Simple rules for discontinuities in finite-temperature field theory

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The discontinuity, or imaginary part, of the self-energy function at $T\neq0$ is found to be of the form $\Gamma=\Gamma_d\mp\Gamma_i$ for bosons and fermions, respectively. The generalized decay rate Γ_d and inverse decay rate Γ_i are recognizable as integrals over phase space of amplitudes squared, weighted with certain statistical factors that account for the possibility of particle absorption from the medium or particle emission into the medium. Nonequilibrium statistical mechanics shows that Γ gives precisely the rate at which the single-particle distribution function approaches the equilibrium form.

I. INTRODUCTION

There are many unanswered questions about the behavior of quantum field theories at nonzero temperature. In particular, not much is known about the imaginary parts of finite-temperature Green's functions. Only the simplest of these, viz., the self-energy Π, will be treated here. At $T=0$, $\text{Im}\Pi$ can be expressed as the square of the decay amplitude, integrated over phase space, and consequently has an obvious physical interpretation as the probability of particle decay. The purpose of this paper is to compute $\text{Im}\Pi$ for $T\neq0$ and organize it into a recognizable form as the square of an amplitude, integrated over phase space but weighted with certain statistical factors appropriate to a thermal distribution.

The field-theory calculations that will be presented are very simple and the results can be nicely accounted for by arguments of statistical mechanics. Consider, for example, a world containing only three species of particles: W, e, and ν. The decay $W\rightarrow e\bar{\nu}$ will have a rate Γ_d in which the phase space is Pauli suppressed by the statistical factors $(1-n_e)(1-n_{\bar{\nu}})$ appropriate to a thermal distribution of e's and $\bar{\nu}$'s, where $n^{-1}=\exp(E/T)+1$. However, since real e's and $\bar{\nu}$'s are present in the heat bath, it is also possible for W's to appear spontaneously by the inverse decay $e\bar{\nu}\rightarrow W$. The rate Γ_i for this process should depend on the product $n_e n_{\bar{\nu}}$ of occupation numbers. It turns out that the W self-energy $\Pi(\omega)$ precisely accounts for both these processes. It has a discontinuity, or imaginary part

$$\text{Im}\Pi(\omega)=-\omega\Gamma(\omega)\ , \tag{1.1}$$

which provides a link between field theory and statistical mechanics in that $\Gamma(\omega)=\Gamma_d-\Gamma_i$.

The above result is for any W boson with $s_W>m_e{}^2$. Suppose, however, that $s_W<m_e{}^2$. Then $W\rightarrow e\bar{\nu}$ is kinematically forbidden and at $T=0$ the self-energy has no imaginary part. But at $T\neq0$ such a virtual W boson can disappear by absorbing a ν from the heat bath. The rate Γ_d for $W\nu\rightarrow e$ should depend on the product $n_\nu(1-n_e)$ for absorption of ν and emission of e. On the other hand, a virtual W boson can spontaneously appear in the medium from the inverse process $e\rightarrow W\nu$, whose rate Γ_i has a statistical weight $n_e(1-n_\nu)$. Both these processes are accounted for by a new cut in the W self-energy for $-m_e{}^2<s_W<m_e{}^2$ that is not present at $T=0$. The discontinuity across this cut is pure imaginary and again given by (1.1) with $\Gamma(\omega)=\Gamma_d-\Gamma_i$ appropriate to $W\nu\rightarrow e$ and $e\rightarrow W\nu$.

There are several ways in which Γ differs from an ordinary decay rate: (1) For fermion self-energies the imaginary part is given by $\Gamma=\Gamma_d+\Gamma_i$ instead of the difference. (2) Since the heat bath provides a special Lorentz frame, a particle moving with energy-momentum $(\omega,\vec{k})$ in this frame is not physically equivalent to a particle at rest. Thus even for a fixed value of $s=\omega^2-\vec{k}^2$, the rate $\Gamma(\omega)$ has nontrivial ω dependence. (3) After a very long time $T\gg1/\Gamma$ it is clearly not true that all the particles in question, e.g., W bosons, have decayed away. Even after an infinite time there will still be a thermal distribution of W's present in the medium.

These three properties show that Γ should not be interpreted as some kind of net decay rate for the particle. Further investigation shows instead that Γ is the rate at which a nonequilibrium distribution $f(\omega,t)$ approaches thermal equilibrium:

$$f(\omega,t)=\frac{1}{\exp(\omega/T)-\sigma}+c(\omega)\exp(-\Gamma t)\ , \tag{1.2}$$

where $\Gamma=\Gamma_d-\sigma\Gamma_i$ is the imaginary part of the self-energy appropriate to bosons $(\sigma=1)$ or fermions $(\sigma=-1)$.

As noted earlier, one could anticipate the form of Γ_d and Γ_i from statistical-mechanical considerations without reference to self-energy graphs. This has been the approach taken in recent papers on neutron decay by Dicus *et al.*[1] and by Cambier, Primack, and Sher[2] and on Higgs boson decay by Donoghue and Holstein.[3] The rather difficult computation of these rates included radiative corrections and cancellation of some new infrared divergences. The emphasis of this paper is not in computing these rates, but rather in relating them to the self-energy function.

The paper is organized as follows: Section II contains analysis of the one-loop contributions to boson and fermion self-energies. From the one-loop results an extrapolation is made to the many-particle case. The justification of the many-particle discontinuity is presented in the Appendix. Section III develops the physical interpretation of Γ [viz., (1.2)] and presents evidence that virtual, off-shell particles will also be in thermal equilibrium in any heat bath with a Breit-Wigner probability of width Γ. Nowhere in the paper are there chemical potentials, but it is straightforward to include them.

II. THE SELF-ENERGY

There exist two formalisms for field-theoretic computations at nonzero temperature. The imaginary-time (or Euclidean) formulation invented by Matsubara[4] quantizes field operators on the imaginary-time interval $0 \le it \le 1/T$. The resulting Green's functions, when Fourier transformed, depend on discrete frequencies. For a relativistic theory of scalar bosons the two-point Green's function is[5]

$$D(\omega_a, \vec{p}) = \frac{1}{\omega_a{}^2 - \vec{p}\,^2 - m^2}, \tag{2.1}$$

$$\omega_a = i\,2\pi a T, \tag{2.2}$$

where a is an integer. The great advantage of this formalism is that perturbation theory may still be organized into a diagrammatic expansion with the same vertices as at $T=0$.[6,7] Computation of a particular diagram will yield a result which depends on various external three-momenta $\vec{p}$ and external frequencies ω_a of the form (2.2). One must then analytically extend the result away from the discrete imaginary frequencies ω_a down to the real ω axis to describe particles with real energy ω. Although this approach is more tedious than the real-time approach, it automatically yields results with the correct analytic properties[8] and will therefore be used throughout this paper.

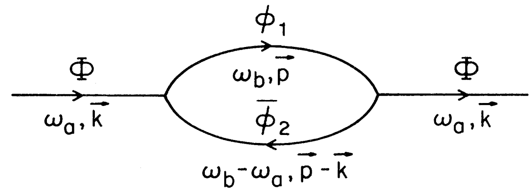

FIG. 1. The general two-particle contribution to the self-energy of Φ.

A. Boson self-energy

(1) Two-boson intermediate states: The simplest self-energy graph is that shown in Fig. 1 when ϕ_1, ϕ_2, and Φ are all scalar bosons with a cubic coupling of strength g. Then

$$\Pi(\omega_a, \vec{k}) = -g^2 T \sum_{b=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} D(\omega_b, \vec{p}) D(\omega_b - \omega_a, \vec{p} - \vec{k}), \tag{2.3}$$

where $\omega_b = i\,2\pi b T$. It is perhaps worth noting that for $T \neq 0$ (2.3) depends separately on ω_a and $\vec{k}$ because the heat bath singles out a preferred frame of reference. However, the $\vec{k}$ dependence on (2.3) will be suppressed from now on. The summation in (2.3) is straightforward and yields

$$\Pi(\omega_a) = g^2 \int \frac{d^3p}{(2\pi)^3} \left[\frac{\coth(E_1/2T)}{4E_1} \left[\frac{1}{(\omega_a - E_1)^2 - E_2{}^2} + \frac{1}{(\omega_a + E_1)^2 - E_2{}^2} \right] + (E_1 \leftrightarrow E_2) \right], \tag{2.4}$$

$$E_1 = (m_1{}^2 + \vec{p}\,^2)^{1/2}, \quad E_2 = [m_2{}^2 + (\vec{p} - \vec{k})^2]^{1/2}. \tag{2.5}$$

It is useful to write $\frac{1}{2}\coth(E/2T) = \frac{1}{2} + n$ where n is the Bose-Einstein distribution

$$n = (e^{E/T} - 1)^{-1} \tag{2.6}$$

and then to separate (2.4) into partial fractions

$$\Pi(\omega_a) = g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_1 2E_2} \left[\frac{1+n_1+n_2}{\omega_a - E_1 - E_2} + \frac{n_1 - n_2}{\omega_a + E_1 - E_2} \frac{n_2 - n_1}{\omega_a - E_1 + E_2} - \frac{1+n_1+n_2}{\omega_a + E_1 + E_2} \right]. \tag{2.7}$$

One can immediately extend this function from the discrete, imaginary values ω_a to the full complex ω plane. There are a few obvious properties of this extension: (a) considered as function of the complex variable ω, Π satisfies

$$\Pi(\omega)^* = \Pi(\omega^*) \tag{2.8}$$

and is thus Hermitian analytic. (b) Π has cuts along the real axis. (c) The discontinuity across these cuts [defined by $\Pi(\omega + i\eta) - \Pi(\omega - i\eta)$ for ω real] is pure imaginary so that

$$\mathrm{Disc}\Pi(\omega) = -2i\,\mathrm{Im}\Pi(\omega). \tag{2.9}$$

The explicit discontinuity of (2.7) is

$$\mathrm{Disc}\Pi = -ig^2 \int \frac{d^3p}{(2\pi)^3} \frac{2\pi}{2E_1 2E_2} \{\delta(\omega - E_1 - E_2)[(1+n_1)(1+n_2) - n_1 n_2] + \delta(\omega + E_1 - E_2)[n_1(1+n_2) - n_2(1+n_1)]$$
$$+\delta(\omega - E_1 + E_2)[n_2(1+n_1) - n_1(1+n_2)]$$
$$+\delta(\omega + E_1 + E_2)[n_1 n_2 - (1+n_1)(1+n_2)]\} \ . \qquad (2.10)$$

Although no products $n_1 n_2$ were present in (2.7) they have been added and subtracted in (2.10) so as to provide a physical interpretation. For example, the first term in (2.10) may be interpreted as the probability for the decay $\Phi \to \phi_1 \phi_2$ with statistical weight $(1+n_1)(1+n_2)$ for stimulated emission minus the probability for the inverse decay $\phi_1 \phi_2 \to \Phi$ with weight $n_1 n_2$ for absorption. Similarly, the second term corresponds to $\Phi \bar{\phi}_1 \to \phi_2$ with weights $n_1(1+n_2)$ minus the probability for $\phi_2 \to \Phi \bar{\phi}_1$ with weight $n_2(1+n_1)$. All eight processes are shown in Fig. 2.

For fixed masses m_1 and m_2 the δ-function constraints can only be satisfied for certain ranges of $s = \omega^2 - k^2$. For definiteness, take $m_1 \le m_2$ and $\omega \ge 0$. Then $\omega = E_1 + E_2$ is possible only if $s \ge (m_1+m_2)^2$; $\omega + E_1 = E_2$ is possible only if $m_1^2 - m_2^2 \le s \le (m_1 - m_2)^2$; but neither $\omega + E_2 = E_1$ nor $\omega + E_1 + E_2 = 0$ can be satisfied. Figure 3 shows the location of the corresponding branch cuts in the complex s plane. The new cut from $m_1^2 - m_2^2 < 0$ to $(m_1 - m_2)^2$ results from the process $\Phi \bar{\phi}_1 \to \phi_2$ and its inverse and has a discontinuity proportional to $n_1(1+n_2) - n_2(1+n_1) = n_1 - n_2$. This is always positive since $E_2 - E_1 = \omega \ge 0$ here.

The $\omega\pi\gamma$ system is a nice example of these analytic properties. The self-energy of a virtual ω has cuts for $-m_\pi^2 < s_\omega < m_\pi^2$ and $m_\pi^2 < s_\omega < \infty$. The same argument shows that the self-energy of a virtual γ has cuts for $-m_\omega^2 + m_\pi^2 < s_\gamma < (m_\omega - m_\pi)^2$ and $(m_\omega + m_\pi)^2 < s_\gamma < \infty$.

(2) Two-fermion intermediate states: Next we consider the self-energy graph shown in Fig. 1, but now take ϕ_1 and ϕ_2 as a fermion-antifermion pair which enjoy a Yukawa coupling to the scalar boson Φ. The imaginary-time propagator for fermions is[5]

$$S(\omega_a, \vec{p}) = \frac{\gamma_0 \omega_a - \vec{\gamma} \cdot \vec{p} + m}{\omega_a^2 - \vec{p}^{\,2} - m^2} \ , \qquad (2.11)$$

$$\omega_a = i 2\pi(a + \tfrac{1}{2})T \ . \qquad (2.12)$$

This gives a one-loop self-energy

$$\Pi(\omega_a, \vec{k}) = g^2 T \sum_{b=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}[S(\omega_b, \vec{p}) S(\omega_b - \omega_a, \vec{p} - \vec{k})] \ , \qquad (2.13)$$

where the extra minus sign relative to (2.3) comes from the closed fermion loop. The trace and summation yield

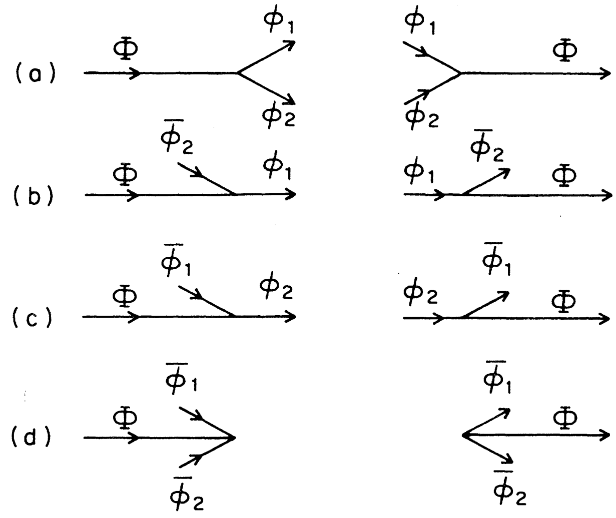

FIG. 2. The eight amplitudes responsible for the disappearance and reappearance of Φ. Only one pair of amplitudes is kinematically possible for given values of ω and s: Reactions (a) for $\omega > 0$ or (d) for $\omega < 0$ require $(m_1+m_2)^2 < s < \infty$; reactions (b) for $\omega(m_1 - m_2) > 0$ or (c) for $\omega(m_1 - m_2) < 0$ require $-|m_1^2 - m_2^2| < s < |m_1 - m_2|^2$.

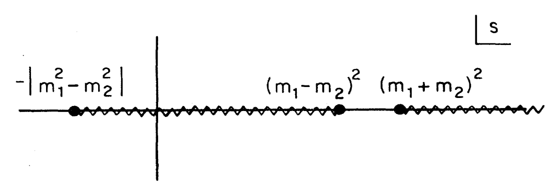

FIG. 3. Location of the branch cuts in the one-loop self-energy as a function of $s = \omega^2 - \vec{k}^2$. The discontinuity across the new cut $-|m_1^2 - m_2^2| < s < (m_1 - m_2)^2$ is produced by reactions (b) or (c) of Fig. 2 and automatically vanishes as $T \to 0$.

$$\Pi(\omega_a)=g^2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\tanh(E_1/2T)}{4E_1}\left[\frac{N}{(\omega_a-E_1)^2-E_2^2}+\frac{N}{(\omega_a+E_1)^2-E_2^2}+2\right]+(E_1\leftrightarrow E_2)\right], \tag{2.14}$$

$$N=2s-2(m_1+m_2)^2, \tag{2.15}$$

where $s=\omega_a^2-\vec{k}^2$ and E_1,E_2 are the same as in (2.5). If we use $\frac{1}{2}\tanh(E/2T)=\frac{1}{2}-n$ where n is now the Fermi-Dirac distribution

$$n=(e^{E/T}+1)^{-1} \tag{2.16}$$

then (2.14) can be separated into partial fractions as

$$\Pi(\omega_a)=g^2\int\frac{d^3p}{(2\pi)^3}\frac{N}{2E_12E_2}\left[\frac{1-n_1-n_2}{\omega_a-E_1-E_2}+\frac{n_2-n_1}{\omega_a+E_1-E_2}+\frac{n_1-n_2}{\omega_a-E_1+E_2}+\frac{-1+n_1+n_2}{\omega_a+E_1+E_2}\right]+C, \tag{2.17}$$

where C is a real constant independent of ω_a and $\vec{k}$:

$$C=g^2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\tanh(E_1/2T)}{2E_1}+\frac{\tanh(E_2/2T)}{2E_2}\right].$$

When the discrete, imaginary frequency ω_a is extended to an arbitrary complex variable ω, then (2.17) is again Hermitian analytic with cuts along the real ω axis.

The discontinuity of (2.17) across the cuts is

$$\begin{aligned}\mathrm{Disc}\Pi=-ig^2\int\frac{d^3p}{(2\pi)^3}\frac{2\pi N}{2E_12E_2}\{&\delta(\omega-E_1-E_2)[(1-n_1)(1-n_2)-n_1n_2]-\delta(\omega+E_1-E_2)[n_1(1-n_2)-n_2(1-n_1)]\\&-\delta(\omega-E_1+E_2)[n_2(1-n_1)-n_1(1-n_2)]\\&+\delta(\omega+E_1+E_2)[n_1n_2-(1-n_1)(1-n_2)]\}.\end{aligned} \tag{2.18}$$

As before, the product n_1n_2 is not present in (2.17) but has been added and subtracted in (2.18). The location of the cuts here are the same as in Fig. 3. The curious alternation of signs in (2.18) has a simple explanation: Since $\omega=\pm(E_1+E_2)$ can only be satisfied for $s\geq(m_1+m_2)^2$, N is positive in this region. Since $\omega=\pm(E_1-E_2)$ can only be satisfied for $-|m_1^2-m_2^2|\leq s\leq|m_1-m_2|^2$, N is negative in this region. Consequently all four terms in (2.18) actually contribute positively.

A more physical representation of (2.18) is obtained by introducing the amplitudes[9,10]

$$\begin{aligned}&M(\Phi\to1,2)=g\bar{u}(p_1)v(p_2),\quad M(\Phi,\bar{2}\to1)=g\bar{u}(p_1)u(p_2),\\&M(\Phi,\bar{1}\to2)=g\bar{v}(p_1)v(p_2),\quad M(\Phi,\bar{1},\bar{2}\to0)=g\bar{v}(p_1)u(p_2),\end{aligned} \tag{2.19}$$

for the fermion ϕ_1 and antifermion ϕ_2. Then

$$\begin{aligned}\mathrm{Disc}\Pi=-i\int dp_1dp_2(2\pi)^4\sum_{s_1s_2}\{&\delta^4(k-p_1-p_2)|M(\Phi\to1,2)|^2[(1-n_1)(1-n_2)-n_1n_2]\\&+\delta^4(k+p_1-p_2)|M(\Phi,\bar{1}\to2)|^2[n_1(1-n_2)-n_2(1-n_1)]\\&+\delta^4(k-p_1+p_2)|M(\Phi,\bar{2}\to1)|^2[n_2(1-n_1)-n_1(1-n_2)]\\&+\delta^4(k+p_1+p_2)|M(\Phi,\bar{1}\bar{2}\to0)|^2[n_1n_2-(1-n_1)(1-n_2)]\},\end{aligned} \tag{2.20}$$

where $dp=d^3p/2E(2\pi)^3$. The probabilistic interpretation of this result is obviously the same as for (2.10).

B. Fermion self-energy

When Φ is a fermion the self-energy graph Fig. 1 still applies if ϕ_1 and ϕ_2 are taken as fermion and boson, respectively, with a Yukawa coupling g. The self-energy matrix $\Sigma(\omega,\vec{k})$ for Φ is obtained by using (2.11) for the ϕ_1 (fermion) propagator and (2.1) for the ϕ_2 (boson) propagator:

$$\Sigma(\omega_a,\vec{k})=-g^2T\sum_{b=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}S(\omega_b,\vec{p})D(\omega_b-\omega_a,\vec{p}-\vec{k})\ , \tag{2.21}$$

where both ω_a and ω_b are of the form (2.12). After computing this sum and extending ω_a to the full complex ω plane, Σ is again a Hermitian analytic function of the complex variable ω. The discontinuity across the real ω axis is

$$\begin{aligned}\mathrm{Disc}\Sigma=-ig^2\int\frac{d^3p}{(2\pi)^3}\frac{2\pi}{2E_12E_2}[&\delta(\omega-E_1-E_2)(\not p_1+m_1)(1-n_1+n_2)+\delta(\omega+E_1-E_2)(\not p_1-m_1)(n_1+n_2)\\&+\delta(\omega-E_1+E_2)(\not p_1+m_1)(n_1+n_2)+\delta(\omega+E_1+E_2)(\not p_1-m_1)(1-n_1+n_2)]\ ,\end{aligned} \tag{2.22}$$

where $p_1^0=E_1>0$ here. Note that n_1 is the fermion distribution function (2.16) and n_2 is the boson distribution function (2.6). This discontinuity is nonvanishing either for $-|m_1{}^2-m_2{}^2|\le s\le(m_1-m_2)^2$ or $(m_1+m_2)^2\le s\le\infty$ as shown in Fig. 3.

In order to interpret (2.22) in terms of probabilities, it is useful to define

$$\Pi(\omega,\vec{k})=\bar{u}(k)\Sigma(\omega,\vec{k})u(k)\ , \tag{2.23}$$

where $u(k)$ is a free particle spinor with effective mass parameter $\sqrt{s}$, i.e.,

$$(\not k-\sqrt{s})u(k)=0,\quad \bar{u}(k)u(k)=2\sqrt{s} \tag{2.24}$$

Then one may rewrite (2.22) as

$$\begin{aligned}\mathrm{Disc}\Pi=-i\int dp_1dp_2\sum_{s_1}(2\pi)^4\{&\delta^4(k-p_1-p_2)|M(\Phi\to1,2)|^2[(1-n_1)(1+n_2)+n_1n_2]\\&+\delta^4(k+p_1-p_2)|M(\Phi,\bar{1}\to2)|^2[n_1(1+n_2)+n_2(1-n_1)]\\&+\delta^4(k-p_1+p_2)|M(\Phi,\bar{2}\to1)|^2[n_2(1-n_1)+n_1(1+n_2)]\\&+\delta^4(k+p_1+p_2)|M(\Phi,\bar{1},\bar{2}\to0)|^2[n_1n_2+(1-n_1)(1+n_2)]\}\ ,\end{aligned} \tag{2.25}$$

where s_1 is the spin of the fermion ϕ_1, $dp=d^3p/2E(2\pi)^3$, and the amplitudes M are given by[10]

$$M(\Phi\to1,2)=g\bar{u}(p_1)u(k)=M(\Phi,\bar{2}\to1),\quad M(\Phi,\bar{1}\to2)=g\bar{v}(p_1)u(k)=M(\Phi,\bar{1},\bar{2}\to0)\ . \tag{2.26}$$

The statistical weights such as $(1-n_1)(1+n_2)$ for $\Phi\to\phi_1\phi_2$ and n_1n_2 for the inverse decay $\phi_1\phi_2\to\Phi$ are just as expected since n_1 refers to a fermion and n_2 to a boson. However, the rates for the disappearance of Φ and for the reappearance of Φ are added in (2.25). This is to be contrasted with the previous results for Φ a boson, in which case the two rates were subtracted as in (2.10) and (2.20). This is a general feature.

C. Multiparticle discontinuities

It is straightforward to generalize the previous results. The probability for a particle Φ to propagate through a $T\neq0$ medium with energy $\omega>0$ and momentum $\vec{k}$ ($s\equiv\omega^2-\vec{k}^2$) will decrease with a rate $\Gamma_d(\omega)$ given by[11]

$$\Gamma_d(\omega)=\frac{1}{2\omega}\sum_{a,b}\int d\Omega_{ab}|M(\Phi,1,\ldots,a\to1',\ldots,b')|^2n_1\cdots n_a(1\pm n_1')\cdots(1\pm n_b') \tag{2.27}$$

and will increase with a rate $\Gamma_i(\omega)$ given by

$$\Gamma_i(\omega)=\frac{1}{2\omega}\sum_{a,b}\int d\Omega_{ab}|M(1',\ldots,b'\to\Phi,1,\ldots,a)|^2n_1'\cdots n_b'(1\pm n_1)\cdots(1\pm n_a)\ . \tag{2.28}$$

Boson emission is enhanced by the statistical factor $1+n(E)$; fermion emission is suppressed by $1-n(E)$. The phase-space integration is

$$\int d\Omega_{ab}\equiv\int dp_1\cdots dp_a dp_{1'}\cdots dp_{b'}(2\pi)^4\delta^4\left[K+\sum_1^a p_i-\sum_1^b p_j'\right]\ , \tag{2.29}$$

where $k^\mu=(\omega,\vec{k})$ and $dp=d^3p/2Ep(2\pi)^3$.

The results of Secs. II A and II B suggest that Γ_d and Γ_i in the general case are also related to the Φ self-energy by

$$\mathrm{Disc}\Pi(\omega)=-i2\omega[\Gamma_d(\omega)-\sigma\Gamma_i(\omega)]\ , \tag{2.30}$$

where $\sigma=1$ if Φ is a boson [as in (2.10) and (2.20)] and $\sigma=-1$ if Φ is a fermion [as in (2.25)]. This conjecture is demonstrated more fully in the Appendix, though not with complete rigor. Similarly, (2.30) can be expressed as

$$\mathrm{Im}\Pi(\omega)=-\omega[\Gamma_d(\omega)-\sigma\Gamma_i(\omega)]\ . \tag{2.31}$$

It is important to point out that the ratio of Γ_d to Γ_i is a universal function of ω, because the single-particle distribution functions $n(E)$ satisfy

$$\begin{aligned}(1+n)/n&=\exp(E/T)\ \ \text{(bosons)}\ ,\\(1-n)/n&=\exp(E/T)\ \ \text{(fermions)}\ .\end{aligned} \tag{2.32}$$

Thus the ratio of the statistical factors in (2.27) to those in (2.28) is given by

$$\exp\left[\left[-\sum_1^a E_i+\sum_1^b E_j'\right]\Big/T\right]=\exp(\omega/T)\ . \tag{2.33}$$

If CP invariance holds, the corresponding $|M|^2$ are equal for direct and inverse reactions so that

$$\frac{\Gamma_d(\omega)}{\Gamma_i(\omega)}=\exp(\omega/T)\ . \tag{2.34}$$

In fact, even if CP invariance is violated, this result still holds because of unitarity.[12,13]

An interesting example of the multiparticle discontinuities occurs in the order-α radiative corrections to the Higgs-boson decay $H\rightarrow e^+e^-$ that were studied by Donoghue and Holstein.[3] They computed rates by squaring amplitudes and integrating over phase space, taking into account the possibility of particle absorption from the heat bath. The same results are obtained by computing the discontinuities in the H self-energy illustrated in Fig. 4. [In the regime $T\ll m_e$ considered by Donoghue and Holstein all of the inverse reactions and $He^\pm\leftrightarrow\gamma e^\pm$ are negligible because of $\exp(-m_e/T)$ suppression factors.]

III. PHYSICAL INTERPRETATION

At $T\neq0$ the self-energy of any particle Φ, stable or unstable, has an imaginary part because of reactions like $\Phi\bar{\phi}_1\leftrightarrow\phi_2\cdots\phi_N$. This, of course, does not mean that all Φ's will disappear from the heat bath after a sufficiently long time. In fact, after a very long time one expects a thermal distribution of all elementary particles, both stable and unstable. Thus a different interpretation of ImΠ is required.

(1) The approach to equilibrium. To interpret ImΠ we note that although $\phi_1,\phi_2,\phi_3,\ldots$ are taken to be in thermal equilibrium, it is not necessary for the previous calculations to assume that Φ itself is in thermal equilibrium. For example, one might specify that at $t=0$ the number of Φ's with energy ω follow an arbitrary nonequilibrium distribution $f_0(\omega)$. The distribution function of the Φ's at any later time will be $f(\omega,t)$. Changes in f come about both from the rate $f\Gamma_d$ for decreasing the number of Φ's and from the rate $(1+\sigma f)\Gamma_i$ for increasing the number of Φ's, where $\sigma=\pm1$ depending on whether Φ is a boson or fermion. Thus $f(\omega,t)$ satisfies

$$\frac{\partial f}{\partial t}=-f\Gamma_d+(1+\sigma f)\Gamma_i\ . \tag{3.1}$$

In higher orders, Φ itself can occur within the self-energy so that Γ_d and Γ_i can depend on f, i.e., (3.1) is nonlinear.[15] However, for small departures from equilibrium one can use the exact equilibrium distributions in Γ_d and Γ_i. Then (3.1) has the solution[16]

$$f(\omega,t)=\frac{\Gamma_i}{\Gamma_d-\sigma\Gamma_i}+c(\omega)e^{-(\Gamma_d-\sigma\Gamma_i)t}\ , \tag{3.2}$$

where $c(\omega)$ is an arbitrary function. Because of (2.34) this may be written

$$f(\omega,t)=\frac{1}{e^{\omega/T}-\sigma}+c(\omega)e^{-\Gamma(\omega)t}\ . \tag{3.3}$$

Thus regardless of the distribution specified at $t=0$, $f(\omega,t)$ inevitably approaches the equilibrium form as $t\rightarrow\infty$. The rate of approach to equilibrium $\Gamma(\omega)$ is related to the self-energy function because of (2.31):

$$\mathrm{Im}\Pi(\omega)=-\omega\Gamma(\omega)\ . \tag{3.4}$$

The distinction between boson self-energies $(\Gamma=\Gamma_d-\Gamma_i)$ and fermion self-energies $(\Gamma=\Gamma_d+\Gamma_i)$ is essential in obtaining the distribution function (3.3).

(2) Thermal properties of virtual particles. The argument leading to the thermal distribution function (3.3) for Φ did not require that Φ be on its mass shell. In fact, no mass value for Φ has ever been introduced. This suggests that the distribution function (3.3) should apply to virtual Φ's with any value of $s=\omega^2-\vec{k}^2$ and that after a time $t\gg1/\Gamma$ the virtual Φ's will reach thermal equilibrium.

Additional evidence for this view is provided by the $T\neq0$ propagator. For example, if Φ is a spinless boson the function

$$\Delta(\omega)=[\omega^2-k^2-M^2-\Pi(\omega)]^{-1} \tag{3.5}$$

is the full propagator in the imaginary-time formalism when ω takes an imaginary value (2.2). The full propagator in the real-time formulation is not just the analytic continuation of (3.5) down to the real ω axis but is given by

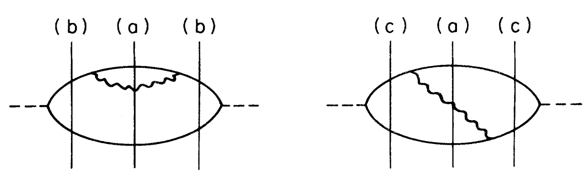

FIG. 4. The order-α radiative corrections to the decay $H\rightarrow e^+e^-$ correspond to the discontinuities in the Higgs-boson self-energy: (a) $H\leftrightarrow e^+e^-\gamma$, $H\gamma\leftrightarrow e^+e^-$, $He^-\leftrightarrow\gamma e^-$, and $He^+\leftrightarrow\gamma e^+$; (b) $H\leftrightarrow e^+e^-$ with fermion self-energy correction; (c) $H\leftrightarrow e^+e^-$ with vertex correction.

$$D(\omega)=\Delta(\omega+i\eta)-\frac{i\rho(\omega)}{\exp(\omega/T)-1}\ , \tag{3.6}$$

$$-i\rho(\omega)\equiv\Delta(\omega+i\eta)-\Delta(\omega-i\eta)\ , \tag{3.7}$$

as shown by Dolan and Jackiw.[5] Although (3.6) holds for ω negative, it is misleading. To rewrite (3.6) we first use the Hermitian analytic property (2.9) to write the spectral function more explicitly,

$$\rho(\omega)=\frac{2\mathrm{Im}\Pi}{(\omega^2-k^2-M^2-\mathrm{Re}\Pi)^2+(\mathrm{Im}\Pi)^2}\ . \tag{3.8}$$

Since $\mathrm{Re}\Pi$ and $\mathrm{Im}\Pi$ are even and odd functions of ω, respectively, the spectral function is odd:

$$\rho(\omega)=\epsilon(\omega)\rho(\,|\,\omega\,|\,)\ . \tag{3.9}$$

The physical value of the variable $s=\omega^2-k^2$ is obtained by approaching the real s axis from above: $s+i\eta$. In the complex ω plane, this limit is equivalent to $\omega+i\eta$ when $\omega>0$ and to $\omega-i\eta$ when $\omega<0$. A simple calculation shows

$$\Delta(\omega+i\eta)=\Delta(\omega+i\epsilon(\omega)\eta)+i\theta(-\omega)\rho(\,|\,\omega\,|\,)\ . \tag{3.10}$$

Using (3.9) and (3.10) allows the full propagator to be written

$$D(\omega)=\Delta(\omega+i\epsilon(\omega)\eta)-\frac{i\rho(\,|\,\omega\,|\,)}{\exp(\,|\,\omega/T\,|\,)-1}\ . \tag{3.11}$$

This form is far more useful: The first term evaluates the appropriate physical boundary value of the self-energy; the second term provides a thermal distribution appropriate to particles of energy $|\,\omega\,|$ regardless of the sign of ω.

In the free-field limit, $\mathrm{Im}\Pi\to 0$ and (3.8) gives $\rho(\,|\,\omega\,|\,)\to 2\pi\delta(\omega^2-k^2-M^2)$. Then (3.11) is the usual free boson propagator[5] in which the thermal distribution factor $[\exp(\,|\,\omega/T\,|\,)-1]^{-1}$ applies only to particles which are on the mass shell (though with either sign for ω).

However, in the presence of interactions $\mathrm{Im}\Pi$ is nonvanishing for any particle, stable or unstable, because of reactions like $\Phi,\bar{1}\leftrightarrow 2,3,\ldots,N$ discussed in Sec. II. Both real and virtual Φ's will be in thermal equilibrium as shown by (3.11) and the probability of an off-shell Φ existing in the heat bath is given by the Breit-Wigner function (3.8).

(3) Further comments. (a) As noted at the beginning of Sec. II, one cannot use the conventional real-time propagators[5] in a diagrammatic expansion and consequently one cannot obtain the results derived here. However, recently a new formulation of finite-temperature field theory has been developed with new real-time propagators that can be used in a diagrammatic expansion.[17–21] It would be interesting to show that this new formulation yields the same imaginary parts as obtained here.

(b) Only normal threshold discontinuities in the self-energy have been investigated here. From the calculations in Sec. II and the Appendix it is clear that discontinuities in the two-to-two scattering amplitude have the same form as (2.27) and (2.28) except that Φ stands for the external two-particle state. (And similarly for all other Green's functions.) However, it is not clear whether the corresponding two-particle Γ_d and Γ_i determine the approach to equilibrium of a two-particle distribution function. This requires generalizing (3.1).

ACKNOWLEDGMENTS

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APPENDIX: MULTIPARTICLE DISCONTINUITIES

The general form for the multiparticle discontinuity that was conjectured in Sec. II C will now be demonstrated. The demonstration emphasizes how the statistical factors arise from the finite-temperature propagators but is not mathematically rigorous.

A single self-energy diagram may have many possible intermediate states. We consider the contribution of a particular N-body state as shown in Fig. 5. For simplicity these N particles are all taken as spinless bosons with propagators

$$D(\omega_j,\vec{p}_j)=(\omega_j{}^2-E_j{}^2)^{-1}\ , \tag{A1}$$

where $\omega_j=i\,2\pi T\times$integer and $E_j{}^2=m_j{}^2+\vec{p}_j{}^2$. The external frequency and momentum are

$$\begin{aligned}\omega&=\omega_1+\cdots+\omega_N\ ,\\ \vec{k}&=\vec{p}_1+\cdots+\vec{p}_N\ .\end{aligned} \tag{A2}$$

The self-energy amplitude in Fig. 5 can be written

$$(-T)^{N-1}\sum_{\omega_1,\ldots,\omega_{N-1}}\int\frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_{N-1}}{(2\pi)^3}\times A(\omega_i)D(\omega_1)\cdots D(\omega_N)B(\omega_j)\ , \tag{A3}$$

where the $\vec{p}$ dependence of the propagators and of the amplitudes A and B is suppressed.

The discontinuity produced by the N-particle state comes from the N propagators D and not from A or B. Consequently, to evaluate the discontinuity one may replace $A(\omega_1,\ldots,\omega_N)$ by $A(E_1,\ldots,E_N)$:

$$(-T)^{N-1}\sum_{\omega_1,\ldots,\omega_{N-1}}\int\frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_{N-1}}{(2\pi)^3}\times A(E_i)D(\omega_1)\cdots D(\omega_N)B(E_j)\ . \tag{A4}$$

A proof that (A3) and (A4) have the same N-particle

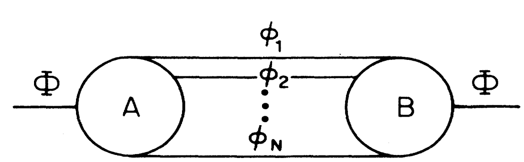

FIG. 5. The contribution of a particular N-body intermediate state to the self-energy of Φ.

discontinuity would presumably follow the zero-temperature argument but is not attempted here.

To evaluate (A4) it is necessary to compute

$$R \equiv (-T)^{N-1} \sum_{\omega_1,\ldots,\omega_{N-1}} D(\omega_1,\vec{p}_1)\cdots D(\omega_N,\vec{p}_N) . \tag{A5}$$

This summation is equivalent to an $(N-1)$-loop diagram and is extremely tedious to evaluate directly. Instead, it is convenient to transform from frequency to time by

$$D(\omega_j,\vec{p}_j) = \int_0^\beta d\tau\, e^{\omega_j\tau}\tilde{D}(\tau,\vec{p}_j) , \tag{A6}$$

$$\tilde{D}(\tau,\vec{p}_j) = \frac{-1}{2E_j}[e^{-E_j\tau}(1+n_j)+e^{E_j\tau}n_j] , \tag{A7}$$

where $0\le\tau\le\beta$ and $n_j=n(E_j)$ is the usual Bose-Einstein distribution function (2.6). The statistical factors in (A7) are just those appropriate to stimulated emission $(1+n_j)$ and absorption (n_j) and will be crucial in the final result. After substituting (A6) the frequency sums in (A5) yield

$$R = (-1)^{N-1}\int_0^\beta d\tau\, e^{\omega\tau}\tilde{D}(\tau,\vec{p}_1)\cdots\tilde{D}(\tau,\vec{p}_N) . \tag{A8}$$

This is a trivial integration which can be performed concisely by writing

$$\tilde{D}(\tau,\vec{p}_j) = \frac{-1}{2E_j}\sum_{\sigma_j=\pm1} e^{-\sigma_j E_j\tau} f_j , \tag{A9}$$

$$f_j \equiv \tfrac{1}{2}(1+\sigma_j)+n_j . \tag{A10}$$

Then (A8) becomes

$$R = \frac{1}{(2E_1)\cdots(2E_N)}\sum_{\sigma_1,\ldots,\sigma_N}\frac{1-e^{-\beta Q}}{\omega-Q} f_1 f_2\cdots f_N , \tag{A11}$$

$$Q \equiv \sigma_1E_1+\sigma_2E_2+\cdots+\sigma_NE_N , \tag{A12}$$

where $\exp(\omega\beta)=1$ has been used. Writing the identity $e^{-\beta E}(1+n)=n$ as

$$e^{-\beta\sigma_jE_j}f_j = g_j , \tag{A13}$$

$$g_j \equiv \tfrac{1}{2}(1-\sigma_j)+n_j , \tag{A14}$$

allows R to be expressed as

$$R = \frac{1}{(2E_1)\cdots(2E_N)}\sum_{\sigma_1,\ldots,\sigma_N}\frac{1}{\omega-Q}(f_1f_2\cdots f_N - g_1g_2\cdots g_N) . \tag{A15}$$

Thus (A4) is given by

$$\int\frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_{N-1}}{(2\pi)^3}A(E_i)RB(E_j) . \tag{A16}$$

Up until now ω has been pure imaginary and discrete. In (A16) one may continue ω down to the real axis and evaluate the discontinuity across the real axis:

$$\text{Disc}\Pi(\omega) = -i\int dp_1\cdots dp_N\sum_{\sigma_1\cdots\sigma_N}2\pi\delta(\omega-Q)(2\pi)^3\delta^2\left[\vec{k}-\sum_1^N p_i\right] \times A(E_i)B(E_j)(f_1f_2\cdots f_N-g_1g_2\cdots g_N) , \tag{A17}$$

where $dp=d^3p/2E(2\pi)^3$. When the contribution of the jth boson to Q is positive (i.e., $\sigma_j=+1$) the statistical factors are $f_j=1+n_j$ appropriate to stimulated emission and $g_j=n_j$ appropriate to absorption. When the contribution of the jth boson to Q is negative (i.e., $\sigma_j=-1$) the statistical factors are automatically reversed: $f_j=n_j$ and $g_j=1+n_j$. Summing the various possible amplitudes A and B then gives the result claimed in Sec. II C.

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