Topology in the Weinberg-Salam theory

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We show that the configuration space of the classical, bosonic Weinberg-Salam theory has a noncontractible loop. This probably implies that there is an unstable, static, finite-energy solution of the field equations. Its energy is the height of the barrier for tunneling between "topologically distinct" vacuums. We establish an upper bound on this energy of order 10 TeV.

I. INTRODUCTION

All experiments to date indicate that the weak and electromagnetic interactions are mediated by the bosonic fields of the gauge theory proposed by Weinberg and Salam.[1] The gauge group is U(2) and it is spontaneously broken to $U(1)_{em}$ by a complex Higgs doublet. Furthermore, all known weak processes, including the recently discovered W-boson production,[2] are well described by a perturbative treatment of the quantized theory. The main purpose of this paper is to argue that nonperturbative phenomena exist in the Weinberg-Salam theory, even though the theory admits neither monopoles nor instantons, and they become important at energies of order 1—10 TeV. Our arguments are primarily topological in nature, so we start with some mathematics.

It has been known for some time that there is a connection between the topology of a smooth manifold and the stationary points of an arbitrary smooth function defined on it. This is the subject of Morse theory.[3] A standard example is a two-torus standing on end, with the height above some reference plane the function defined on it (see Fig. 1). The existence of points where the height is minimal (P_0) and maximal (P_3) is a consequence simply of the compactness of the torus, and such points must also occur on a two-sphere, for example. What is surprising is that the topology of the torus requires that there must be at least two saddle points (P_1 and P_2). On a two-sphere there need be no further stationary points.

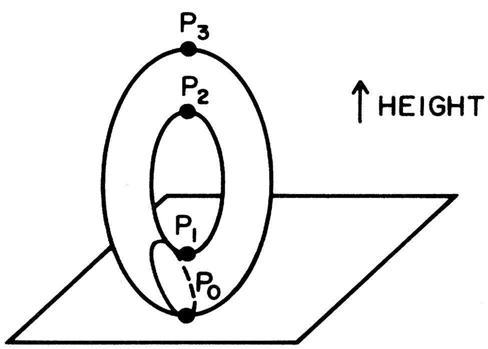

FIG. 1. Stationary points of the height on a torus.

Ljusternik and Šnirelman use the following minimax idea to prove the existence of these saddle points.[4] Suppose that the minimal height occurs at a single point P_0. Consider all loops on the torus which pass through P_0 and are homotopic to the loop shown in Fig. 1. On each of these loops there is a point where the height is maximal. Now consider the infimum over all loops of these maximal heights. It can be shown that there is a loop whose maximal height is precisely this minimal possible value, and P_1, the point on it where this height is attained, is distinct from P_0 and is a saddle point. The maximal height on any other loop provides an upper bound to the height at P_1.

On a noncompact manifold, this reasoning can break down. For example, consider the two-dimensional manifold shown in Fig. 2, and the loops homotopic to the one indicated. The infimum of the maximal heights on these loops exists, but there is no saddle point because it "escapes to infinity."

By a bold extension of these ideas, one can study the static classical solutions in field theories. The manifold here, which we shall refer to as the configuration space, is the function space which consists of all finite energy, static field configurations, and the function(al) defined on it is the energy. Suppose that there is a unique vacuum configuration of minimal energy E_0 and that there are noncontractible loops in the configuration space beginning and ending at the vacuum. Let us restrict attention just to the loops in a particular homotopy class. Each of these loops has a configuration on it of maximal energy, and let E_1 be the infimum of these energies. By analogy with the situation on the torus, it is a reasonable hypothesis that there is

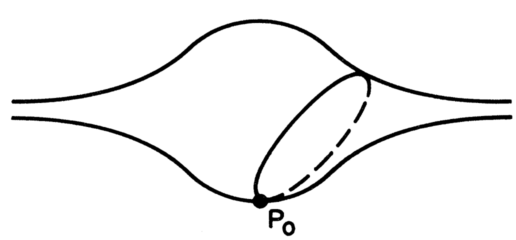

FIG. 2. A manifold with noncontractible loops but no saddle points.

a loop whose maximal energy configuration has an energy precisely equal to E_1, and that this configuration is a saddle point of the energy functional, and therefore a solution of the field equations. Since the vacuum is unique, a necessary condition for the existence of such a configuration is that $E_1 > E_0$.

By developing rigorously an infinite-dimensional version of Ljusternik-Šnirelman theory for this field-theory application, Taubes has recently obtained a remarkable proof of the existence of a new type of static solution in a Yang-Mills-Higgs model.[5] The model is the Bogomolny-Prasad-Sommerfield (BPS) limit of the SO(3) Yang-Mills theory minimally coupled to a triplet (adjoint) Higgs field. Finite-energy configurations have a quantized topological charge which measures the net magnetic charge. It was already known that this model has many static solutions, namely, the Prasad-Sommerfield monopole and the large class of static multimonopoles, all of which are local minima of the energy. But Taubes' new solution is in the vacuum sector, where the topological charge is zero, and where the only previously known solution was the vacuum itself. It is associated with a noncontractible loop in this sector of the configuration space, and it is a saddle point of the energy functional, and therefore unstable. Physically, it can probably be visualized as a monopole-antimonopole pair in unstable equilibrium, at a separation where the long-range Coulomb attraction is balanced by a short-range repulsion of essentially topological origin. This configuration is unstable against a rotation of one monopole relative to the other about the line joining them.

The application of Ljusternik-Šnirelman theory to any Yang-Mills-Higgs model is hard, for several reasons. First, the manifold is infinite dimensional, so there are analytic difficulties. Second, the manifold is noncompact. Third, one must deal with the gauge invariance of the energy functional.

The gauge invariance can be handled by regarding the configuration space as the space of gauge orbits.[6] That is, one regards a field configuration not simply as a gauge potential and Higgs field $\{A,\Phi\}$ defined everywhere in space, but as the set of all $\{A',\Phi'\}$ gauge equivalent to $\{A,\Phi\}$. It helps if one can find a completely unambiguous partial or complete gauge-fixing procedure because this reduces the size of the gauge orbits. Note that only the spatial components of the gauge potential appear here. Time, and the time component of the gauge potential, play no role. The field energy is given by the spatial part of the Hamiltonian and there is no kinetic contribution.

The reason for worrying about gauge invariance is that, depending on the boundary conditions, the gauge orbits themselves could have some nontrivial topology unrelated to the existence of stationary points of the energy. One wants, in particular, that the vacuum is a single point in the configuration space, and this requires the removal of the gauge freedom.

We shall consider just one aspect of the noncompactness problem, namely, the effect of scale transformations.[7] Suppose $\{A,\Phi\}$ is some field configuration, and $\{A_{(\nu)},\Phi_{(\nu)}\}$ the same configuration rescaled by ν. That is,

$$A_{(\nu)}(x)=\nu A(\nu x)\ ,\quad \Phi_{(\nu)}(x)=\Phi(\nu x)\ . \qquad (1.1)$$

The energy of $\{A,\Phi\}$ has three non-negative contributions: from the field tensor, from the covariant derivative of Φ, and from the Higgs potential. In three dimensions, these are multiplied by ν, ν^{-1}, and ν^{-3}, respectively, when $\{A,\Phi\}$ is replaced by $\{A_{(\nu)},\Phi_{(\nu)}\}$. The energy diverges both as $\nu\to 0$ and $\nu\to\infty$, and has a unique minimum with respect to ν at some finite value. Scale transformations are therefore no obstacle to the existence of classical solutions in a coupled Yang-Mills-Higgs model. This contrasts with the situation in pure Yang-Mills or pure scalar models in three dimensions, where the energy varies monotonically with ν, and there are no static solutions other than the vacuum.

In this paper we look at the topology of the configuration space of the classical Weinberg-Salam theory.[1] Only the gauge and Higgs fields are considered, and the fermion fields set to zero. We shall show that here too there are noncontractible loops in the configuration space, beginning and ending at the vacuum, and it seems likely that the minimax principle applied to these loops again implies the existence of a static, unstable, finite-energy solution. As before, its energy is the infimum of the maximal energies on the loops. We explicitly describe the field configurations on a restricted class of noncontractible loops. These fields have a high degree of symmetry. By computing the maximal energy on a suitably chosen loop in this class, we obtain an upper bound on the actual energy of the solution. Both our bound and the actual energy depend on such details as the weak mixing angle and the mass of the Higgs particle, but their order of magnitude is M_W/α. Here M_W is the semiclassical mass of the W boson and α is the fine-structure constant. We make no attempt to prove rigorously that a solution exists, but hope that Taubes' methods can be adapted to this case.

There are important physical consequences of the existence of noncontractible loops, whether or not the associated static solution exists. In general, it is likely that the infimum of the maximal energies on such loops defines the energy scale where perturbation theory breaks down. The analogy here is with the quantum pendulum, described by the Mathieu equation

$$-\frac{d^2\psi}{d\theta^2}+q(1-\cos\theta)\psi=E\psi\ . \qquad (1.2)$$

This is the simplest model of a physical system where the configuration space has a noncontractible loop. If the pendulum is long (q large), the ground state and low-lying excited states are approximately harmonic-oscillator states near $\theta=0$. One can develop perturbation series in $q^{-1/2}$ to take into account the anharmonicity, but such series for the eigenvalues do not converge.[8] Loosely speaking, this is because they ignore tunneling between $\theta=0$ and $\theta=2\pi$. If one is interested in the high-energy states ($E \gtrsim 2q$), these series are clearly inappropriate. There a better first approximation is to regard the pendulum as freely rotating.

In the Weinberg-Salam theory, conventional weak-coupling perturbation theory provides series expansions about the harmonic-oscillator states of free fields. For phenomena at or below the energy scale M_W these series should be useful up to high order. However, we expect the nontrivial topology of the configuration space ultimately to make them diverge. More importantly, for phenomena at the energy scale M_W/α or higher, conventional perturbation theory is probably completely unreliable.

Let us consider what the most interesting nonperturbative phenomena would be. We suppose now that the unstable solution does exist, although this is not crucial. Since the solution is at a saddle point of the field potential energy, its energy represents the height of a barrier between lower energy regions of field configuration space. We shall show that this barrier in fact separates what can be regarded as topologically distinct vacuums. Such vacuums are really identical field configurations, but continuity considerations imply that they must be described in different gauges.

In a pure SU(2) gauge theory, instantons control tunneling over a barrier between topologically distinct vacuums, and their Euclidean action is $8\pi^2/g^2$, where g is the coupling constant. However, because of scale invariance, the height of the barrier is indeterminate. The Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton[9] climbs to a height inversely proportional to its scale size, so this height is arbitrarily small.

In the Weinberg-Salam theory, on the other hand, the barrier height is well defined but there are no true instantons. A scaling argument shows that in any finite-action solution of the four-dimensional Euclidean field equations, the Higgs field cannot differ from its vacuum value in any finite region, so the only instantons are singular. Nevertheless, there exist smooth dynamical fields in Euclidean space which connect topologically distinct vacuums and also satisfy Gauss's law, and whose action is greater than $8\pi^2/g^2$ by an arbitrarily small amount. Consequently, one still expects the tunneling amplitude between these vacuums to be of order $\exp(-8\pi^2/g^2)$. Since $g^2 \sim 10^{-2}$ in the Weinberg-Salam theory, this amplitude is vanishingly small and the dramatic effects associated with the tunneling are usually assumed to be unobservable. Because of anomalies in the baryon and lepton currents, these effects include violation of baryon and lepton number.[10]

But are these effects negligibly small? We suggest that the amplitude for passing over the barrier could be greatly enhanced by simply increasing the real energy in the gauge and Higgs fields so as to be comparable with, or greater than, the height of the barrier. This could be achieved in $p\bar{p}$ or e^+e^- collisions, for example, at center-of-mass energies of order 1–10 TeV. Whether the field energy would ever be in a coherent enough form remains a problem though.

In an earlier version of this paper,[11] we stated that the unstable static solution might correspond to a resonance. We no longer believe this is likely. Our idea was based on the fact that in finite-dimensional quantum mechanics a smooth wave function peaked around a saddle point of the potential energy is not a stationary state, but the decay rate can be slow if the saddle region is large. Such a wave function(al) in field theory would correspond to an unstable, but relatively long-lived particle. The relevant saddle region is large in the Weinberg-Salam theory, and one would estimate that although the mass of the unstable particle was $O(M_W/g^2)$, its width would be only $O(M_W)$. However, more structure in the potential barrier is necessary before resonance behavior occurs. In one dimension, a square potential barrier or one with a dip in the middle exhibits resonance, but a smooth bell-shaped barrier does not. It is well known that the transmission amplitude rises and then has multiple dips as the energy of an incoming particle increases above the level of a square barrier, but for the potential $V_0(1-\tanh^2(x/a))$ the transmission amplitude increases monotonically with energy.[12]

II. A CLASS OF NONCONTRACTIBLE LOOPS

For static classical fields in the Weinberg-Salam theory, one form of the energy functional is

$$E=\int\left[-\frac{1}{2g^2}\mathrm{Tr}(F_{ij}F_{ij})-\left[\frac{1}{4g'^2}-\frac{1}{4g^2}\right](\mathrm{Tr}F_{ij})(\mathrm{Tr}F_{ij})+\frac{v^2}{2}(D_i\Phi)^\dagger D_i\Phi+\frac{\lambda v^4}{4}(\Phi^\dagger\Phi-1)^2\right]d^3x\ , \tag{2.1}$$

where

$$\begin{aligned}F_{ij}&=\partial_iA_j-\partial_jA_i+[A_i,A_j]\ ,\\ D_i\Phi&=\partial_i\Phi+A_i\Phi\ .\end{aligned} \tag{2.2}$$

The gauge potential is an anti-Hermitian 2×2 matrix and the Higgs field is a two-component complex vector

$$\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}.$$

Although the fields are normalized in an unconventional way, the parameters g, g', λ, and v have their conventional meaning, as in Ref. 13. We denote by Φ_{Re} the four-component real vector

$$\Phi_{\mathrm{Re}}=\begin{bmatrix}\mathrm{Re}\Phi_1\\ \mathrm{Im}\Phi_1\\ \mathrm{Re}\Phi_2\\ \mathrm{Im}\Phi_2\end{bmatrix}. \tag{2.3}$$

Let us introduce spherical polar coordinates r, θ, ϕ. The associated covariant components of the gauge potential $\{A_r,A_\theta,A_\phi\}$ are related to the Cartesian components by

$$A_r dr+A_\theta d\theta+A_\phi d\phi=A_i dx^i\ . \tag{2.4}$$

We are interested in fields which are smooth in their Cartesian form. In spherical polars, ensuring smoothness at $r=0$ requires care.

Let us next impose the polar gauge condition[14] $A_r=0$. By doing so we avoid having to work explicitly with the space of full gauge orbits. Any field configuration with $A_r\neq0$ can be put in this gauge via the gauge transformation

$$U(r,\theta,\phi)=\mathscr{P}\exp\left[\int_0^1 A_r(\sigma r,\theta,\phi)r\,d\sigma\right]. \tag{2.5}$$

Moreover, when $A_r=0$, there is no further local gauge freedom. In principle, a gauge transformation $U(\theta,\phi)$ would preserve the polar gauge condition, but this is ill defined at the origin and leads to a singular gauge potential there, unless U is independent of θ and ϕ. The residual global gauge freedom will be dealt with later.

Asymptotically, the magnitude of the Higgs field of a finite-energy configuration must tend to 1. More precisely, we suppose that in the polar gauge there exists a limiting field

$$\Phi^\infty(\theta,\phi)=\lim_{r\to\infty}\Phi(r,\theta,\phi)\ , \tag{2.6}$$

which is a smooth function of θ and ϕ, and which satisfies

$$|\Phi^\infty|=1\ . \tag{2.7}$$

(Sufficient conditions for the existence of Φ^∞ are established in Ref. 15. These are in fact slightly stronger than simply the condition of finite energy.) Let us use the global gauge freedom to fix

$$\Phi^\infty(\theta=0)=\begin{bmatrix}0\\1\end{bmatrix}\ .$$

There still remains a global U(1) freedom corresponding to the unbroken gauge group. The vacuum configuration is, however, completely fixed. It is

$$\Phi_{\rm vac}(x)=\begin{bmatrix}0\\1\end{bmatrix}\ ,\quad A_{\rm vac}(x)=0\ . \tag{2.8}$$

Equation (2.7) implies that we can regard the field $\Phi^\infty_{\rm Re}$ as a map $\Phi^\infty_{\rm Re}$: $S^2\to S^3$, where S^2 is the two-sphere at spatial infinity and S^3 is the vacuum manifold of the Higgs field. Because of the polar gauge condition, this map is a physical property of the field configuration and not a gauge artifact. The homotopy group $\Pi_2(S^3)$ is trivial, so the map $\Phi^\infty_{\rm Re}$ is contractible. It follows that any finite-energy field configuration can be continuously transformed to the vacuum, and for this reason there are no magnetic monopoles in the Weinberg-Salam theory.

We are interested in loops in the configuration space, beginning and ending at the vacuum. Let $\mu\in[0,\pi]$ be the parameter along one such loop. The asymptotic Higgs fields of the configurations on the loop define a family of maps $\Phi^\infty_{\rm Re}(\mu)$: $S^2\to S^3$, varying continuously with μ. $\Phi^\infty_{\rm Re}(0)$ and $\Phi^\infty_{\rm Re}(\pi)$ are identical constant maps, mapping all of S^2 to the point (0,0,1,0) on S^3, because they correspond to vacuum configurations. For all μ, $\Phi^\infty_{\rm Re}$ maps the point $(\theta=0)$ on S^2 to the point (0,0,1,0) on S^3, because of our gauge choice.

We wish to show that such a family of maps is topologically equivalent to a single map Ψ: $S^3\to S^3$. To avoid confusion, we shall denote the domain of Ψ by $S^3_{\rm dom}$ and its target by $S^3_{\rm Higgs}$. We proceed by associating with each triple (μ,θ,ϕ) a point $p(\mu,\theta,\phi)$ on $S^3_{\rm dom}$, which we write as a four-component unit vector

$$p(\mu,\theta,\phi)=(\sin\mu\,\sin\theta\cos\phi,\sin\mu\,\sin\theta\,\sin\phi,\sin^2\mu\,\cos\theta+\cos^2\mu,\sin\mu\,\cos\mu(\cos\theta-1))\ . \tag{2.9}$$

This identification has the following desired properties: (i) $p(\mu,\theta,\phi)$ is continuous in its arguments; (ii) the dependence of p on θ and ϕ is consistent with these angles being spherical polar coordinates—that is, p is unchanged if $\phi\to\phi+2\pi$, and p is independent of ϕ when $\theta=0,\pi$; (iii) for all μ, $p(\mu,\theta=0,\phi)=(0,0,1,0)$; (iv) for $\mu=0$ and $\mu=\pi$, $p(\mu,\theta,\phi)=(0,0,1,0)$ for all θ,ϕ; (v) each point p on $S^3_{\rm dom}$ occurs for at least one triple $(\mu=\mu(p),\theta=\theta(p),\phi=\phi(p))$, and if p is not the point (0,0,1,0), then $\mu(p)$ is unique [if restricted to the range $0<\mu(p)<\pi$] and $(\theta(p),\phi(p))$ represents a unique point of S^2.

The map Ψ may now be defined by

$$\Psi(p)=\Phi^\infty_{\rm Re}(\mu(p),\theta(p),\phi(p))\ . \tag{2.10}$$

This is an unambiguous definition for $p\neq(0,0,1,0)$ because of property (v) above, and unambiguous for $p=(0,0,1,0)$ because of the special properties of the maps $\Phi^\infty_{\rm Re}(\mu)$ mentioned earlier.

The geometric meaning of (2.9) is simple. For given μ between 0 and π, the points p lie on the two-sphere which is at the intersection of the unit three-sphere with the hyperplane $p_3\cos\mu-p_4\sin\mu=\cos\mu$ (see Fig. 3). As μ varies these two-sphere sections swing over the whole three-sphere.

We have now associated with a loop in the configuration space, beginning and ending at the vacuum, a map Ψ: $S^3\to S^3$. Ψ is important because if its degree is nonzero, then the loop is noncontractible. To verify this statement, we suppose the loop is contractible. Then the fields on it can be simultaneously and continuously transformed to the vacuum. In the process, Ψ is continuously transformed to the trivial map of degree zero, where all of S^3 is mapped to one point. The degree must therefore have been zero initially, because it is unchanged by a continuous change of Ψ.[16]

Recall that we still have a global U(1) gauge freedom. Under a gauge transformation of this type

$$\Phi^\infty(\mu)\to\begin{bmatrix}e^{i\alpha(\mu)}&0\\0&1\end{bmatrix}\Phi^\infty(\mu)\ , \tag{2.11}$$

where $\alpha(\mu)$ is any continuous function. Ψ will vary as $\alpha(\mu)$ varies, but in a continuous way, so its degree is unchanged. This gauge freedom does not therefore affect the relationship between a nonzero degree and a noncontractible loop.

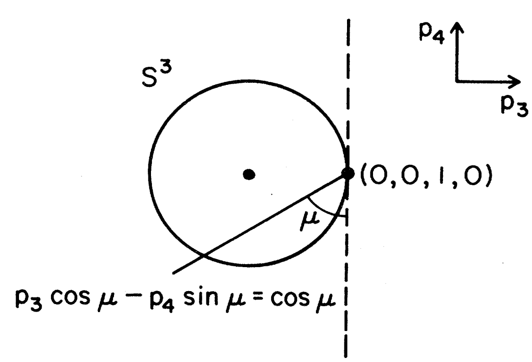

FIG. 3. Two-sphere sections of a three-sphere.

A simple map of nonzero degree is the identity map, which has degree one. With this choice for Ψ, the asymptotic Higgs fields are

$$\Phi^{\infty}_{\rm Re}(\mu,\theta,\phi)=\begin{bmatrix}\sin\mu\,\sin\theta\cos\phi\\ \sin\mu\,\sin\theta\sin\phi\\ \sin^2\mu\,\cos\theta+\cos^2\mu\\ \sin\mu\,\cos\mu(\cos\theta-1)\end{bmatrix} \tag{2.12}$$

or equivalently

$$\Phi^{\infty}(\mu,\theta,\phi)=\begin{bmatrix}\sin\mu\,\sin\theta e^{i\phi}\\ e^{-i\mu}(\cos\mu+i\,\sin\mu\,\cos\theta)\end{bmatrix}\,. \tag{2.13}$$

A loop of finite-energy field configurations, whose asymptotic Higgs fields are given by (2.13), will be noncontractible.

A suitable ansatz for the asymptotic gauge potential is

$$A^{\infty}_{\{\theta \atop \phi\}}=-\partial_{\{\theta \atop \phi\}}U^{\infty}(U^{\infty})^{-1}\,, \tag{2.14}$$

where U^{∞} is the U(2) matrix

$$U^{\infty}=\begin{bmatrix}\Phi_2^{\infty *} & \Phi_1^{\infty}\\ -\Phi_1^{\infty *} & \Phi_2^{\infty}\end{bmatrix}\,. \tag{2.15}$$

Since Φ^{∞} varies with μ, θ, and ϕ, so do U^{∞} and $A^{\infty}_{\{\theta \atop \phi\}}$. U^{∞} has the property that

$$\Phi^{\infty}=U^{\infty}\begin{bmatrix}0\\1\end{bmatrix} \tag{2.16}$$

and for this reason, the covariant derivatives $D_\theta\Phi$ and $D_\phi\Phi$ vanish asymptotically, as they must for a finite-energy configuration.

Consider now the following field configurations:

$$\begin{aligned}&\Phi(\mu,r,\theta,\phi)=[1-h(r)]\begin{bmatrix}0\\ e^{-i\mu}\cos\mu\end{bmatrix}+h(r)\Phi^{\infty}(\mu,\theta,\phi)\,,\\ &A_{\{\theta \atop \phi\}}(\mu,r,\theta,\phi)=f(r)A^{\infty}_{\{\theta \atop \phi\}}(\mu,\theta,\phi)\,,\\ &A_r(\mu,r,\theta,\phi)=0\end{aligned} \tag{2.17}$$

with Φ^{∞} given by (2.13) and A_θ,A_ϕ given in terms of Φ^{∞} by (2.14) and (2.15). These fields are defined over all of space, are smooth, and have finite energy for suitable radial functions f and h. To ensure smoothness at the origin and to ensure that the fields have the desired asymptotic behavior, f and h must satisfy the boundary conditions

$$\begin{aligned}&\lim_{r\to 0}h(r)=0\,,\quad \lim_{r\to\infty}h(r)=1\,,\\ &\lim_{r\to 0}\frac{1}{r}f(r)=0\,,\quad \lim_{r\to\infty}f(r)=1\,.\end{aligned} \tag{2.18}$$

For $\mu=0$ and $\mu=\pi$ the fields are those of the vacuum.

We conclude that the fields (2.17) represent a noncontractible loop in the configuration space of the classical Weinberg-Salam theory, beginning and ending at the vacuum. Note that U^{∞} is not just unitary, but has determinant equal to one. Both the gauge potential and the field tensor are pure SU(2) and therefore traceless, and there is no contribution to the energy from the second term in (2.1).

Finally, let us note that by relaxing the polar gauge condition, a noncontractible loop connecting the vacuum to itself can be cast in a form where it becomes a path connecting "topologically distinct" vacuums. If one imposes the gauge condition that at spatial infinity all fields must approach the unitary vacuum (2.8), and also that the configuration corresponding to $\mu=0$ is still the unitary vacuum, then the configuration corresponding to $\mu=\pi$ will have the form

$$\Phi(x)=U(x)\begin{bmatrix}0\\1\end{bmatrix}\,,\quad A_i(x)=-(\partial_i UU^{-1})(x)\,, \tag{2.19}$$

where $U\to 1$ at spatial infinity, and U cannot be continuously transformed to $U\equiv 1$ while preserving this boundary condition.

Topologically distinct vacuums are not really physically distinct, just as little as points on a circle whose angular coordinates differ by 2π are geometrically distinct, but the notion is useful if one considers noncontractible loops in configuration space. The notion will probably cease to be useful when one attempts to deal with the nontrivial higher-dimensional topology of the configuration space (i.e., noncontractible spheres). This higher-dimensional topology certainly exists.[5,6]

III. AN ENERGY BOUND

We have argued that the existence of noncontractible loops in the Weinberg-Salam theory may imply the existence of an unstable classical solution, whose energy is the infimum of the maximal energies on these loops. In this section we use the loop of field configurations of Eq. (2.17), with particular forms for the radial functions f and h, to obtain an upper bound on this energy. Recall that μ is the parameter along the loop.

Let us first rewrite the energy functional (2.1) in spherical polar coordinates, taking into account the polar gauge condition $A_r=0$ and the tracelessness of the field tensor:

$$\begin{aligned}E=\int\Bigg\{&\frac{-1}{g^2}\left[\frac{1}{r^2}\mathrm{Tr}(\partial_r A_\theta\partial_r A_\theta)+\frac{1}{r^2\sin^2\theta}\mathrm{Tr}(\partial_r A_\phi\partial_r A_\phi)+\frac{1}{r^4\sin^2\theta}\mathrm{Tr}(F_{\theta\phi}F_{\theta\phi})\right]\\ &+\frac{v^2}{2}\left[(\partial_r\Phi)^\dagger\partial_r\Phi+\frac{1}{r^2}(D_\theta\Phi)^\dagger D_\theta\Phi+\frac{1}{r^2\sin^2\theta}(D_\phi\Phi)^\dagger D_\phi\Phi\right]+\frac{\lambda v^4}{4}(\Phi^\dagger\Phi-1)^2\Bigg\}r^2\sin\theta\,dr\,d\theta\,d\phi\,.\end{aligned} \tag{3.1}$$

Note that

$$F_{\theta\phi}=-\frac{1-f}{f}[A_\theta,A_\phi] \tag{3.2}$$

because the gauge potential is proportional to a pure gauge.

A direct computation shows that the fields (2.17) have a spherically symmetric energy density. This is rather remarkable, since the fields themselves are not spherically symmetric, as we shall show in Sec. IV. The total energy is

$$\begin{aligned}E(\mu)=\int\Bigg\{&\frac{4}{g^2r^2}\left[\left(\frac{df}{dr}\right)^2\sin^2\mu+\frac{2}{r^2}[f(1-f)]^2\sin^4\mu\right]\\&+\frac{v^2}{2}\left[\left(\frac{dh}{dr}\right)^2\sin^2\mu+\frac{2}{r^2}\{[h(1-f)]^2\sin^2\mu-2fh(1-f)(1-h)\cos^2\mu\sin^2\mu+[f(1-h)]^2\cos^2\mu\sin^2\mu\}\right]\\&+\frac{\lambda v^4}{4}(h^2-1)^2\sin^4\mu\Bigg\}4\pi r^2dr\ .\end{aligned} \tag{3.3}$$

A simple choice for f and h, satisfying the boundary conditions (2.18), is

$$h(r)=\begin{cases}\dfrac{r}{R_0} & r\le R_0\\ 1\ , & r\ge R_0\ ,\end{cases}\qquad f(r)=\begin{cases}\dfrac{r^2}{R_0{}^2}\ , & r\le R_0\\ 1\ , & r\ge R_0\ .\end{cases} \tag{3.4}$$

R_0 is an arbitrary parameter which represents the core size of the configuration. The energy density is identically zero outside the core. If desired, f and h can be made smooth near R_0 with an infinitesimal change in the energy.

With the above radial functions the maximal energy on the loop occurs when $\mu=\frac{1}{2}\pi$. The only term in the energy density for which this is not obvious is

$$\frac{v^2}{r^2}\{[h(1-f)]^2\sin^2\mu-2fh(1-f)(1-h)\cos^2\mu\sin^2\mu+[f(1-h)]^2\cos^2\mu\sin^2\mu\}\ . \tag{3.5}$$

However, if $0\le f\le 1$ and $0\le h\le 1$, then this term has its maximum at $\mu=\frac{1}{2}\pi$ provided

$$(1+\sqrt{2})h(1-f)\ge f(1-h)\ . \tag{3.6}$$

This condition is satisfied by the radial functions we have chosen, for all r.

The field configuration with $\mu=\frac{1}{2}\pi$ has energy

$$E=\frac{4\pi}{210}\left[\frac{1248}{g^2R_0}+51v^2R_0+4\lambda v^4R_0{}^3\right]\ . \tag{3.7}$$

This shows that a change of R_0 corresponds to a rescaling of the fields in the canonical sense. Our upper bound on the energy of the expected unstable solution of the field equations is the minimum of (3.7) with respect to R_0. The dependence of this bound on λ is not very informative, so we just give it for two values of λ. For $\lambda=0$ we find

$$E=2.4\times\frac{4\pi v}{g}\ , \tag{3.8}$$

and for $\lambda=g^2$,

$$E=3.2\times\frac{4\pi v}{g}\ . \tag{3.9}$$

One could lower these bounds by choosing optimal forms for the radial functions, but that would require a numerical solution of coupled ordinary differential equations. Since the angular dependence of the trial configurations probably differs considerably from that of the solution, this effort does not seem worthwhile.

In physical terms

$$\frac{4\pi v}{g}=2\frac{M_W}{\alpha}\sin^2\theta_W\ , \tag{3.10}$$

where θ_W is the weak mixing angle. If $\sin^2\theta_W=0.23$ and $M_W=79$ GeV, the energies (3.8) and (3.9) have values of 12 and 16 TeV, respectively. The values $\lambda=0$ and $\lambda=g^2$ correspond to Higgs-particle masses $M_H=0$ and $M_H=2\sqrt{2}M_W$. The actual energy of the solution is probably in the range 1—10 TeV.

IV. MORE ON CLASSICAL SOLUTIONS

There are clearly many questions left unanswered by the preceding analysis. Most important is whether the classical solution that we expect does in fact exist. While Taubes' methods may lead to an affirmative answer, they give no detailed description of the solution. Our present ansatz (2.17), with $\mu=\frac{1}{2}\pi$, is probably far from an accurate picture. There are some ideas in the literature which may be more helpful, however.

Let us first note that the Weinberg-Salam theory has a vortex solution. This vortex is simply the Nielsen-Olesen vortex[17] of the Abelian Higgs model embedded in the non-Abelian theory. One can truncate the Weinberg-Salam theory by assuming that the Higgs field has the form $\Phi=\binom{0}{\phi}$ and that the gauge potential has the form

$$A_i=\begin{bmatrix}0 & 0\\ 0 & a_i\end{bmatrix}\ . \tag{4.1}$$

a_i is the field corresponding to the Z boson. The energy functional in terms of ϕ and a_i is precisely the Abelian Higgs model. The argument which showed that the vortex was topologically stable in the Abelian model fails in the larger theory, so there it is probably unstable. In any case it has infinite energy, because of its infinite length, so it does not correspond directly to the solution we seek.

However, there have been two proposals for using the vortex. First, Nambu studied a configuration where a finite piece of the vortex is terminated by monopoles.[18] Second, Huang and Tipton considered joining the ends of the vortex to form a vortex ring.[19] They argued that quantum effects could turn this into a metastable particle, dubbed a "vorticon." The classical vortex ring, just because it is unstable, may be an approximation to the solution we seek.

We stated earlier that the field configurations of Eq. (2.17) are not spherically symmetric, even though the energy density is. The reason for this is rather interesting. There is a simple way to regard the bosonic fields of the Weinberg-Salam theory as embedded in an SO(4) gauge theory. One just forgets the complex structure and writes everything in real form. A complex doublet becomes a real four-component vector, as in (2.3) for the Higgs field. An element of the U(2) Lie algebra

$$u=\begin{bmatrix} ia & be^{i\tau} \\ -be^{-i\tau} & ic \end{bmatrix} \tag{4.2}$$

is identified with the element

$$u_{\rm Re}=\begin{bmatrix} 0 & -a & b\cos\tau & -b\sin\tau \\ a & 0 & b\sin\tau & b\cos\tau \\ -b\cos\tau & -b\sin\tau & 0 & -c \\ b\sin\tau & -b\cos\tau & c & 0 \end{bmatrix} \tag{4.3}$$

of the SO(4) Lie algebra. If $c=-a$ then the U(2) matrix is traceless and hence in the Lie algebra of SU(2). The corresponding SO(4) matrix is anti-self-dual.

SO(4) is generated by the anti-self-dual and self-dual 4×4 matrices, which commute, and therefore SO(4) $\cong \mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. (We ignore central elements in what follows.) We see from (4.3) that U(2) can be identified with $\mathrm{SU}(2)_L\times\mathrm{U}(1)_R\subset\mathrm{SO}(4)$, where $\mathrm{U}(1)_R$ is a subgroup of $\mathrm{SU}(2)_R$ which depends on how the four real variables are paired to make two complex ones.

Let us just discuss the symmetries of the fields (2.17) for $\mu=\frac{1}{2}\pi$. This is the most important case, and the simplest. Let us also revert to Cartesian coordinates. One can show that the real forms of the Higgs field and gauge potential are

$$\Phi_{\rm Re}=\frac{h(r)}{r}\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}, \tag{4.4}$$

$$A_{\rm Re}=A_{{\rm Re},i}dx^i=-\frac{f(r)}{r^2}\begin{bmatrix} 0 & & & y\,dz-z\,dy \\ & 0 & & z\,dx-x\,dz \\ & & 0 & x\,dy-y\,dx \\ & & & 0 \end{bmatrix}. \tag{4.5}$$

The unmarked entries in $A_{\rm Re}$ are determined by anti-self-duality, since the gauge potential is pure SU(2).

Both $\Phi_{\rm Re}$ and $A_{\rm Re}$ are manifestly rotationally symmetric. A rotation R acting on $\Phi_{\rm Re}$ is compensated by the SO(4) gauge transformation $\left(\begin{smallmatrix} R & 0 \\ 0 & 1\end{smallmatrix}\right)$, which is identified with the element (R,R) of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. The rotation R acting on $A_{\rm Re}$ is compensated by the gauge transformation R in $\mathrm{SU}(2)_L$, or more generally by any element (R,R') in $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. If R' is position independent it has no effect on $A_{\rm Re}$. $\Phi_{\rm Re}$ and $A_{\rm Re}$ are simultaneously invariant under the rotation R if we choose $R'=R$.

If the fields of the Weinberg-Salam theory are in real form, then the field energy has a form which would be standard in an SO(4) Yang-Mills-Higgs theory. Since the SO(4) fields are spherically symmetric, the energy density is also spherically symmetric.

On the other hand, the fields in their original complex form are not spherically symmetric. There is no way a rotation R can be compensated by a global gauge transformation in U(2). Trying to use position-dependent gauge transformations does not help either. It follows that even for $\mu=\frac{1}{2}\pi$ the fields (2.17) cannot satisfy the field equations for any nontrivial choice of the radial functions. The U(1) current

$$j_i^{\mathrm{U}(1)}=\Phi^\dagger D_i\Phi-(D_i\Phi)^\dagger\Phi \tag{4.6}$$

is nonvanishing and not spherically symmetric, although it is part of a spherically symmetric $\mathrm{SU}(2)_R$ current. It is therefore inconsistent with the field equations to have a vanishing U(1) gauge potential, as in (2.17). However, if there were a nonvanishing U(1) gauge potential, it would not be spherically symmetric, and without further modifications the field equations would still not be satisfied.

In conclusion, we expect the actual solution of the field equations to have even less symmetry than the fields we have been discussing. Possibly there is a residual axial symmetry. A physical consequence is that the solution, while not being a magnetic monopole, would have a definite magnetic moment determined by the leading behavior of the asymptotic electromagnetic gauge fields.

V. CONCLUSIONS

We have shown that there are noncontractible loops in the field-configuration space of the Weinberg-Salam theory. If some analog of Morse theory applies to this infinite-dimensional configuration space, then associated with these loops there is a static, unstable solution of the field equations. In a related gauge theory, Taubes has proved that such a solution does occur, and it is likely that his methods can be applied in the Weinberg-Salam case. It would also be worthwhile to seek the relevant solution numerically, and find its energy.

The solution, if it exists, is at a saddle point of the field potential energy, so its energy is the barrier height for vacuum-to-vacuum tunneling along these topologically nontrivial paths. Such tunneling leads to baryon- and lepton-number violation. Although the tunneling amplitude appears to be negligibly small, we suggest it could be enhanced by pumping enough energy into the fields. Particle collisions at 1 TeV and above may achieve this.

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