Non-Abelian solutions of Yang-Mills equations in the strong-coupling limit

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Using the $1/g$ expansion we find a new class of non-Abelian solutions to the Yang-Mills equations with sources in the strong-coupling limit. They appear to have a nontrivial topology and they can also be viewed as the $g\to\infty$ limit of a wider class of potentials. The external source, which was chosen to have a one-maximum strong-coupling limit, has to have a two-maxima weak-coupling distribution to make the solution exact for an arbitrary value of the coupling constant.

I. INTRODUCTION

There are several reasons to investigate the classical Yang-Mills (YM) theory: a possibility of semiclassical approximations for strong couplings,[1] a common belief that topologically nontrivial solutions to the YM equations can survive after quantization,[2] or, last but not least, our ignorance about the nonperturbative region of quantum chromodynamics (QCD).

However, even the classical system of the coupled Yang-Mills-Dirac equations is highly complicated and little is known about their solutions. For that reason attention has been paid to solutions of the YM equations with c-number sources.[3]

The source may provide an approximate description of a system to which the YM fields are coupled, and whose dynamics may be ignored. Hence, it is obvious that the external source and potential can be complicated functions of the space-time variable and the coupling constant g.

A significant step in *classical chromodynamics* (CCD) was taken by Mandula[4] who showed that the Abelian solutions to the YM equations becomes unstable above the critical value of the coupling constant $g=g_{\text{crit}}\approx 1$ and, because of it, they seem to be physically irrelevant.

Some of the new bifurcating solutions were investigated numerically and in a vicinity of the bifurcation point.[5] Nevertheless, even CCD remains unknown in the strong-coupling region. We can only guess that in this region the most nonlinear (cubic) terms in the YM equations become dominant. Hence, owing to the nonlinearity, we expect the YM potentials to have a nontrivial topology and to be essentially non-Abelian in this region.

Recently, the $1/g$ expansion was proposed[6] to investigate CCD for large values of the coupling constant. It was also shown that the equations for arbitrary-order coefficients in this expansion are gauge covariant order by order and they imply the gauge-invariant total and local screening effect in the strong-coupling limit: $g\to\infty$.

Here, using the proposed expansion, we investigate the strong-coupling limit of CCD. In the following section the $1/g$ expansion is introduced and briefly discussed. In Sec. III a new class of solutions to the YM equations is found. Section IV is devoted to the analysis of special solutions from the new class. They appear to have a non-Abelian holonomy group and nontrivial topology. The last section contains final remarks. In the Appendix a class of solutions to the YM equations, dual to those of Sec. III, is briefly discussed.

II. STRONG-COUPLING EXPANSION

The YM equations are[7]

$$\partial^\mu\vec{F}_{\mu\nu}+g\vec{A}^\mu\times\vec{F}_{\mu\nu}=g\,\vec{j}_\nu\,, \tag{2.1}$$

where the field strength tensor is defined by

$$\vec{F}_{\mu\nu}\equiv\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu+g\vec{A}_\mu\times\vec{A}_\nu\,. \tag{2.1'}$$

We are interested in the YM potentials and sources with a finite strong-coupling limit. Hence, in this limit, they are to have the form

$$\vec{A}_\mu(x,g)=O(g^\alpha)\quad\text{as } g\to\infty\ , \tag{2.2}$$

where

$$\alpha\leq 0\ . \tag{2.3a}$$

Let us notice that to make the most nonlinear (cubic) terms in the YM equation dominant we also have to impose, according to naive power counting, the following *lower* limit for α:

$$-1 \leq \alpha \ . \tag{2.3b}$$

In order to satisfy the conditions (2.3), potentials (and sources) should have, as functions of g, the general form

$$\vec{A}_\mu(x,g) = \sum_n \vec{A}^n_\mu(x) g^{\alpha_n} \ , \tag{2.4}$$

where

$$\alpha \equiv \text{maximum}\{\alpha_n\} \tag{2.5}$$

should satisfy (2.3).

We choose $\alpha=0$ and $\alpha_n=-n$, as in Ref. 6, which seems to be the simplest and the most plausible choice (at least from the mathematical point of view). Then (2.4) has the form

$$\vec{A}_\mu(x,g) = \sum_{n=0}^{\infty} \vec{A}^n_\mu(x) g^{-n} \tag{2.6}$$

and, respectively,

$$\vec{j}_\mu(x,g) = \sum_{n=0}^{\infty} \vec{j}^n_\mu(x) g^{-n} \ . \tag{2.6$'$}$$

The lowest-order coefficient in (2.6) has been shown to be[6]

$$\vec{A}^0_\mu(x) = \vec{A}(x)\alpha_\mu(x) \ , \tag{2.7}$$

where $\vec{A}(x), \alpha_\mu(x)$ are arbitrary functions. Then the first- and second-order equations are

$$\vec{A} \times (\alpha_\mu \alpha^\nu \partial_\nu \vec{A} - \alpha_\nu \alpha^\nu \partial_\mu \vec{A}) + \text{terms with } \vec{A}^1_\mu = \vec{j}^0_\mu \ , \tag{2.8a}$$

$$\Delta(\vec{A}\alpha_\mu) - \partial_\mu \partial^\nu (\vec{A}\alpha_\nu) + \text{terms with } \vec{A}^1_\mu \text{ and } \vec{A}^2_\mu = \vec{j}^1_\mu \ . \tag{2.8b}$$

Here the point is that if we impose on the coefficients $\vec{A}^1_\mu$ and $\vec{A}^2_\mu$ additional restrictions of the form

$$\vec{A} \times \vec{A}^1_\mu = 0 \ , \tag{2.9a}$$

$$\vec{A} \times \vec{A}^2_\mu = 0 \ , \tag{2.9b}$$

$$\vec{A}^1_\mu \times \vec{A}^2_\nu = 0 \ , \tag{2.9c}$$

then the terms in (2.8) which contain $\vec{A}^1_\mu$ and $\vec{A}^2_\mu$ coefficients vanish (they are not written explicitly for brevity). In fact, weaker but more complicated restrictions are sufficient.

Now Eqs. (2.8) can be be solved if sources are given. The solution $\vec{A}^0_\mu(x) = \vec{A}\alpha_\mu$ approximates the full potential $\vec{A}_\mu(x,g)$ in the $g \to \infty$ limit. However, the same $\vec{A}^0_\mu$ can also be viewed as an *exact* solution to the YM equations with the external source

$$\vec{j}_\mu = \vec{j}^0_\mu + \frac{1}{g} \vec{j}^1_\mu \ , \tag{2.10}$$

where $\vec{j}^1_\mu$ is not arbitrary if $\vec{j}^0_\mu$ is given. As can be seen from (2.10) the necessary correction to the source is of order $1/g$ and it vanishes in the $g \to \infty$ limit.

In the following section a class of solutions to (2.8) with the above twofold interpretation will be given and adequate corrections to the source will be computed.

III. NEW STRONG-COUPLING SOLUTIONS

Before solving Eqs. (2.8) consistency conditions for (2.8a) will be given. They are

$$\vec{j}^0_\mu \cdot \vec{A} = 0 \ , \tag{3.1a}$$

$$\vec{j}^0_\mu \alpha^\mu = 0 \ . \tag{3.1b}$$

Equations (3.1) will be useful in solving (2.8a). It is interesting to notice that (3.1) imposes *restrictions on the source term* $\vec{j}^0_\mu$ of the form

$$\text{range} \, || j^{0a}_\mu || < 3 \ , \tag{3.2}$$

where j^{0a}_μ is treated as a 3×4 matrix with the indices a, μ. As can be checked the condition (3.2) is gauge invariant.

To solve Eq. (2.8a) generally (i.e., without specification of the external source) we assume that α_μ is of the simple form

$$\alpha_\mu(x) = \delta_{\mu 0} \ . \tag{3.3}$$

An analogous *Ansatz* leading to the purely chromomagnetic YM field is briefly discussed in the Appendix.

With (3.1) and (3.3) we have, instead of (2.8a), the following equations:

$$\vec{j}^0_0 = 0 \tag{3.4}$$

and

$$\partial_i \vec{A} \times \vec{A} = \vec{j}^0_i \ , \tag{3.5}$$

with consistency conditions

$$\vec{A} \cdot \vec{j}^0_i = 0 \ , \tag{3.6a}$$

$$\partial_i \vec{A} \cdot \vec{j}^0_i = 0 \ , \tag{3.6b}$$

where from now on there is no summation over repeated Latin indices i,j,k.

Notice that, in accordance with the general result of Ref. 6, the source term $\vec{j}^0_\mu$ is totally and locally screened by the field current and Eq. (3.5) is exactly the screening condition.

It is easy to show that Eqs. (3.6) are equivalent to

$$\vec{A} \cdot \vec{j}^0_i = 0 \ , \tag{3.7a}$$

$$\vec{A}\cdot\vec{j}^{\,0}_{i,i}=0\,, \tag{3.7b}$$

where $\vec{j}_{i,i}\equiv(\partial/\partial x^i)\vec{j}_i$, no summation. Hence the general solution to (2.8a) should be of the form

$$\vec{A}=\lambda_i\,\vec{j}^{\,0}_i\times\vec{j}^{\,0}_{i,i}\,. \tag{3.8}$$

The scalar factors $\lambda_i(x)$ are easy to calculate using (2.8a). We have

$$\lambda_i=\left[\vec{j}^{\,0}_{i,i}\cdot\left[\vec{j}^{\,0}_i\times\vec{j}^{\,0}_{i,ii}\right]\right]^{-1/2}\,. \tag{3.9}$$

As (3.8) should be valid for $i=1,2,3$ it imposes additional restrictions on $\vec{j}^{\,0}_i$ of the form

$$\lambda_i\,\vec{j}^{\,0}_i\times\vec{j}^{\,0}_{i,i}=\lambda_k\,\vec{j}^{\,0}_k\times\vec{j}^{\,0}_{k,k} \tag{3.10}$$

for each $i,k=1,2,3$, unless

$$\vec{j}^{\,0}_i\times\vec{j}^{\,0}_{i,i}=0\,. \tag{3.11}$$

If some of the space components of $\vec{j}^{\,0}_i$ vanish and (3.11) is satisfied,[8] the potential (3.8) is Abelian with respect to the space variable x_i, i.e., the x_i dependence of the potential is through a trivial, color-independent factor.

Solutions (3.8) and (3.9) are quite general because they contain an arbitrary external-source term $\vec{j}^{\,0}_i$. Moreover, a source dependence of potentials is very simple and local (i.e., there is no integration as there is, for example, in the classical electrodynamics for smooth sources). This is possible thanks to the non-Abelian and nonlinear nature of the YM equations.

The obtained solution (3.8) is an exact solution to the YM equations in the $g\rightarrow\infty$ limit only. However, as was mentioned earlier, if we add an extra term to the source $\vec{j}^{\,0}_\mu$ then the higher-order equations (2.7), as well as the full YM equations, are satisfied with the same potential.

The additional source term should be of the form

$$\vec{j}^{\,i}_i=0\,, \tag{3.12}$$

$$\vec{j}^{\,1}_0=\Delta\vec{A}^{0}_0\equiv\Delta\vec{A}\,, \tag{3.13}$$

with the additional *Ansatz* for $\vec{A}^{1}_\mu,\vec{A}^{2}_\mu$ of the form (2.9). Then the full external current is (2.10) and our solution is exact. But it is approximate (in the $g\rightarrow\infty$ limit) for other sources, which tend to $\vec{j}^{\,0}_\mu$ in this limit.

IV. PROPERTIES OF THE NEW SOLUTIONS

In this section properties of the new solutions are investigated and illustrated on a special example.

Let us start with the *Ansatz* for the strong-coupling limit of the external source of a simple Gaussian but non-Abelian form

$$\vec{j}^{\,0}_\mu=\delta_{\mu 3}\frac{1}{\lambda^3}\vec{j}\,, \tag{4.1a}$$

where

$$\vec{j}\equiv\psi^2(x,y)e^{-\alpha z^2/\lambda^2}\begin{bmatrix} a\\ bz/\lambda\\ cz^2/\lambda^2\end{bmatrix}, \tag{4.1b}$$

$\psi=\psi(x,y)$ is a scalar function vanishing at infinity, α,a,b,c are real parameters satisfying conditions

$$\alpha>0,\quad abc<0\,, \tag{4.2}$$

and λ is a dimensional parameter: $[\lambda]=$length. Introduction of at least one dimensional parameter is necessary if the source is to be localized, with finite size and energy. It is easy to check that the current (4.1) satisfies conditions (3.1b) and (3.2).

Using (3.8) and (3.9) we have immediately the following potential generated by our source:

$$\vec{A}_\mu=\delta_{\mu 0}\frac{1}{\lambda}(-2abc)^{-1/2}\psi(x,y)\exp\left[-\frac{\alpha}{2}z^2/\lambda^2\right]\begin{bmatrix} bcz^2/\lambda^2\\ -2acz/\lambda\\ ab\end{bmatrix}. \tag{4.3}$$

To make the solution (4.3) exact, according to (3.12), and (3.13), an additional term should be added to the external current. It is

$$\vec{j}^{\,1}_\mu=\delta_{\mu 0}\frac{1}{\lambda^3}(-2abc)^{-1/2}\exp\left[-\frac{\alpha}{2}z^2/\lambda^2\right]\begin{bmatrix}\alpha^2bc\psi z^4/\lambda^4+bc(\Delta_2\psi-5\alpha\psi)z^2/\lambda^2+2bc\\ -2\alpha^2ac\psi z^3/\lambda^3-2ac(\Delta_2\psi-3\alpha\psi)z/\lambda\\ \alpha^2ab\psi z^2/\lambda^2+ab(\Delta_2\psi-\alpha\psi)\end{bmatrix}, \tag{4.4}$$

where

$$\Delta_2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\,.$$

Hence potentials and sources are smooth, localized functions vanishing at infinity provided that ψ is regular and tends to zero when $x,y \to \pm\infty$.

The field tensor is purely chromoelectric and its only nonzero components are

$$\vec{F}_{0i} \equiv \vec{E}_i = -\partial_i \vec{A}_0 \, . \tag{4.5}$$

From (4.3) we have

$$\vec{E}_{x,y} = -\frac{1}{\lambda^2}(-2abc)^{-1/2}\partial_{x,y}\psi \exp\left[-\frac{\alpha}{2}z^2/\lambda^2\right]\begin{bmatrix} bcz^2/\lambda^2 \\ -2acz/\lambda \\ ab \end{bmatrix} , \tag{4.6a}$$

$$\vec{E}_z = \frac{1}{\lambda^2}(-2abc)^{-1/2}\psi \exp\left[-\frac{\alpha}{2}z^2/\lambda^2\right]\begin{bmatrix} \alpha bcz^3/\lambda^3 - 2bcz/\lambda \\ -2\alpha acz^2/\lambda^2 + 2ac \\ \alpha abz/\lambda \end{bmatrix} , \tag{4.6b}$$

i.e., $\vec{E}_x$ and $\vec{E}_y$ are parallel in group (color) space to $\vec{A}_0$, and $\vec{E}_z$ is parallel to $\partial_z \vec{A}_0$. Analogous solutions with purely chromomagnetic field are given in the Appendix.

A. Special solution

Here we fix free parameters and function ψ in (4.1b) to analyze the new solutions and their properties. Let us choose for them the following values:

$$\alpha = +\tfrac{1}{2}, \quad a = -1, \quad b = +1, \quad c = +\tfrac{1}{2} \tag{4.7}$$

and for function $\psi = \psi(x,y)$ the axially symmetric function of the form

$$\psi(x,y) = \exp(-\tfrac{1}{4}\rho^2/\lambda^2) \, , \tag{4.7'}$$

where $\rho^2 \equiv x^2 + y^2 \equiv r^2 - z^2$.

Then the solution and source are axially symmetric and they have the simple form

$$\vec{j} = \exp(-\tfrac{1}{2}r^2/\lambda^2)\begin{bmatrix} -1 \\ z/\lambda \\ \tfrac{1}{2}z^2/\lambda^2 \end{bmatrix} \tag{4.8}$$

and, using (4.3),

$$\vec{A}_0 = \frac{1}{\lambda}\exp(-\tfrac{1}{4}r^2/\lambda^2)\begin{bmatrix} \tfrac{1}{2}z^2/\lambda^2 \\ z/\lambda \\ -1 \end{bmatrix} . \tag{4.9}$$

The additional source term (4.4) is

$$\vec{j}^{\,1}_{\mu} = \delta_{\mu 0}\frac{1}{\lambda^3}\frac{1}{8}\exp(-\tfrac{1}{4}r^2/\lambda^2)\begin{bmatrix} z^4/\lambda^4 + (\rho^2/\lambda^2 - 14)z^2/\lambda^2 + 8 \\ 2z^3/\lambda^3 + 2(\rho^2/\lambda^2 - 20)z/\lambda \\ -2z^2/\lambda^2 - (2\rho^2/\lambda^2 - 12) \end{bmatrix} \tag{4.10}$$

and it breaks the local screening effect.

The leading term in the source for strong couplings, $\vec{j}^{\,0}_{\mu}$, is suppressed for weak couplings by the term $(1/g)\,\vec{j}^{\,1}_{\mu}$. However, the total color charge defined as

$$\vec{Q} \equiv \int d\vec{x}\; \vec{j}_0(\vec{x}) \tag{4.11}$$

remains unchanged and it vanishes:

$$\vec{Q} = 0 \, . \tag{4.12}$$

To visualize the g dependence of the external source the Lorentz- and gauge-invariant $q = \vec{j}_{\mu} \cdot \vec{j}^{\,\mu}$ is plotted as a function of the variable

z for strong and weak couplings [Figs. 1(A) and 1(B), respectively]. We set ρ equal to zero. The invariant q is of the Gaussian form in the strong-coupling limit [Fig. 1(A)], whereas it has apparent two-maxima distribution for weak couplings (at the symmetry axis).

The field energy density, defined as usual by

$$\mathscr{E} \equiv \tfrac{1}{2}(\vec{E}_k \cdot \vec{E}_k + \vec{B}_k \cdot \vec{B}_k) \ , \tag{4.13}$$

is independent of the coupling constant and it is concentrated near the 0z axis. For $\rho=0$ it has the form plotted in Fig. 2. The total field energy is

$$E = \frac{1}{\lambda} \frac{45}{8\sqrt{2}} \pi^{3/2} \ . \tag{4.14}$$

B. Holonomy group and topology

Now we are going to prove that the solution (4.9) is essentially non-Abelian, i.e., it is not gauge equivalent to an Abelian one. At first a holonomy group will be investigated.

The holonomy group is non-Abelian if at least one of the commutators of the $F_{\mu\nu}$ components is nonzero[9]:

$$[F_{\mu\nu}, F_{\alpha\beta}] \neq 0 \ . \tag{4.15}$$

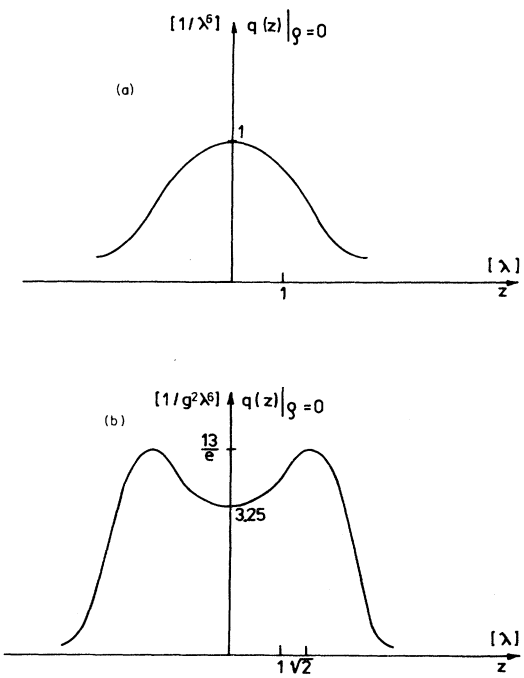

FIG. 1. Distribution of the invariant q at the 0z axis (a) for strong couplings, $g \gg 1$, (b) for weak couplings, $g \ll 1$.

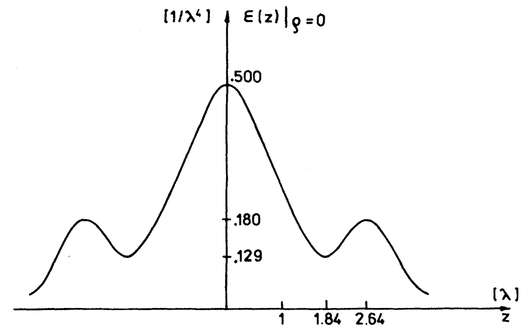

FIG. 2. The field energy density distribution at the 0z axis.

The only nonzero commutators of the type (4.15) are in our case

$$[E_{x,y}, E_z] \neq 0 \ . \tag{4.15$'$}$$

Owing to the field equation (3.5) the condition (4.15′) is fulfilled when we have

$$\partial_x \psi(x,y) \neq 0 \text{ or } \partial_y \psi(x,y) \neq 0 \ , \tag{4.16}$$

where ψ was introduced in (4.1b).

If (4.16) is satisfied the commutator (4.15′) is proportional to the external source term $\vec{j}$. Hence the non-Abelian character of the new solutions is proved.

Our solutions also have interesting topological properties. Discussed potentials and sources normalized to the unit (color) vectors have well-defined limit when $z \to \pm\infty$. We have

$$\lim_{z\to+\infty} \vec{A}/|\vec{A}| = \lim_{z\to-\infty} \vec{A}/|\vec{A}| = \text{const} \tag{4.17}$$

and the same for the source $\vec{j} = \vec{j}(z)$. This enables us to compactify the real axis $R \ni z$ into the one-dimensional sphere S^1. Then $\vec{A}(z)$ and $\vec{j}(z)$ can be viewed as a mapping: $z \in S^1 \to S^1 \subset SU_2$. Owing to this property potentials and sources can be classified by a topological invariant called the winding number.[10]

It is easy to check that for our potentials and sources the winding number is equal to 1. Hence they cannot be continuously deformed (e.g., by a gauge transformation) to the topologically trivial ones with one group (color) direction and the zero winding number, as is the case for the Abelian potentials and sources.

V. FINAL REMARKS

Within the framework of the $1/g$ expansion the strong-coupling limit of the YM potentials obeying

(2.6) was investigated. Their explicit form in this limit was obtained with the use of additional simplifying *Ansätze* (2.9) and (3.3). They appeared to be local functions of the smooth external current (here *local* means that there is no integration).

Moreover, they obey the full YM equations for arbitrary values of the coupling constant g if the extra terms (3.12) and (3.13), vanishing in the $g \to \infty$ limit, are added to the source. A special class of localized, axially symmetric, non-Abelian and topologically nontrivial solutions of such kind is discussed in Sec. IV. The solutions also exhibit the local screening effect.

It is interesting to note that the external current (more precisely the invariant q built of the current) has the one-maximum form in the strong-coupling limit [Fig. 1(a)], whereas for weak couplings it has the distinct two-maximum distribution [Fig. 1(b)]. This is due to the nonzero charge density (but with zero total charge) from the correction (4.10) to the external source. This correction becomes the leading term for $g \ll 1$.

In this place we are tempted to speculate that the source from Sec. IV mimics hadrons, which are also seen as the one-particle objects for low energies (strong couplings) but as two- or three-particle objects for high energies (weak couplings) in the quark model.

Further investigations can be continued in at least two directions. More general and physically relevant solutions can be searched analytically and numerically within the proposed approach. Their quantum meaning and possible applications in QCD should also be considered.

APPENDIX

Using the *Ansatz*

$$\alpha_\mu(x) = \delta_{\mu 1} \quad \text{(A1)}$$

instead of (3.3), chromomagnetic solutions are obtained and have the following form:

$$\vec{E}_i = 0, \quad \vec{B}_1 = 0 , \quad \text{(A2)}$$

$$\vec{B}_{2,3} = \pm \partial_{3,2} \vec{A} , \quad \text{(A3)}$$

where $\vec{B}_i$ is defined as usual by

$$\vec{B}_i = \tfrac{1}{2} \epsilon_{imn} \vec{F}^{mn} . \quad \text{(A4)}$$

Then the field equations, analogous to (3.4) and (3.5), are

$$\vec{j}^{\,0}_0 = \vec{j}^{\,0}_1 = 0 \quad \text{(A5)}$$

and

$$\partial_2 \vec{A} \times \vec{A} = \vec{j}^{\,0}_2 , \quad \text{(A6)}$$

$$\partial_3 \vec{A} \times \vec{A} = \vec{j}^{\,0}_3 . \quad \text{(A7)}$$

Equations (A6) and (A7) are the same as (3.5). Hence the chromomagnetic solutions have properties analogous to the chromoelectric solutions of Sec. IV. To have the exact correspondence we should choose $\vec{j}^{\,0}_2 = 0$.

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[6]A. Górski, J. Phys. A 16, 849 (1983).

[7]Here vectors are in the group (color) space. We limit our discussion to the gauge group $G = SU_2$ and to the time-independent potentials and sources. The metric tensor signature is chosen to be $(+\ +\ +\ -)$.

[8]The special case of $\vec{j}^{\,0}_i \neq 0$ but (3.11) is satisfied is not discussed here for the sake of simplicity.

[9]H. G. Loos, J. Math. Phys. 8, 2114 (1967).

[10]See, for example, H. Flanders, *Differential Forms with Applications to the Physical Sciences* (Academic, New York, 1963), Chap. 6.