t')+\delta F(t)-igN(t) \ , \tag{6a}$$

$$\frac{da^\dagger(t)}{dt}=i\omega a^\dagger(t)-\gamma a^\dagger(t)+ig\int dt' M^\dagger(t-t')a^\dagger(t')+\delta F^\dagger(t)-igN^\dagger(t) \ . \tag{6b}$$

For definiteness, let us imagine that our model Hamiltonian describes a system of mass m and stiffness κ, so that the displacement

$$q=[\hbar/2(m\kappa)^{1/2}]^{1/2}(a^\dagger+a) \ ,$$

while an external force F adds a term $H_F=-Fq$ to the Hamiltonian. Then it is straightforward to show that Eqs. (6) imply a response function

$$\frac{\tilde{q}(\Omega)}{\tilde{F}(\Omega)}=\tfrac{1}{2}(m\kappa)^{-1/2}\left[\frac{1}{\Omega+\omega+g\tilde{M}(\Omega)-i\gamma}-\frac{1}{\Omega-\omega-g\tilde{M}^\dagger(\Omega)-i\gamma}\right] \ , \tag{7}$$

where $\tilde{q}(\Omega)=\int dt\, e^{i\Omega t}q(t)$, etc., while the Langevin terms in Eqs. (6) correspond to an effective noise force with spectral density

$$S_F^{\rm eff}(\Omega)=2\hbar(m\kappa)^{1/2}\int dt\, e^{i\Omega t}[\langle\delta F^\dagger(t)\delta F(0)\rangle+\langle\delta F(t)\delta F^\dagger(0)\rangle+g^2\langle N^\dagger(t)N(0)\rangle+g^2\langle N(t)N^\dagger(0)\rangle] \ . \quad (8)$$

It may be verified from Eqs. (3) that the force noise in the absence of feedback is given by

$$S_F^{(0)}(\Omega)=2\hbar(m\kappa)^{1/2}\int dt\, e^{i\Omega t}[\langle\delta F^\dagger(t)\delta F(0)\rangle+\langle\delta F(t)\delta F^\dagger(0)\rangle]=4\gamma(m\kappa)^{1/2}[2\bar{\nu}(\Omega)+1] \ , \quad (9)$$

in accord with the fluctuation-dissipation theorem.[6]

The correlation functions $\langle N^\dagger(t)N(0)\rangle$ may be bounded by applying the quantum-mechanical consistency condition that the commutation relations of the input and output modes be the same:

$$[b^\dagger(t),b(t')]=[a^\dagger(t),a(t')] \ . \quad (10)$$

Application of this constraint to Eqs. (5) is simplified by passing to the Fourier representation; in this representation one obtains

$$[\tilde{N}^\dagger(\Omega),\tilde{N}(\Omega')]=[1-\tilde{M}^\dagger(\Omega)\tilde{M}(\Omega')][\tilde{a}^\dagger(\Omega),\tilde{a}(\Omega')] \ . \quad (11)$$

Because all the noise processes are stationary, we must have

$$\langle[a^\dagger(\Omega),a(\Omega')]\rangle=\tilde{C}(\Omega)2\pi\delta(\Omega-\Omega') \ .$$

The function $\tilde{C}(\Omega)$ will be peaked near resonances of the response function in Eq. (7), and will be concentrated in a range of frequencies about these resonances comparable to their bandwidths. $\tilde{C}(\Omega)$ must also be normalized to $\int(d\Omega/2\pi)\tilde{C}(\Omega)=1$ to preserve the equal-time commutator $[a(t),a^\dagger(t)]=1$. These relations, together with the "generalized uncertainty principle"

$$\tfrac{1}{2}\langle R^\dagger R+RR^\dagger\rangle\geqslant\tfrac{1}{2}|\langle[R^\dagger,R]\rangle| \ , \quad (12)$$

imply that the spectral density of the amplifier noise N is

$$S_N(\Omega)\geqslant\tfrac{1}{2}|1-\tilde{M}^\dagger(\Omega)\tilde{M}(\Omega)||\tilde{C}(\Omega)| \ . \quad (13)$$

From Eq. (8) we then obtain the effective spectral density of the force noise,

$$S_F^{\rm eff}(\Omega)=S_F^{(0)}(\Omega)+(m\kappa)^{1/2}\hbar|\tilde{C}(\Omega)|g^2|1-\tilde{M}^\dagger(\Omega)\tilde{M}(\Omega)| \ . \quad (14)$$

In the limit of a high-gain amplifier ($|\tilde{M}|>>1$), we find that the added noise is just

$$\Delta S_F^{\rm eff}(\Omega)=(m\kappa)^{1/2}\hbar|\tilde{C}(\Omega)||g\tilde{M}(\Omega)|^2 \ . \quad (15)$$

This is a particularly simple result because, from Eq. (7), $g\tilde{M}$ is just the "self-energy term" in the response function; that is, $g\tilde{M}$ determines the change in the frequency response of the system when the feedback is applied. We thus arrive at the central conclusion: *The minimum noise added upon feedback from a high-gain amplifier is uniquely related to the change in frequency response achieved by the feedback.*

We can now answer the three questions outlined at the outset:

(1) If we want to shift the resonance frequency from ω to ω', then we must have $|g\tilde{M}|\approx|\omega-\omega'|$, so that the frequency-integrated force noise is

$$\int\frac{d\Omega}{2\pi}\Delta S_F^{\rm eff}(\Omega)\approx(m\kappa)^{1/2}\hbar(\omega-\omega')^2 \ , \quad (16)$$

assuming that the resonance in the presence of feedback is reasonably narrow and making use of the normalization condition on $\tilde{C}$. Thus we see that if feedback shifts the resonance down in frequency by a large amount, then the force noise added by the feedback amplifier becomes $\approx\hbar(m\kappa)^{1/2}\omega^2=\kappa\hbar(\kappa/m)^{1/2}$, independent of the resonance frequency in the presence of feedback. Had we tried to detect the signal at $\omega'<<\omega$ in the absence of feedback, there would have been a quantum displacement noise (zero-point motion) of $(\hbar/2\kappa)(\kappa/m)^{1/2}$, while a force F at ω' would produce a displacement F/κ, for an effective force noise of $(\kappa\hbar/2)(\kappa/m)^{1/2}$. It is apparent that the quantum noise in the amplifier "puts back" all of this quantum noise which was present before the feedback, and which was (nominally) removed by going to a much lower operating frequency. In fact the amplifier results in net doubling of the quantum noise which was present in the absence of feedback, and this factor of 2 is discussed by Caves.[1]

(2) If we want to narrow the bandwidth of the detector from γ to $\gamma-\eta$, then we must have $|g\tilde{M}|\approx\eta$. The frequency-integrated force noise from the amplifier is therefore $\approx(m\kappa)^{1/2}\hbar\eta^2$. Again we can compare to the situation in the absence of feedback, where the effective quantum force noise for a signal at resonance is $(\hbar\gamma^2/2)(m\kappa)^{1/2}$, so that in the narrow-band ($\eta\to\gamma$) limit the amplifier noise once again "puts back" twice the quantum noise which it filtered out.

(3) In the "bandwidth narrowing" configuration, instability results as soon as $\eta\geqslant\gamma$. Beyond this point the system is not a filter but an oscillator, emitting some stable signal of frequency ω and bandwidth $\Delta\omega$. The effective spectral density of the force noise contributed by the amplifier which powers the oscillator is then $S_F\geqslant(\hbar\gamma^2/\Delta\Omega)(m\kappa)^{1/2}$. If the oscillator is consuming a power P, then it is as if there were a force $F_{\rm eff}=(\gamma mP)^{1/2}$ across the dissipative element which determines the bandwidth γ in the absence of feedback. If there were no noise, then the zero crossings of the effective force would simply be spaced by the period of the oscillator, but the finite noise force results in a jitter in the timing of these zero crossings, and hence a finite bandwidth of the oscillator. It is easy to see that $\Delta\omega/\omega\approx(S_F\Delta\omega/F_{\rm eff})^{1/2}$, so that the quantum limit to oscillator stability is given by $\Delta\omega/\omega\geqslant(\hbar\omega\gamma/P)^{1/2}$. This result may be understood as follows: The energy E stored in the oscillator is dissipated in a mean time γ^{-1}, which means that $E=P/\gamma$; thus the frequency stability is given by $\Delta\omega/\omega\geqslant(\hbar\omega/E)^{1/2}$. But the frequency stability is just a measure of the phase noise, $\Delta\omega/\omega\approx\Delta\phi$, so that we obtain $\Delta\phi\geqslant N^{-1/2}$, where N is the number of quanta stored in the oscillator; this of course is precisely a factor of 2 more noise than the "standard quantum limit," $\Delta\phi\geqslant\frac{1}{2}N^{-1/2}$. Once again the effect of the amplifier is simply to double the quantum noise, even when it serves to qualitatively change the dynamics of the system

from small amplitude stability to instability.

The physical picture which emerges from this analysis is straightforward. Active feedback can be used to manipulate the frequency response of a system, and thereby improve the ratio of signal to thermal noise. Nominally we expect, from a Langevin approach, that these changes in frequency response could also reduce the effects of quantum noise, but if this were true we could use active feedback to circumvent the quantum limits to measurement. In fact, the self-consistent inclusion of the quantum limit on amplifier noise corrects this error, and yields the same quantum limits as would be obtained if the amplifier followed the detector with no feedback. Thus the limits to measurement imposed by the need to amplify a quantum signal up to the classical level are the same whether the amplifier is used in series with the detector or as an essential part of the detector (in feedback), and this is as it must be.

Similarly, a Langevin approach would suggest that, as the threshold of oscillation is approached and the bandwidth of the system narrows to zero, all the noise would be filtered away and the frequency stability of the oscillation could be infinite. Again this would conflict with the uncertainty princple, which dictates a minimum phase noise, and again the self-consistent inclusion of the amplifier noise rectifies this error.

These conclusions emphasize that, although quantum noise may sometimes be effectively described by a Langevin "force operator," one must be careful in applying this approach to the quantum noise of amplifiers. Essentially this is because the quantum noise of amplifiers arises from a consistency condition between input and output—unitarity—and the apparent Langevin force changes whenever either input or output mode dynamics are changed. This may be contrasted with the Langevin forces describing thermal noise, which arise from coupling between the system and the heat bath, and therefore change only if this coupling is changed.

In practical terms, these results imply that amplifiers can be used in lieu of refrigerators—we can take many broadband detectors and apply feedback to synthesize narrowband detectors, thereby reducing the effects of thermal noise—but that, as in normal refrigerators, we can never freeze out the quantum fluctuations whose presence is dictated by the uncertainty principle.

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[1]C. M. Caves, Phys. Rev. D 26, 1817 (1982).

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[3]Dissipation is essential in any treatment of thermal noise, but many situations where quantum noise is relevant involve little or no damping (e.g., the resonant bar gravitational antennae). Applications of active feedback almost always involve some source of damping which is compensated by the amplifier, however, so we must be careful to include dissipation explicitly. Interestingly, this means that the analysis involves only a single mode of the harmonic oscillator, as is evident from Eq. (1), but in fact takes account of the large number of modes which form the heat bath; as a result, the creation and annihilation operators of the oscillator can have time dependencies significantly different from $e^{\pm i\omega t}$. This approach therefore combines aspects of the single- and multimode cases treated separately by Caves (Ref. 1).

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