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Theory and renormalization of the gauge-invariant effective action

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The different methods for constructing a gauge-invariant effective action (GIEA) for quantum non-Abelian gauge field theories proposed by 't Hooft, DeWitt, Boulware, and Abbott are all shown to be equivalent. In the course of proving this equivalence we show how to extend the usual background-field method so as to construct what may be considered the prototypical GIEA and discuss in some detail the invariance and gauge transformation properties of both the usual theory and the new theory using the GIEA. All solutions to the GIEA field equations are shown to be physical—being solutions to the usual field equations with an arbitrary gauge condition. The renormalization program based upon the GIEA is shown to differ from the standard theory and we outline the modifications which are needed in the present proof of renormalizability. In particular we prove that the physical renormalization is independent of any gauge-fixing choice. Finally, we prove that the *S*-matrix elements derived from the GIEA for an arbitrary background-field solution to the field equations are the same as those derived using the usual effective action.

I. INTRODUCTION

In quantum field theory physical predictions are plagued by ultraviolet infinities which must be exorcised by the techniques of renormalization theory. The renormalization of non-Abelian gauge theories is considerably more complex than that of quantum electrodynamics and it is crucial for the proof of renormalizability that all invariances of the theory be fully exploited. To quantize a classically gauge-invariant Lagrangian field theory a gauge condition must be imposed, thereby breaking the gauge invariance, in order to secure a well-defined propagator. The background-field method was originally introduced by DeWitt[1,2] as a method for retaining a residual gauge invariance in the theory. This simplified calculations by maintaining manifest covariance under background-field gauge transformations; however, in its original formulation, the method worked well only for one-loop calculations. Attempts to extend the method to higher orders were criticized on various grounds until Kluberg-Stern and Zuber,[3] using the supergauge transformations of Becchi, Rouet, and Stora (BRS), reformulated the method in such a way as to be valid to all orders in perturbation theory. The problem with their reformulation was that the original simplicity of the background-field method was lost. In order to retain the simplicity of the original method in higher orders, 't Hooft[4] proposed an alternate method in which gauge-covariant and background-field-dependent sources are introduced. His proposal was never implemented in actual calculations and the equivalence of his formulation with the usual field theory methods was uncertain since much of his paper was left somewhat schematic.

Recently there have appeared several discussions of the multiloop extension of the background-field method based on 't Hooft's idea. Independently, DeWitt,[5] Boulware,[6] and Abbott[7] have proposed methods for constructing a manifestly gauge-invariant effective action (GIEA) and have, to varying degrees, discussed how to utilize their results in order to calculate physical quantities and to implement the renormalization program. In addition, using these extensions of the background-field method, explicit two-loop calculations of the renormalization constants and β function for pure Yang-Mills theory in the Feynman gauge have been made by Abbott,[7] the author,[8] and by Ichinose and Omote.[9] These calculations have recently been extended to the general-gauge case by Capper and MacLean.[10]

In this paper we show that the different procedures used by the aforementioned authors to construct a GIEA are in fact equivalent. In the process of proving equivalence we construct what may be considered the prototypical GIEA and formulate the calculational rules to be used with the GIEA. It is shown that all solutions to the new GIEA field equations are physical and some mistakes and subtleties of these earlier papers are clarified. It is also the purpose of this paper to discuss the renormalization program based upon the GIEA. This has not been discussed in any detail in the literature, and since the usual iterative proof of renormalizability cannot be applied to the GIEA a different approach must be taken. In particular we show why the physical renormalization is independent of any gauge-fixing choice and we give a brief proof of renormalizability for the GIEA of pure Yang-Mills theory using the renormalization theory of Caswell and Kennedy.[11] Finally we answer in the affirmative a conjecture made by DeWitt[5] as to whether the S-matrix elements derived using the GIEA for a nonvacuum background field are the same as those derived using the usual effective action.

The organization of this paper is as follows. The standard functional quantization of non-Abelian gauge theories using the background-field method is briefly reviewed in Sec. II. Important invariance and gauge transformation properties are derived without using BRS techniques. Since it is necessary to introduce several different effective actions, we try to be very explicit about their functional dependence on fields and distinguish between total and partial functional derivatives. The GIEA $\Gamma[\varphi]$ is constructed in Sec. III and we show that previous GIEA's are equivalent to $\Gamma[\varphi]$. The renormalization program using the GIEA is sketched in Sec. IV and the equivalence of S-matrix elements is proven in Sec. V. A summary and discussion of the paper is given in the last section where some further points regarding the uses of the GIEA are made.

II. BACKGROUND-FIELD QUANTIZATION

In this section we derive standard results of the usual background-field method giving particular emphasis to gauge transformation and invariance properties of the effective action. This material is needed for the derivation of the GIEA. Most of these results are well known and have also been covered in some detail by Boulware[6] and DeWitt[5]; however, the present derivations make no use of the BRS techniques as used by Boulware and many of the equations are written in such a way as to facilitate the procedure of Sec. III. The condensed notation and conventions of DeWitt[1,2,5] are used throughout this paper.

A. The effective action

It is well known that the generating functional for connected Green's functions may be written as[12]

$$e^{iW[J,P]} \equiv N \int d\underline{A}\, \exp\{i(S[\underline{A}]+\tfrac{1}{2}\underline{P}^\alpha \underline{P}_\alpha + J_i \underline{A}^i)\} \times \det \frac{\delta \underline{P}^\alpha}{\delta \underline{A}^i} Q^i{}_\beta[\underline{A}] \, , \tag{1}$$

where the action $S[\underline{A}]$ is given by $S[\underline{A}]=\int d^4x\, \mathscr{L}[\underline{A},\partial_\mu \underline{A},\dots]$ for $\mathscr{L}$ the Lagrangian. The action is invariant under gauge transformations of the field $\underline{A}^i \to \underline{A}^i + Q^i{}_\alpha[\underline{A}]\delta\xi^\alpha$:

$$\frac{\delta S[\underline{A}]}{\delta \underline{A}^i} Q^i{}_\alpha[\underline{A}] \equiv S_{,i}[\underline{A}] Q^i{}_\alpha[\underline{A}] = 0 \, , \tag{2}$$

where the gauge transformations are assumed linear in $\underline{A}$:

$$Q^i{}_\alpha[\underline{A}] = Q^i{}_\alpha[0] + Q^i{}_{\alpha,j}\underline{A}^j \, , \quad Q^i{}_{\alpha,jk} = 0 \, . \tag{3}$$

Since these transformations are also assumed to form a group we have

$$Q^i{}_{\alpha,j} Q^j{}_\beta - Q^i{}_{\beta,j} Q^j{}_\alpha = Q^i{}_\gamma c^\gamma{}_{\alpha\beta} \, , \tag{4}$$

where the $c^\alpha{}_{\gamma\beta}$ are the $\underline{A}$-independent structure constants satisfying

$$c^\delta{}_{\alpha\epsilon} c^\epsilon{}_{\beta\gamma} + c^\delta{}_{\beta\epsilon} c^\epsilon{}_{\gamma\alpha} + c^\delta{}_{\gamma\epsilon} c^\epsilon{}_{\alpha\beta} = 0 \, . \tag{5}$$

The $\underline{P}^\alpha = P^\alpha[\underline{A}]$ are gauge-fixing constraints introduced in (1) so as to remove the redundancy due to gauge invariance of the action and N is a normalization constant. Since $\underline{A}$ is a dummy variable of integration in (1) we may divide the quantum operator field $\underline{A}^i$ into an arbitrary classical field φ^i plus a quantum piece $\underline{\phi}^i$: $\underline{A}^i = \varphi^i + \underline{\phi}^i$ (Ref. 13) and define $\overline{W}[\varphi,J,P]$ via

$$e^{i\overline{W}[\varphi,J,P]} \equiv N' \int d\underline{\phi}\, \exp\{i(S[\varphi+\underline{\phi}] + \tfrac{1}{2}(P^\alpha{}_i \underline{\phi}^i - \zeta^\alpha)(P_{\alpha j}\underline{\phi}^j - \zeta_\alpha) + J_i \underline{\phi}^i)\} \, \det P^\alpha{}_i Q^i{}_\beta[\varphi+\underline{\phi}] \, . \tag{6}$$

Note that in (6) we have chosen to couple the classical external source J_i only to the quantum field $\underline{\phi}^i$ and have chosen the linear gauge-fixing condition to be a condition on $\underline{\phi}^i$ only:

$$P^\alpha[\underline{\phi}] \equiv P^\alpha{}_i \underline{\phi}^i - \zeta^\alpha = 0 \, , \tag{7}$$

where the $P^\alpha{}_i$ will later on be taken to depend on the field φ^i but for now are not so constrained. The ζ^α are arbitrary constants introduced for later use.

The time-ordered quantum expectation value (chronological average) of a quantum operator functional of φ and $\underline{\phi}$, $\mathcal{O}[\varphi,\underline{\phi}]$, is defined as

$$\langle \mathcal{O}[\varphi,\underline{\phi}]\rangle \equiv e^{-i\overline{W}[\varphi,J,P]}N' \int d\underline{\phi}\, \mathcal{O}[\varphi,\underline{\phi}] \exp\{i(S[\varphi+\underline{\phi}]+\tfrac{1}{2}\underline{P}_\alpha \underline{P}^\alpha + J_i\underline{\phi}^i)\} \det P^\alpha{}_i Q^i{}_\beta[\varphi+\underline{\phi}] \; . \tag{8}$$

Of particular importance is

$$\bar{\phi}^i \equiv \langle \underline{\phi}^i\rangle = \frac{\partial \overline{W}[\varphi,J,P]}{\partial J_i} \tag{9}$$

and

$$\begin{aligned}
&\langle \underline{\phi}^i\underline{\phi}^j\rangle = \bar{\phi}^i\bar{\phi}^j - i\bar{\Gamma}^{ij} \; ,\\
&\langle \underline{\phi}^i\underline{\phi}^j\underline{\phi}^k\rangle = \bar{\phi}^i\bar{\phi}^j\bar{\phi}^k - iP_3\bar{\phi}^i\bar{\Gamma}^{jk} + (-i)^2\bar{\Gamma}^{ijk} \; ,\\
&\langle \underline{\phi}^i\underline{\phi}^j\underline{\phi}^k\underline{\phi}^l\rangle = \bar{\phi}^i\bar{\phi}^j\bar{\phi}^k\bar{\phi}^l - iP_6\bar{\phi}^i\bar{\phi}^j\bar{\Gamma}^{kl} + (-i)^2P_4\bar{\phi}^i\bar{\Gamma}^{jkl} + (-i)^2P_3\bar{\Gamma}^{ij}\bar{\Gamma}^{kl} + (-i)^3\bar{\Gamma}^{ijkl}, \text{ etc. } ,
\end{aligned} \tag{10}$$

where

$$\bar{\Gamma}^{i_1\cdots i_n} \equiv \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_n}} \overline{W}[\varphi,J,P] \tag{11}$$

and "P_n" in (10) indicates that all distinct permutations of the indices should be added while the subscript n on P_n indicates the number of these required. As will be shown presently the Jacobian matrix

$$\bar{\Gamma}^{ij} = \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\overline{W}[\varphi,J,P] = \frac{\partial \bar{\phi}^i}{\partial J_j} = \frac{\partial \bar{\phi}^j}{\partial J_i} = \bar{\Gamma}^{ji} \tag{12}$$

is nonsingular and we will be able to solve (9) for the J_i as a functional of φ, P, and $\bar{\phi}$: $J_i = J_i[\varphi,P,\bar{\phi}]$. The $\bar{\Gamma}^{i_1\cdots i_n}$ may also be regarded as functionals of φ, P, and $\bar{\phi}$. It is then easy to derive the following very useful series expansion for the chronological average of an arbitrary operator[2,5,8]:

$$\begin{aligned}
\langle \mathcal{O}[\varphi,\underline{\phi}]\rangle &= :\!\exp\left\{ i\sum_{n=2}^{\infty} \frac{(-i)^n}{n!}\bar{\Gamma}^{i_1\cdots i_n}[\varphi,P,\bar{\phi}]\frac{\partial}{\partial\bar{\phi}^{i_1}}\cdots\frac{\partial}{\partial\bar{\phi}^{i_n}}\right\}\!:\mathcal{O}[\varphi,\bar{\phi}]\\
&= \left[1 + \frac{(-i)}{2}\bar{\Gamma}^{ij}\frac{\partial}{\partial\bar{\phi}^i}\frac{\partial}{\partial\bar{\phi}^j} + \frac{(-i)^2}{3!}\bar{\Gamma}^{ijk}\frac{\partial}{\partial\bar{\phi}^i}\frac{\partial}{\partial\bar{\phi}^j}\frac{\partial}{\partial\bar{\phi}^k} \right.\\
&\quad \left. + \frac{(-i)^3}{4!}(\bar{\Gamma}^{ijkl} + 3i\bar{\Gamma}^{ij}\bar{\Gamma}^{kl})\frac{\partial}{\partial\bar{\phi}^i}\frac{\partial}{\partial\bar{\phi}^j}\frac{\partial}{\partial\bar{\phi}^k}\frac{\partial}{\partial\bar{\phi}^l} + \cdots \right]\mathcal{O}[\varphi,\bar{\phi}] \; .
\end{aligned} \tag{13}$$

The colons indicate that all $\bar{\Gamma}$'s stand to the left of the $\partial/\partial\bar{\phi}$'s and $\mathcal{O}[\varphi,\bar{\phi}]$ is $\mathcal{O}[\varphi,\underline{\phi}]$ with $\underline{\phi}$ replaced by its average $\bar{\phi}$.

The quantum field equations follow from (6) with the condition that the functional integral of a total functional derivative vanish:

$$\langle S_{,i}[\varphi+\underline{\phi}]\rangle + i\langle \mathfrak{G}^\alpha{}_\beta[\varphi,\underline{\phi}]\rangle V^\beta{}_{\alpha i} + P^\alpha{}_i P_{\alpha j}\bar{\phi}^j - \zeta^\alpha P_{\alpha i} = -J_i \; , \tag{14}$$

where we have defined $\mathfrak{G}^\alpha{}_\beta[\varphi,\underline{\phi}]$ to be the negative inverse of the operator $\mathfrak{F}^\alpha{}_\beta[\varphi,\underline{\phi}]$:

$$\begin{aligned}
&\mathfrak{F}^\alpha{}_\beta[\varphi,\underline{\phi}] \equiv P^\alpha{}_i Q^i{}_\beta[\varphi+\underline{\phi}] \; ,\\
&\mathfrak{F}^\alpha{}_\beta[\varphi,\underline{\phi}]\mathfrak{G}^\beta{}_\gamma[\varphi,\underline{\phi}] = -\delta^\alpha{}_\gamma \; ,
\end{aligned} \tag{15}$$

and the vertex $V^\alpha{}_{\beta i}$ is defined by

$$V^\alpha{}_{\beta i} \equiv P^\alpha{}_j Q^j{}_{\beta,i} \; . \tag{16}$$

Remembering that the term $\det P^\alpha{}_i Q^i{}_\beta[\varphi+\underline{\phi}] = \det \mathfrak{F}^\alpha{}_\beta[\varphi,\underline{\phi}]$ is often written as a functional integral over anticommuting fields we note that $\mathfrak{G}^\alpha{}_\beta[\varphi,\underline{\phi}]$ is the usual ghost propagator in the background-field method while $V^\alpha{}_{\beta i}$ is the usual ghost-ghost-quantum field vertex. If we now differentiate both sides of (14) with respect to J_j and use (12) we have

$$\left\{ \frac{\delta}{\delta\bar{\phi}^k}\left[\langle S_{,i}[\varphi+\underline{\phi}]\rangle + i\langle \mathfrak{G}^\alpha{}_\beta[\varphi,\underline{\phi}]\rangle V^\beta{}_{\alpha i}\right] + P^\alpha{}_i P_{\alpha k}\right\}\bar{\Gamma}^{kj} = -\delta_i{}^j \; . \tag{17}$$

Since $\bar{\Gamma}^{kj}$ is symmetric in its indices and the quantity inside the brackets is the negative inverse of $\bar{\Gamma}^{kj}$ which must also be symmetric, there exists a functional $\bar{\Gamma}[\varphi,P,\bar{\phi}]$, the effective action, which satisfies

$$\frac{\partial\bar{\Gamma}[\varphi,P,\bar{\phi}]}{\partial\bar{\phi}^i}=-J_i\ ,\qquad \frac{\partial^2\bar{\Gamma}}{\partial\bar{\phi}^i\partial\bar{\phi}^j}\bar{\Gamma}^{jk}=-\delta_i{}^k\ . \tag{18}$$

Further functional derivatives of (18) give relations between the $\bar{\Gamma}^{i_1\cdots i_n}$ and the vertex functions $\bar{\Gamma}_{,j_1\cdots j_m}$.

The effective action $\bar{\Gamma}$ is related to $\bar{W}$ via the Legendre transform

$$\bar{\Gamma}[\varphi,P,\bar{\phi}]\equiv\bar{W}[\varphi,P,J]-J_i\bar{\phi}^i+\text{constant}\ . \tag{19}$$

One can, using (14), (18), and (13), easily derive the well-known graphical expansion given in Fig. 1,[5,8] in terms of bare propagators and bare vertices. Here the solid lines represent the bare propagator $G^{ij}[\varphi,\bar{\phi}]$ where

$$F_{ij}[\varphi,\bar{\phi}]\equiv S_{,ij}[\varphi+\bar{\phi}]+P^{\alpha}{}_iP_{\alpha j}\ ,\qquad G^{ij}[\varphi,\bar{\phi}]F_{jk}[\varphi,\bar{\phi}]=-\delta^i{}_k\ , \tag{20}$$

and the dashed lines represent the ghost propagator $\mathfrak{G}^{\alpha}{}_{\beta}[\varphi,\bar{\phi}]$ gotten by replacing $\underline{\phi}$ by $\bar{\phi}$ in (15). The vertices where n solid lines meet are given by $S_{,i_1\cdots i_n}$ and the vertices where two dashed lines meet one solid line are given by the $V^{\beta}{}_{\alpha i}$ of Eq. (16). Note that the dashed lines always appear in closed, oriented loops. Also note that the loop expansion of Fig. 1 is entirely in terms of one-particle-irreducible diagrams.

B. Invariance properties of the theory

We now derive several important relationships satisfied by $\bar{W}$ and $\bar{\Gamma}$. It is straightforward to show from (8) that

$$\frac{\partial\bar{W}}{\partial P^{\alpha}{}_i}=P_{\alpha j}\langle\underline{\phi}^j\underline{\phi}^i\rangle-\zeta_{\alpha}\bar{\phi}^i+i\langle Q^i{}_{\beta}[\varphi+\underline{\phi}]\mathfrak{G}^{\beta}{}_{\alpha}[\varphi,\underline{\phi}]\rangle\ ,\qquad \frac{\partial\bar{W}}{\partial\zeta_{\alpha}}=-P^{\alpha}{}_i\bar{\phi}^i+\zeta^{\alpha}\ . \tag{21}$$

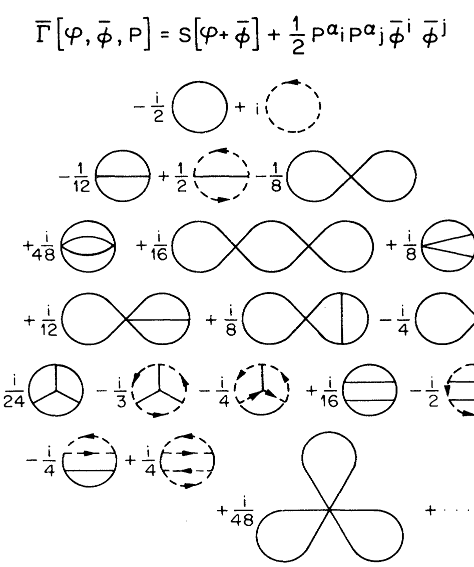

FIG. 1. Diagrammatic loop expansion of the effective action $\bar{\Gamma}[\varphi,\bar{\phi},P]$. See Sec. II for definition of lines and vertices. The GIEA has the same diagrammatic expansion, except $\bar{\phi}$ is set equal to zero and the lines and vertices are reinterpreted as described in Sec. III (adapted from DeWitt, Ref. 5).

If we now make the following change of variables on the dummy variable of integration $\underline{\phi}$ in (8),

$$\underline{\phi}^i=\underline{\phi}'^i+Q^i{}_{\alpha}[\varphi+\underline{\phi}']\mathfrak{G}^{\alpha}{}_{\beta}[\varphi,\underline{\phi}']\delta\xi^{\beta}\ , \tag{22}$$

compute the required Jacobian, take into account the gauge invariance of the action, and use (3), (4), and (5),[14] we arrive at

$$P_{\alpha i}\bar{\phi}^i=\zeta_{\alpha}+J_i\langle Q^i{}_{\beta}[\varphi+\underline{\phi}]\mathfrak{G}^{\beta}{}_{\alpha}[\varphi,\underline{\phi}]\rangle\ . \tag{23}$$

Equation (23) gives the important result that when the external source J_i vanishes, the expectation value of the field $\bar{\phi}^i$ satisfies the same linear gauge condition (7) used to break gauge invariance

$$P^{\alpha}{}_i\bar{\phi}^i\underset{J\to 0}{\longrightarrow}\zeta^{\alpha}\ . \tag{24}$$

Taking the derivative of both sides of (23) with respect to J gives

$$P_{\alpha i}\langle\underline{\phi}^i\underline{\phi}^j\rangle=P_{\alpha i}\bar{\phi}^i\bar{\phi}^j-i\langle Q^j{}_{\beta}[\varphi+\underline{\phi}]\mathfrak{G}^{\beta}{}_{\alpha}[\varphi,\underline{\phi}]\rangle-J_i\langle Q^i{}_{\beta}[\varphi+\underline{\phi}]\mathfrak{G}^{\beta}{}_{\alpha}[\varphi,\underline{\phi}]\rangle\bar{\phi}^j+J_i\langle Q^i{}_{\beta}[\varphi+\underline{\phi}]\mathfrak{G}^{\beta}{}_{\alpha}[\varphi,\underline{\phi}]\underline{\phi}^j\rangle\ . \tag{25}$$

We can use (23) and (25) to rewrite (21) as

$$\frac{\partial \overline{W}}{\partial P^\alpha{}_i}=J_j\langle Q^j{}_\beta[\varphi+\underline{\phi}]\mathfrak{G}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^i\rangle ,$$
$$\frac{\partial \overline{W}}{\partial \xi^\alpha}=-J_i\langle Q^i{}_\beta[\varphi+\underline{\phi}]\mathfrak{G}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle , \tag{26}$$

both of which vanish for $J=0$. This means that $\overline{W}$ is independent of the gauge-fixing condition when the source vanishes. It is also important to understand how $\overline{W}[\varphi,P,J]$ depends on the arbitrary classical field φ. Taking the derivative of (8) with respect to φ^i and using the equations of motion (14) as well as (23) and (27) gives

$$\frac{\partial \overline{W}}{\partial \varphi^i}=-J_i-J_j\langle Q^j{}_\beta[\varphi+\underline{\phi}]\mathfrak{G}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle P^\alpha{}_i \tag{27}$$

which implies that $\overline{W}[\varphi,P,J]$ is also independent of φ for $J=0$.

The relations derived up until now have assumed that $P^\alpha{}_i$ is not a functional of φ. In the background-field method,[1,4] however, we choose $P^\alpha{}_i=\hat{P}^\alpha{}_i[\varphi]$, a functional of the background field, in such a way that $\hat{P}^\alpha{}_i[\varphi]$ obeys the gauge transformation rule

$$\delta\hat{P}^\alpha{}_i[\varphi]\equiv\hat{P}^\alpha{}_{i,j}[\varphi]Q^j{}_\beta[\varphi]\delta\xi^\beta$$
$$=(c^\alpha{}_{\beta\gamma}\hat{P}^\gamma{}_i[\varphi]-\hat{P}^\alpha{}_j[\varphi]Q^j{}_{\beta,i})\delta\xi^\beta . \tag{28}$$

A suitable choice would be $\hat{P}^\alpha{}_i[\varphi]=Q_i{}^\alpha[\varphi]$. Unless specifically noted otherwise we assume from here on that the $\hat{P}^\alpha{}_i[\varphi]$ are chosen in just this way and therefore $\overline{W}[\varphi,P,J]\to\overline{W}[\varphi,\hat{P}[\varphi],J]\equiv\hat{W}[\varphi,J]$. The total variation of $\hat{W}$ is now given by

$$\frac{\delta\hat{W}}{\delta\varphi^i}=\frac{\partial\hat{W}}{\partial\varphi^i}+\frac{\partial\hat{W}}{\partial\hat{P}^\alpha{}_j}\frac{\partial\hat{P}^\alpha{}_j}{\partial\varphi_i}$$
$$=-J_i-J_j\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle\hat{P}^\alpha{}_i[\varphi]$$
$$+J_j\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\alpha{}_{k,i}[\varphi] , \tag{29}$$

where $\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]$ and $\hat{\mathfrak{F}}^\alpha{}_\beta[\varphi,\underline{\phi}]$ are given as in (15) with $P^\alpha{}_i\to\hat{P}^\alpha{}_i[\varphi]$.

$\hat{W}[\varphi,J]$ satisfies an important invariance relationship which we now derive. If the background field undergoes a gauge transformation: $\varphi^i\to\varphi^i+Q^i{}_\alpha[\varphi]\delta\xi^\alpha$ then by (29), $\hat{W}$ transforms as

$$\frac{\delta\hat{W}}{\delta\varphi^i}Q^i{}_\gamma[\varphi]=-J_j(Q^j{}_\gamma[\varphi]+\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle\hat{F}^\alpha{}_\gamma[\varphi,0]-\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\alpha{}_{k,i}Q^i{}_\gamma[\varphi]) .$$

Using the transformation law for $\hat{P}^\alpha{}_i$, Eq. (28), as well as the antisymmetry of $c^\alpha{}_{\beta\gamma}$ and Eq. (25) we can rewrite this as

$$\frac{\delta\hat{W}}{\delta\varphi^i}Q^i{}_\alpha[\varphi]-J_jQ^j{}_{\alpha,i}\frac{\partial\hat{W}}{\partial J_i}=0 . \tag{30}$$

This means that $\hat{W}[\varphi,J]$ is invariant under the combined transformations

$$\delta\varphi^i=Q^i{}_\alpha[\varphi]\delta\xi^\alpha ,$$
$$\delta J_j=-Q^i{}_{\alpha,j}J_i\delta\xi^\alpha . \tag{31}$$

We can now proceed to convert the preceding properties of $\overline{W}$ and $\hat{W}$ into relationships satisfied by $\hat{\Gamma}[\varphi,\hat{\phi}]$ where $\hat{\Gamma}$ is $\overline{\Gamma}$ with $P=\hat{P}[\varphi]$. From (19) and (26) we have

$$\frac{\partial\hat{\Gamma}}{\partial\xi^\alpha}=\frac{\partial\hat{\Gamma}}{\partial\hat{\phi}^i}\langle Q^i{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle\underset{J\to0}{\longrightarrow}0 \tag{32}$$

and if the functional form of $\hat{P}^\alpha{}_i$ is changed, while still satisfying (28), keeping φ fixed we have

$$\delta_P\hat{\Gamma}=-\frac{\partial\hat{\Gamma}}{\partial\hat{\phi}^j}\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^i\rangle\delta\hat{P}^\alpha{}_i[\varphi]$$
$$\underset{J\to0}{\longrightarrow}0 . \tag{33}$$

These relations will be important in discussing the physical significance and renormalization of the GIEA.

If we vary both sides of (19) (with $P=\hat{P}[\varphi]$) with respect to φ keeping J fixed we have

$$\frac{\delta\hat{\Gamma}}{\delta\varphi^i}=\left.\frac{\partial\hat{\Gamma}}{\partial\varphi^i}\right|_{\hat{\phi}}+\left.\frac{\partial\hat{\Gamma}}{\partial\hat{\phi}^j}\right|_{\varphi}\frac{\partial\hat{\phi}^j}{\partial\varphi^i}$$
$$=\left.\frac{\partial\hat{W}}{\partial\varphi^i}\right|_J-J_j\frac{\partial\hat{\phi}^j}{\partial\varphi^i} ,$$

where $\hat{\phi}^i\equiv\partial\hat{W}/\partial J_i$, which implies that

$$\left.\frac{\partial\hat{\Gamma}}{\partial\varphi^i}\right|_{\hat{\phi}}=\left.\frac{\partial\hat{W}}{\partial\varphi^i}\right|_J \tag{34}$$

and therefore we have the important invariance for the effective action which follows from (30)

$$\frac{\partial\hat{\Gamma}}{\partial\varphi^i}Q^i{}_\alpha[\varphi]+\frac{\partial\hat{\Gamma}}{\partial\hat{\phi}^j}Q^j{}_{\alpha,i}\hat{\phi}^i=0\ . \tag{35}$$

Equation (35) shows that the effective action is invariant under the combined transformations

$$\begin{aligned}\delta\varphi^i&=Q^i{}_\alpha[\varphi]\delta\xi^\alpha\ ,\\ \delta\hat{\phi}^i&=Q^i{}_{\alpha,j}\hat{\phi}^j\delta\xi^\alpha\ .\end{aligned} \tag{36}$$

The invariance (35) holds for every diagram individually in Fig. 1.[1,5] Finally (29) and (34) give

$$\frac{\partial\hat{\Gamma}}{\partial\varphi^i}=\frac{\partial\hat{\Gamma}}{\partial\hat{\phi}^j}\left[\delta^j{}_i+\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle\hat{P}^\alpha{}_i[\varphi]-\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\alpha{}_{k,i}\right]. \tag{37}$$

III. THE GAUGE-INVARIANT EFFECTIVE ACTIO

A. Theory of the GIEA

We begin our derivation of the GIEA by first defining a new functional $\tilde{W}[\varphi,J]$ as

$$\tilde{W}[\varphi,J]\equiv\hat{W}[\varphi,J]+J_i\varphi^i\ . \tag{38}$$

Using Eq. (6) and letting $\underline{A}^i=\varphi^i+\underline{\phi}^i$ we have

$$e^{i\tilde{W}[\varphi,J]}=N'\int d\underline{A}\,\exp\{i(S[\underline{A}]+\tfrac{1}{2}(\hat{P}^\alpha{}_i[\varphi](\underline{A}^i-\varphi^i)-\xi^\alpha)(\hat{P}_{\alpha j}[\varphi](\underline{A}^j-\varphi^j)-\xi_\alpha)+J_i\underline{A}^i)\}\ \det\hat{P}^\alpha{}_i[\varphi]Q^i{}_\beta[\underline{A}]\ . \tag{39}$$

We see that $\tilde{W}$ is in fact equivalent to the W introduced in Eq. (1) where, however, we have the unusual gauge condition[7]

$$P^\alpha[\underline{A}]=\hat{P}^\alpha{}_i[\varphi]\underline{A}^i-(\hat{P}^\alpha{}_i[\varphi]\varphi^i+\xi^\alpha)=0\ . \tag{40}$$

It is important to note that

$$\det\frac{\delta P^\alpha[\underline{A}]}{\delta\underline{A}^i}Q^i{}_\beta[\underline{A}]$$

is correctly given by $\det\hat{P}^\alpha{}_i[\varphi]Q^i{}_\beta[\underline{A}]$ in (39). From (38) we also have

$$\tilde{A}^i\equiv\frac{\partial\tilde{W}}{\partial J_i}=\varphi^i+\hat{\phi}^i \tag{41}$$

and Eq. (30) gives the following invariance for $\tilde{W}$:

$$\begin{aligned}\frac{\partial\tilde{W}}{\partial\varphi^i}Q^i{}_\alpha[\varphi]&=\frac{\partial\hat{W}}{\partial\varphi^i}Q^i{}_\alpha[\varphi]+J_iQ^i{}_\alpha[\varphi]\\&=J_jQ^j{}_{\alpha,i}\hat{\phi}^i+J_jQ^j{}_\alpha[\varphi]\\&=J_jQ^j{}_\alpha[\underline{A}]\ .\end{aligned} \tag{42}$$

We can now define another effective action $\tilde{\Gamma}[\varphi,\tilde{A}]$ via

$$\begin{aligned}\hat{\Gamma}[\varphi,\hat{\phi}]&=\hat{W}[\varphi,J]-J_i\hat{\phi}^i+\text{constant}\\&=\tilde{W}[\varphi,J]-J_i\varphi^i-J_i\hat{\phi}^i+\text{constant}\\&=\tilde{W}[\varphi,J]-J_i\tilde{A}^i+\text{constant}\\&\equiv\tilde{\Gamma}[\varphi,\tilde{A}]\end{aligned} \tag{43}$$

and $\tilde{\Gamma}$ satisfies

$$\frac{\partial\tilde{\Gamma}[\varphi,\tilde{A}]}{\partial\tilde{A}^i}=-J_i \tag{44}$$

and analogously to (34)

$$\left.\frac{\partial\tilde{\Gamma}}{\partial\varphi^i}\right|_{\tilde{A}}=\left.\frac{\partial\tilde{W}}{\partial\varphi^i}\right|_J\ . \tag{45}$$

Using (45) and (42) we find that $\tilde{\Gamma}$ satisfies the invariance relation

$$\frac{\partial\tilde{\Gamma}}{\partial\varphi^i}Q^i{}_\alpha[\varphi]+\frac{\partial\tilde{\Gamma}}{\partial\tilde{A}^i}Q^i{}_\alpha[\tilde{A}]=0\ . \tag{46}$$

Suppose we now choose to constrain either $J=J[\varphi]$ or $\varphi=\varphi[J]$ in such a way that $\hat{\phi}^i=0$ or equivalently $\tilde{A}^i=\varphi^i$. Since $\tilde{\Gamma}[\varphi,\tilde{A}]$ has free variables φ and $\tilde{A}$ we can vary these independently, choosing $\tilde{A}=\varphi$ and fixing φ via $\partial\tilde{\Gamma}/\partial\tilde{A}^i|_{\tilde{A}=\varphi}=-J_i$. We eventually set $J=0$. This is the same as defining

$$\tilde{\Gamma}[\varphi,\tilde{A}[J[\varphi],\varphi]]=\tilde{W}[\varphi,J[\varphi]]-J_i[\varphi]\tilde{A}^i[\varphi,J[\varphi]]$$

which gives

$$\begin{aligned}\frac{\delta\tilde{\Gamma}}{\delta\varphi^i}&=\frac{\partial\tilde{\Gamma}}{\partial\varphi^i}+\frac{\partial\tilde{\Gamma}}{\partial\tilde{A}^j}\frac{\partial\tilde{A}^j}{\partial\varphi^i}\\&=\frac{\partial\tilde{W}}{\partial\varphi^i}+\frac{\partial\tilde{W}}{\partial J_j}\frac{\partial J_j}{\partial\varphi^i}-\frac{\partial J_j}{\partial\varphi^i}\tilde{A}^j-J_j\frac{\partial\tilde{A}^j}{\partial\varphi^i}\ .\end{aligned}$$

This implies that

$$\frac{\partial\tilde{\Gamma}}{\partial\tilde{A}^j}\frac{\partial\tilde{A}^j}{\partial\varphi^i}=-J_j\frac{\partial\tilde{A}^j}{\partial\varphi^i}\ .$$

However, since $\partial\tilde{A}/\partial\varphi$ is singular, which follows from (29), we impose the condition that $\partial\tilde{\Gamma}/\partial\tilde{A}^j=-J_j$ to determine $\varphi[J]$.[15] It is clear that with this choice of φ or J that

$$\frac{\delta\tilde{\Gamma}[\varphi,\varphi]}{\delta\varphi^i}=\left[\frac{\partial\tilde{\Gamma}[\varphi,\tilde{A}]}{\partial\varphi^i}+\frac{\partial\tilde{\Gamma}[\varphi,\tilde{A}]}{\partial\tilde{A}^i}\right]_{\tilde{A}=\varphi} \tag{47}$$

and therefore from (46) that $\tilde{\Gamma}[\varphi,\varphi]$ is manifestly gauge invariant:

$$\frac{\delta\tilde{\Gamma}[\varphi,\varphi]}{\delta\varphi^i}Q^i{}_\alpha[\varphi]\equiv\Gamma_{,i}[\varphi]Q^i{}_\alpha[\varphi]=0\ . \tag{48}$$

The effective action $\Gamma[\varphi]\equiv\tilde{\Gamma}[\varphi,\varphi]$ is our GIEA, and we can easily derive its graphical loop representation and Feynman rules from Fig. 1. From the procedure outlined above, we set $\hat{\phi}=0$ and take $P^\alpha{}_i=\hat{P}^\alpha{}_i[\varphi]$ which gives the new bare propagators $\tilde{G}^{ij}=\hat{G}^{ij}[\varphi,0]$ where $\tilde{F}_{ij}=S_{,ij}[\varphi]+\hat{P}^\alpha{}_i[\varphi]\hat{P}_{\alpha j}[\varphi]$ and $\tilde{\mathfrak{G}}^\alpha{}_\beta=\hat{\mathfrak{G}}^\alpha{}_\beta[\varphi,0]$ where $\tilde{\mathfrak{F}}^\alpha{}_\beta=\hat{P}^\alpha{}_i[\varphi]Q^i{}_\beta[\varphi]$. The new vertices are $S_{,i_1\cdots i_n}[\varphi]$ and $\tilde{V}^\alpha{}_{\beta i}=\hat{P}^\alpha{}_j[\varphi]Q^j{}_{\beta,i}$. It is extremely important to realize that functional derivatives of $\Gamma[\varphi]$ introduce new vertices which are not present in Fig. 1 with $\hat{\phi}=0$ and $P=\hat{P}[\varphi]$, and we must therefore distinguish between internal and external lines. Internal lines which do not join up to external lines have the propagators and vertices the same as before, whereas external lines meet internal lines in the vertices: (1) $\tilde{F}_{jk,i_1\cdots i_n}[\varphi]$, where n external lines meet two internal solid lines; (2) $\tilde{\mathfrak{F}}^\alpha{}_{\beta,i_1\cdots i_n}[\varphi]$, where n external lines meet two ghost lines; (3) $\tilde{V}^\alpha{}_{\beta j,i_1\cdots i_n}[\varphi]$, where n external lines meet two ghost lines and one solid line; and (4) $S_{,j_1\cdots j_n i_1\cdots i_m}$, where n external lines meet $m\geq 3$ internal lines. Explicit Feynman rules for pure Yang-Mills theory with $\hat{P}[\varphi]=Q[\varphi]$ and $\varphi=0$ are given in Refs. 7 and 8.

The GIEA $\Gamma[\varphi]$ satisfies a different field equation from the usual one, Eq. (14), which is easily derived. Equations (48), (45), (44), and (29) give

$$\begin{aligned}\Gamma_{,i}[\varphi]&=\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}=\frac{\partial\tilde{W}}{\partial\varphi^i}\bigg|_{J=J[\varphi]}+\frac{\partial\tilde{\Gamma}}{\partial\tilde{A}^i}\bigg|_{J=J[\varphi]}\\&=\frac{\partial\hat{W}}{\partial\varphi^i}\bigg|_{\substack{J\ \text{fixed}\\ \hat{\phi}=0}}-J_i[\varphi]+J_i[\varphi]\\&=-J_j[\varphi](\delta^j{}_i+\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle\hat{P}^\alpha{}_i[\varphi]-\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\alpha{}_{k,i}[\varphi])_{\hat{\phi}=0}\ ,\end{aligned} \tag{49}$$

where in the last line we evaluate the term in parentheses using (13) and set $\hat{\phi}=0$. We must next determine whether solutions to (49) are physical solutions and understand how they relate to solutions of the usual field equations (44), (18), or (14). Physical solutions to the field equations are determined by the condition $J=0$. We have, however, placed a condition on J. The source $J=J[\varphi]$ is required to be that functional of the background field which gives $\hat{\phi}=0$ or $\tilde{A}=\varphi$. The condition $J_i[\varphi]=0$ is then a condition on the background field. If $J_i[\varphi]=0$, solutions to (44), (18), or (14) are the usual ones and we see from (49) that this also implies that $\Gamma_{,i}=0$. Hence all solutions to the usual field equations are solutions to the GIEA field equation. Conversely, what if φ is such that $\Gamma_{,i}[\varphi]=0$—does this imply $J_i[\varphi]=0$? The answer is no as can easily be shown.[8] Suppose that $J[\varphi]$ can be written in the form

$$J_i[\varphi]=\mathscr{J}_i[\varphi]+j_\alpha\hat{P}^\alpha{}_i[\varphi] \tag{50}$$

then from (49)

$$\begin{aligned}\Gamma_{,i}[\varphi]=&-\mathscr{J}_j[\varphi](\delta^j{}_i+\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\rangle\hat{P}^\gamma{}_i[\varphi]-\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\gamma{}_{k,i}[\varphi])_{\hat{\phi}=0}\\&-j_\alpha(\hat{P}^\alpha{}_i[\varphi]+\hat{P}^\alpha{}_j[\varphi]\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\rangle\hat{P}^\gamma{}_i[\varphi]\\&\qquad-\hat{P}^\alpha{}_j[\varphi]\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\gamma{}_{k,i}[\varphi])_{\hat{\phi}=0}\ .\end{aligned} \tag{51}$$

The term multiplying j_α is zero as we now show. Note that from (15)

$$\hat{P}^\alpha{}_j[\varphi]\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\rangle = \langle \hat{\mathfrak{F}}^\alpha{}_\beta[\varphi,\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\rangle = -\delta^\alpha\gamma$$

and

$$\hat{P}^\alpha{}_j[\varphi]\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\gamma[\varphi,\underline{\phi}]\underline{\phi}^k\rangle = -\delta^\alpha{}_\gamma\langle\underline{\phi}^k\rangle = -\delta^\alpha{}_\gamma\hat{\phi}^k .$$

Therefore the term multiplying j_α can be rewritten as

$$-j_\alpha(\hat{P}^\alpha{}_i[\varphi]-\hat{P}^\alpha{}_i[\varphi]+\hat{\phi}^k\hat{P}^\alpha{}_{k,i}[\varphi])_{\hat{\phi}=0}=0 .$$

We see that for $\mathscr{J}_i[\varphi]=0$ but $j_\alpha\neq 0$ that $\Gamma_{,i}=0$. These solutions, which have not been discussed in the literature, are nevertheless physical. The source term $j_\alpha\hat{P}^\alpha{}_i\underline{\phi}^i$ can be absorbed into the gauge-fixing term in (6) such that the new gauge-fixing condition is

$$P'_\alpha[\underline{\phi}]=(\hat{P}_{\alpha i}[\varphi]\underline{\phi}^i-\zeta'_\alpha)=0 ,$$

where

$$\zeta'_\alpha=\zeta_\alpha-j_\alpha .$$

This gives

$$e^{i\hat{W}[\varphi,\mathscr{J}+jP,P]}=\exp\left[\hat{W}[\varphi,J,P']-i\zeta_\alpha j^\alpha+\frac{i}{2}j_\alpha j^\alpha\right] ,$$

however, in chronological averages the factors of $\zeta j+\frac{1}{2}j^2$ cancel between numerator and denominator and the averages therefore remain unaffected when the source vanishes.[16] The only change is that for $\mathscr{J}_i=0$ the gauge condition on the field is changed to

$$\hat{P}^\alpha{}_i[\varphi]\hat{\phi}^i=\zeta'^\alpha .$$

We therefore see that solutions to $\Gamma_{,i}[\varphi]=0$ are solutions to the equivalence class of problems given by $\delta\hat{\Gamma}/\delta\hat{\phi}^i=0$ where the gauge condition on the solution $\hat{\phi}_{J=0}$ is $\hat{P}_{\alpha j}[\varphi]\hat{\phi}^j=\zeta'_\alpha$. It is of course very reasonable that the GIEA field equations do not place any gauge condition on $\hat{\phi}$ since Γ does not depend on $\hat{\phi}$. Note also that since Γ is gauge invariant there is no gauge condition on the background field and hence we are free to choose any condition which is convenient.[5,6,8]

Although it is possible that there are other solutions to $\Gamma_{,i}=0$ which are not of the form (50) with $\mathscr{J}=0$ this is unlikely (none occur in perturbation theory). In particular the spurious solutions found by Boulware[6] will not occur. These come about if, when $\hat{\phi}=0$ in (23) we, at the same time, were to have $\hat{P}_{\alpha i}[\varphi]\hat{\phi}^i=\zeta_\alpha$ for $J\neq 0$, for then one can show that $J_i\langle Q^i{}_\alpha[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\alpha{}_\beta[\varphi,\underline{\phi}]\rangle=0$ gives rise to unphysical solutions. If we remember that $\hat{P}^\alpha{}_i[\varphi]$ will, in general, involve space-time derivatives and that ζ_α is arbitrary, it is clear that we can adjust ζ_α so as to avoid this possibility.[17]

B. Equivalence with other methods

The method originally suggested by 't Hooft[4] was to choose a $J_i=\bar{J}_i[\varphi]$ in such a way that $\delta\hat{W}[\varphi,\bar{J}[\varphi]]/\delta\varphi^i=-\bar{J}_i[\varphi]$ and then to solve for the φ such that $\bar{J}[\varphi]=0$. $\hat{W}[\varphi,\bar{J}[\varphi]]$ can be easily shown to be gauge invariant under gauge transformations of φ, using (30), if $\bar{J}[\varphi]$ transforms as in (31). From (29) and (14) we have the explicit equation

$$\begin{aligned}\frac{\delta\hat{W}[\varphi,\bar{J}[\varphi]]}{\delta\varphi^i} &= -\bar{J}_j\{\delta^j{}_i+\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\rangle\hat{P}^\alpha{}_i[\varphi] \\ &\quad -\langle Q^j{}_\beta[\varphi+\underline{\phi}]\hat{\mathfrak{G}}^\beta{}_\alpha[\varphi,\underline{\phi}]\underline{\phi}^k\rangle\hat{P}^\alpha{}_{k,i}\}+\hat{\phi}^j\frac{\partial\bar{J}_j[\varphi]}{\partial\varphi^i} \\ &= -\bar{J}_i[\varphi]=-\langle S_{,i}[\varphi+\underline{\phi}]\rangle+i\langle\hat{\mathfrak{G}}^\alpha{}_\beta[\varphi,\underline{\phi}]\rangle\hat{V}^\beta{}_{\alpha i}+\hat{P}^\alpha{}_i\hat{P}_{\alpha j}\hat{\phi}^j-\zeta_\alpha\hat{P}^\alpha{}_i .\end{aligned} \tag{52}$$

't Hooft proposes that this equation be solved for $\bar{J}$ iteratively via perturbation theory in which case it is clear that $\hat{\phi}=0$ is a solution. Note that since we have not imposed the condition (52) on the $J[\varphi]$ used earlier that $\bar{J}[\varphi]$ is not necessarily the same as $J[\varphi]$. From our earlier results we know that when $\hat{\phi}=0$, Eq. (52) is exactly the field equation for the GIEA equation (49) and hence for $\hat{\phi}=0$, 't Hooft's proposal is equivalent to that outlined earlier. There is really no need to actually compute $\bar{J}[\varphi]$, as 't Hooft suggests, and our procedure is simpler.

DeWitt's[5,8] procedure for computing a GIEA is as follows. If we look at Fig. 1 and drop the term $\frac{1}{2}\hat{P}\hat{\phi}\hat{P}\hat{\phi}$ and replace every $\hat{P}[\varphi]$ by $\hat{P}[\varphi+\hat{\phi}]$ we see that the resulting expansion is a functional of φ and $\hat{\phi}$ in the combination $\varphi+\hat{\phi}\equiv\bar{\varphi}$ only. DeWitt then defines

$$\Gamma[\bar{\varphi}] \equiv (\hat{\Gamma}[\varphi,\hat{\phi}] - \tfrac{1}{2}\hat{P}\hat{\phi}\hat{P}\hat{\phi})_{\rightarrow} = S[\bar{\varphi}] + \Sigma[\bar{\varphi}] \,,$$

where "$\rightarrow$" indicates the aforementioned replacement of $\hat{P}[\varphi] \rightarrow \hat{P}[\varphi+\hat{\phi}]$. He then shows that $\Sigma[\bar{\varphi}]$ is a gauge-invariant functional of $\bar{\varphi}$: $\Sigma_{,i}[\bar{\varphi}]Q^{i}{}_{\alpha}[\bar{\varphi}]=0$. Since $S[\bar{\varphi}]$ is also gauge invariant this implies that $\Gamma[\bar{\varphi}]$ is a GIEA. It is clear by comparing this expansion for $\Gamma[\bar{\varphi}]$ to that for $\Gamma[\varphi]$ that $\Gamma[\bar{\varphi}]$ is identical to $\Gamma[\varphi]$ if we replace $\bar{\varphi} \rightarrow \varphi$ and that DeWitt's procedure yields the same GIEA.

The method for constructing the GIEA used in this paper can be considered a synthesis of those used by Abbott[7] and Boulware.[6] The suggestion of viewing the GIEA as the usual effective action with an unusual gauge condition is due to Abbott. Boulware defines a GIEA by $\Gamma[\varphi]=\tilde{W}[\varphi,J[\varphi]]$ where $J[\varphi]$ is chosen such that $\hat{\phi}=0$ which is from (43) equivalent to our procedure for constructing $\Gamma[\varphi]$. It is therefore clear that the construction of $\Gamma[\varphi]$ presented here can be considered the prototypical GIEA.

IV. RENORMALIZATION

Even though, through the introduction of the GIEA, we have added new bare vertex functions to the theory and conceivably made calculations more difficult, there is, in actuality, a tremendous simplification in the calculation of renormalization constants. The simplification arises because all of the full radiatively corrected vertex functions $\Gamma_{,i_1 \cdots i_n}$ are now related through the simple Ward identities obtained by differentiating Eq. (48):

$$\begin{aligned}
&\Gamma_{,ij}Q^{j}{}_{\alpha} = -\Gamma_{,j}Q^{j}{}_{\alpha,i} \,,\\
&\Gamma_{,ijk}Q^{k}{}_{\alpha} = -\Gamma_{,kj}Q^{k}{}_{\alpha,i} - \Gamma_{,ki}Q^{k}{}_{\alpha,j} \,, \qquad (53)\\
&\Gamma_{,ijkl}Q^{l}{}_{\alpha} = -\Gamma_{,ljk}Q^{l}{}_{\alpha,i} - \Gamma_{,ilk}Q^{l}{}_{\alpha,j} - \Gamma_{,ijl}Q^{l}{}_{\alpha,k}, \text{ etc. },
\end{aligned}$$

instead of the more complicated Ward-Takahashi-Slavnov-Taylor identities. This implies that for renormalizable theories only a small number of adjustable constants are present and that only a small number of counterterms are needed. All counterterms, except those involved in gauge fixing, will be gauge invariant and must be constructed from the few, local, integral invariants of the correct dimension.

There is, however, a subtlety in the renormalization theory based on $\Gamma[\varphi]$ in that subdiagrams of the full vertex functions $\Gamma_{,i_1 \cdots i_n}$ at a given loop order are not just insertions of lower-order vertices and propagators. Instead, the free lines of these subdiagrams are internal lines which meet internal vertices, and not external lines which meet external vertices. In other words, these renormalization parts cannot be derived from the GIEA by functional differentiation.[3] One cannot, therefore, use the usual iterative proof of renormalizability and must, instead, proceed in the following manner. We first claim that all divergences of $\tilde{W}[\varphi,J]$ can be made finite by subtracting a (possibly infinite) set of purely local Hermitian counterterms. This necessarily implies that $\tilde{\Gamma}[\varphi,\tilde{A}]$ is also made finite and, in particular, that the renormalized GIEA $\Gamma[\varphi]$ is finite. Since $\Gamma[\varphi]$ is also manifestly gauge invariant this will imply, as mentioned earlier, that we need only the few local integral invariants of the correct dimension plus a possible gauge-fixing renormalization. Since there exists only a finite number of such counterterms for Yang-Mills theory we will have then proven renormalizability.

To prove that $\tilde{W}$ can be made finite, we briefly sketch the renormalization theory of Caswell and Kennedy[11] and apply their results to our case (the reader is referred to this paper for further details). The factor of $\det \hat{\mathfrak{F}}^{\alpha}{}_{\beta}[\varphi,\underline{A}]$ is first rewritten as

$$\begin{aligned}
\det \hat{\mathfrak{F}}^{\alpha}{}_{\beta}[\varphi,\underline{A}] &= \exp\{\mathrm{Tr}\ln \hat{\mathfrak{F}}^{\alpha}{}_{\beta}[\varphi,\underline{A}]\} = \exp\{\mathrm{Tr}\ln(\hat{\mathfrak{F}}^{\alpha}{}_{\beta}[\varphi,0] + \hat{V}^{\alpha}{}_{\beta i}[\varphi]\underline{A}^{i})\}\\
&= \exp\{\mathrm{Tr}\ln \hat{\mathfrak{F}}^{\alpha}{}_{\beta}[\varphi,0] - \hat{\mathfrak{G}}^{\alpha}{}_{\beta}[\varphi,0]\hat{V}^{\beta}{}_{\alpha j}[\varphi]\underline{A}^{j} - \tfrac{1}{2}\hat{\mathfrak{G}}^{\alpha}{}_{\beta}[\varphi,0]\hat{V}^{\beta}{}_{\delta j}\underline{A}^{j}\hat{\mathfrak{G}}^{\delta}{}_{\eta}[\varphi,0]\hat{V}^{\eta}{}_{\alpha k}\underline{A}^{k} - \cdots\} \,. \qquad (54)
\end{aligned}$$

The integrand of Eq. (1) with $P^{\alpha}[\underline{A}]$ chosen as in (40) is now a pure exponential with argument a polynomial in $\underline{A}$. If we denote this argument by $\bar{S}[\varphi,\underline{A}]$ then

$$\begin{aligned}
\bar{S}[\varphi,\underline{A}] \equiv S[\underline{A}] &+ \tfrac{1}{2}(\hat{P}^{\alpha}{}_{i}[\varphi]\underline{A}^{i} - \zeta^{\alpha})(\hat{P}_{\alpha j}[\varphi]\underline{A}^{j} - \zeta_{\alpha}) - \hat{\mathfrak{G}}^{\alpha}{}_{\beta}[\varphi,0]\hat{V}^{\beta}{}_{\alpha j}\underline{A}^{j} - \tfrac{1}{2}\hat{\mathfrak{G}}^{\alpha}{}_{\delta}[\varphi,0]\hat{V}^{\delta}{}_{\beta j}\underline{A}^{j}\hat{\mathfrak{G}}^{\beta}{}_{\eta}[\varphi,0]\hat{V}^{\eta}{}_{\alpha k}\underline{A}^{k}\\
&- \cdots \qquad (55)
\end{aligned}$$

and

$$e^{i\widetilde{W}}=N\,\det\widetilde{F}^{\alpha}{}_{\beta}[\varphi,0]\int d\underline{A}\,e^{i(\bar{S}[\varphi,\underline{A}]+J_i\underline{A}^i)}\ . \tag{56}$$

Next we use the usual trick of splitting $\bar{S}$ into a quadratic part $\bar{S}_0$ and an interaction part $\bar{S}_I$ to obtain

$$\widetilde{W}[\varphi,J]=\left[\sum_{n=0}^{\infty}\frac{(i\bar{S}_I[\varphi,\delta/\delta J])^n}{n!}\right]\widetilde{W}_0[\varphi,J]\ ,$$

$$\widetilde{W}_0[\varphi,J]\equiv\exp\left[\frac{i}{2}J_i\hat{G}^{ij}[\varphi,0]J_j\right]\ . \tag{57}$$

Caswell and Kennedy introduce an operation R which acts on dimensionally regularized graphs to render them finite. They show that the action of R on $\widetilde{W}[\varphi,J]$, where we consider $\bar{S}$ to be the renormalized action, is

$$\mathrm{R}\widetilde{W}[\varphi,J]=\exp[\Lambda(e^{i\bar{S}_I[\varphi,\delta/\delta J]}-1)]\widetilde{W}_0[\varphi,J]\ , \tag{58}$$

where Λ is an operator which acts on generalized vertices in a particular way which we need not go into here. We now define the interaction part of the bar action to be $\Lambda[1-\exp(i\bar{S}_I)]$. This gives an explicit expression for the bare Lagrangian because $\mathrm{R}\widetilde{W}[\varphi,J]$ generates physical (subtracted) graphs and the counterterm we compute is simply $\bar{S}_{I\,\mathrm{bare}}-\bar{S}_I$, which is composed of purely Hermitian polynomials in $\underline{A}$. This proves that the resulting bare action gives finite results and that $\widetilde{\Gamma}[\varphi,\widetilde{A}]$ calculated using this bare action is finite. In particular $\Gamma_{\mathrm{bare}}[\varphi]=\widetilde{\Gamma}_{\mathrm{bare}}[\varphi,\widetilde{A}]$ is finite. This does not however prove that $\Gamma[\varphi]$ is a renormalizable theory—we must prove that there are only a finite number of counterterms.

Since $\Gamma[\varphi]$ is gauge invariant so must all divergences be gauge invariant.[8] In fact this must hold order by order in a loop expansion and also each term in the dimensional-regularization Laurent series in $\epsilon^{-n}\equiv(N-4)^{-n}$ must be separately gauge invariant. Since the divergences of $\Gamma[\varphi]$ must result from purely local Hermitian counterterms as just discussed, we need only enumerate those which have the correct dimension. For pure Yang-Mills theory the only term with the correct dimension is a multiple of the action $S[\varphi]$ and therefore the divergent part of $\Gamma[\varphi]$ is given by $\Gamma_{\mathrm{DIV}}=\delta ZS[\varphi]$. This obviously comes from a term in the bare action of the form $(1+\delta Z)S[\underline{A}]\equiv ZS[\underline{A}]$. We further note that Z does not depend on φ, for if it did, the resulting divergence could not be a multiple of the action. An easy way to calculate Z is to first calculate $\Gamma_{,ij}$ and then evaluate it for $\varphi=0$. Since $\varphi=0$ is a solution to the field equations (i.e., the vacuum), Eq. (53) gives $\Gamma_{,ij}[0]Q^j{}_\alpha[0]=0$. In other words, the divergent part of $\Gamma_{,ij}[0]$ will also be gauge invariant and must, in fact, give $\delta ZS_{,ij}[0]$. There is, however, one subtlety which must be understood. Equation (33) is easily shown to be applicable to $\Gamma[\varphi]$.[8] Therefore at a solution to the field equations, $J=0$, we have

$$\delta_P\Gamma[\varphi]=0\ . \tag{59}$$

This does not imply that $\Gamma_{,ij}$ is independent of the gauge-fixing term P—the finite part of $\Gamma_{,ij}$ will depend on P. The actual dependence of $\Gamma_{,ij}$ on P can be found by taking derivatives of Eq. (33) (modified so as to apply to Γ) with respect to φ. The divergent part of Γ is independent of P from (59) and therefore $\delta_P Z=0$. The actual computation of Z is done via the loop expansion in perturbation theory. First the one-loop contribution to $\Gamma_{,ij}$ is calculated and then the two-loop, etc. The one-loop divergences are single poles in ϵ^{-1} while, in general, two-loop divergences give ϵ^{-1} and ϵ^{-2} terms. Also note that sets of diagrams will be separately gauge invariant because of the diagram-by-diagram invariance of Fig. 1 mentioned following Eq. (36). This provides a useful check on calculations. What one finds if a gauge parameter α is included in the definition of P, as is usually done, is that a naive computation of $\Gamma_{,ij}$ gives a (gauge-invariant) divergent term proportional to a function of powers of α.[10] In other words a P-dependent divergence. What we have not taken into account is that the original renormalization of $\widetilde{W}$, Eq. (58), also adds counterterms to the P-dependent part of the action $\bar{S}$—the gauge-fixing term is renormalized. These P-dependent counterterms give second-order contributions to the one-loop diagrams and these counterterm divergences from the one-loop diagrams *exactly* cancel the P-dependent divergences at two loops thereby making Z P-independent as it must be.[7–10] This is clearly evident in the general gauge two-loop calculation of Capper and MacLean.[10] If one is only interested in computing Z, the above procedure is followed and any P-dependent terms are simply ignored.

There are several methods for calculating the gauge-fixing renormalization. Abbott[7] and Capper and MacLean[10] calculate the counterterm by demanding that the one-loop contribution of the quantum field self-energy be finite. By this we mean the subdivergences of Γ which have two internal solid lines connected to a one-loop self-energy insertion. Since we cannot generate these subdiagrams via functional differentiation of Γ (we can only generate terms with external field lines) a method more in the spirit of the GIEA is the following.[8] We add counterterms to the renormalized action and rescale the fields according to

$$\underline{A}^B \equiv Z^{1/2}\underline{A}, \quad \varphi^B \equiv Z^{1/2}\varphi ,$$
$$g_B \equiv Z^{-1/2} g \mu^{-\epsilon}, \quad \hat{P}^B \equiv Z_\alpha{}^{1/2} Z^{-1/2} \hat{P} , \tag{60}$$
$$J^B \equiv Z^{-1/2} J ,$$

and define

$$\Gamma^R[\varphi,g] \equiv \Gamma_{\text{bare}}[\varphi^B, g_B] \tag{61}$$

which is now a finite functional of finite arguments. The Z_α renormalization comes from the gauge-fixing counterterm in (58). Since $\hat{\mathfrak{G}}^{R\alpha}{}_\beta[\varphi,0] = \hat{P}^\alpha{}_i[\varphi] Q^i{}_\beta[\varphi]$ is a finite nonsingular operator we can determine Z_α from the condition that the (matrix) functional derivative of $\Gamma^R[\varphi,g]$ with respect to $\mathfrak{G}^R$ be finite: $\partial\Gamma^R / \partial\hat{\mathfrak{G}}^{R\alpha}{}_\beta = \text{finite}$. This can easily be computed from the expansion for Γ given in Fig. 1 with $\hat{\phi}=0$ and $P=\hat{P}[\varphi]$ and generates the expansion of the full ghost propagator $\hat{G}^\alpha{}_\beta[\varphi] \equiv \langle \hat{\mathfrak{G}}^\alpha{}_\beta[\varphi,\underline{\phi}] \rangle_{\hat{\phi}=0}$.

The renormalization program using the GIEA is therefore seen to be far simpler than that of the usual theory. The proof of the renormalizability of Γ is perhaps less straightforward than that for $\hat{\Gamma}$; however, it is no more difficult, especially using the method of Caswell and Kennedy. For pure Yang-Mills theory we need only calculate one set of diagrams, $\Gamma_{,ij}$ to find the physical renormalization Z. As was shown earlier, the calculation of Z_α is unnecessary unless one wants it for some specific reason. In the usual theory four counterterms (really only three after using Ward-Takahashi-Slanvov-Taylor identity) had to be calculated which involved computing the field self-energy, ghost self-energy, field three-point vertex, and ghost-ghost-field vertex diagrams.[18,19]

V. S MATRIX

As has been shown by DeWitt,[5] the GIEA $\Gamma[\varphi]$ may be used to generate S-matrix elements in a manner analogous to that which uses $\tilde{\Gamma}[\varphi,\tilde{A}]$.[1,2] His construction however applied only to the case where φ is taken to be the classical vacuum φ_0. Using the results of Sec. III we can extend his proof so as to apply to any arbitrary field[12] which solves the GIEA field equation $\Gamma_{,i}[\varphi]=0$. This is important in that a field which is not the vacuum field φ_0 may be considered a relative vacua and therefore is a coherent state with respect to the absolute vacuum. The S matrix then describes transitions between coherent superpositions of particle states in the Fock space based on the absolute vacuum.

Suppose $\tilde{A}_0$ is a solution to $\partial\tilde{\Gamma}[\varphi,\tilde{A}_0]/\partial\tilde{A}^i = 0$ where there is as yet no condition on φ. It is then trivial to show that $\tilde{A}_0 + \Delta\tilde{A}$ satisfies

$$\frac{\partial\Gamma[\varphi,\tilde{A}_0+\Delta\tilde{A}]}{\partial\tilde{A}^i} = 0 , \tag{62}$$

where $\Delta\tilde{A}^i$ is defined by

$$\begin{aligned}\Delta\tilde{A}^i &\equiv \Delta\tilde{A}^i_0 + \tilde{\Gamma}^{ij}[\varphi,\tilde{A}_0]\left\{\frac{\partial\tilde{\Gamma}[\varphi,\tilde{A}_0+\Delta\tilde{A}]}{\partial\tilde{A}^j} - \frac{\partial^2\tilde{\Gamma}[\varphi,\tilde{A}_0]}{\partial\tilde{A}^j\partial\tilde{A}^k}\Delta\tilde{A}^k\right\} \\ &= \Delta\tilde{A}^i_0 + \tilde{\Gamma}^{ij}[\varphi,\tilde{A}_0]\left\{\frac{1}{2}\frac{\partial^3\tilde{\Gamma}[\varphi,\tilde{A}_0]}{\partial\tilde{A}^j\partial\tilde{A}^k\partial\tilde{A}^l}\Delta\tilde{A}^k\Delta\tilde{A}^l + \cdots\right\}\end{aligned} \tag{63}$$

and $\Delta\tilde{A}^j_0$ is an arbitrary solution to

$$\frac{\partial^2\tilde{\Gamma}[\varphi,\tilde{A}_0]}{\partial\tilde{A}^i\partial\tilde{A}^j}\Delta\tilde{A}^j_0 = 0 . \tag{64}$$

We have previously shown in Eq. (49) that if $\varphi=\tilde{A}_0$ and $\partial\tilde{\Gamma}[\varphi,\tilde{A}_0]/\partial\tilde{A}^i=0$ that this means that $J_i[\tilde{A}_0]=0$. Since

$$\frac{\partial\tilde{\Gamma}[\tilde{A}_0,\tilde{A}_0+\Delta\tilde{A}]}{\partial\tilde{A}^i} = -J_i[\tilde{A}_0] = 0$$

we see that both $\tilde{A}_0$ and $\tilde{A}_0+\Delta\tilde{A}$ are solutions to the GIEA field equation $\Gamma_{,i}=0$. We can use a construction similar to that of Eq. (63) to generate other solutions to the GIEA field equation once we have found one solution, i.e., $\tilde{A}_0$. Since both $\tilde{A}_0$ and $\tilde{A}_0+\Delta\tilde{A}$ are solutions to $\Gamma_{,i}=0$ it is easy to show[20,2] that $\Delta\tilde{A}$ is also given by

$$\begin{aligned}\Delta\tilde{A}^i &= \Delta\tilde{A}^i_{00} + G^{ij}[\tilde{A}_0]\{\Gamma_{,j}[\tilde{A}_0+\Delta\tilde{A}] - \Gamma_{,jk}[\tilde{A}_0]\Delta\tilde{A}^k\} \\ &= \Delta\tilde{A}^i_{00} + G^{ij}[\tilde{A}_0]\{\tfrac{1}{2}\Gamma_{,jkl}[\tilde{A}_0]\Delta\tilde{A}^k\Delta\tilde{A}^l + \cdots\} ,\end{aligned} \tag{65}$$

where $\Delta\widetilde{A}^{j}_{00}$ is an arbitrary solution to

$$\Gamma_{,ij}[\widetilde{A}_0]\Delta\widetilde{A}^{j}_{00}=0 \,. \tag{66}$$

Here $G^{ij}[\widetilde{A}_0]$ is the inverse (Green's function) to $F_{ij}[\widetilde{A}_0]$ where

$$F_{ij}[\widetilde{A}_0]\equiv\Gamma_{,ij}[\widetilde{A}_0]+P^{\alpha}{}_{i}P_{\alpha j},\quad F_{ij}G^{ik}=-\delta_i{}^{k}\,, \tag{67}$$

and the $P^{\alpha}{}_{i}$ are arbitrary operators chosen such that F_{ij} is nonsingular and $P^{\alpha}{}_{i}Q^{i}{}_{\beta}[\widetilde{A}_0]$ is nonsingular. Equation (66) does not place any gauge condition on $\Delta\widetilde{A}_{00}$ since $\Gamma_{,ij}Q^{j}{}_{\alpha}=0$ for $J=0$ and $\Delta\widetilde{A}_{00}$ is not necessarily equal to $\Delta\widetilde{A}_0$.

If we now expand $\Gamma_{,i}[\widetilde{A}_0+\Delta\widetilde{A}]=0$ and insert (63) for $\Delta\widetilde{A}$ we have

$$\begin{aligned}\Gamma_{,i}[\widetilde{A}_0+\Delta\widetilde{A}]=0&=\Gamma_{,i}[\widetilde{A}_0]+\Gamma_{,ij}[\widetilde{A}_0]\Delta\widetilde{A}^{j}+\tfrac{1}{2}\Gamma_{,ijk}[\widetilde{A}_0]\Delta\widetilde{A}^{j}\Delta\widetilde{A}^{k}+\cdots\\ &=\Gamma_{,ij}[\widetilde{A}_0]\left\{\Delta\widetilde{A}^{j}_{0}+\widetilde{\Gamma}^{jk}\sum_{n=2}^{\infty}\frac{1}{n!}\widetilde{T}_{ki_1\cdots i_n}[\widetilde{A}_0,\widetilde{A}_0]\Delta\widetilde{A}^{i_1}_{0}\cdots\Delta\widetilde{A}^{i_n}_{0}\right\}\\ &\quad+\tfrac{1}{2}\Gamma_{,ijk}[\widetilde{A}_0]\left\{\Delta\widetilde{A}^{j}_{0}+\widetilde{\Gamma}^{jl}\sum_{n=2}^{\infty}\frac{1}{n!}\widetilde{T}_{li_1\cdots i_n}[\widetilde{A}_0,\widetilde{A}_0]\Delta\widetilde{A}^{i_1}_{0}\cdots\Delta\widetilde{A}^{i_n}_{0}\right\}\\ &\quad\times\left\{\Delta\widetilde{A}^{k}_{0}+\widetilde{\Gamma}^{km}\sum_{r=2}^{\infty}\frac{1}{r!}\widetilde{T}_{mj_1\cdots j_r}[\widetilde{A}_0,\widetilde{A}_0]\Delta\widetilde{A}^{j_1}_{0}\cdots\Delta\widetilde{A}^{j_r}_{0}\right\}+\cdots\,,\end{aligned} \tag{68}$$

where the $\widetilde{T}_n[\widetilde{A}_0,\widetilde{A}_0]$ are the sum of tree diagrams with external lines removed which result from the iteration of (63) expressing $\Delta\widetilde{A}$ in terms of $\Delta\widetilde{A}_0$. Since the $\Delta\widetilde{A}_0$ are arbitrary solutions to (64), Eq. (68) implies the sequence of identities[8]

$$\Gamma_{,ij}[\widetilde{A}_0]\Delta\widetilde{A}^{j}_{0}=0,\quad (\Gamma_{,ij}[\widetilde{A}_0]\widetilde{\Gamma}^{jk}[\widetilde{A}_0,\widetilde{A}_0]\widetilde{T}_{klm}[\widetilde{A}_0,\widetilde{A}_0]+\Gamma_{,ilm}[\widetilde{A}_0])\Delta\widetilde{A}^{l}_{0}\Delta\widetilde{A}^{m}_{0}=0,\ \text{etc.} \tag{69}$$

The first identity in (69) means that $\Delta\widetilde{A}^{j}_{0}$ is equal to (a constant times[21]) $\Delta\widetilde{A}^{j}_{00}$ modulo an arbitrary gauge term $Q^{i}{}_{\alpha}[\widetilde{A}_0]\delta\xi^{\alpha}$. This, of course, corresponds to the fact that the GIEA field equation does not fix the gauge of the solution.

One can show that the structural elements of the S matrix may be obtained by repeatedly differentiating the following expressions with respect to the a 's and a^* 's and then setting these coefficients equal to zero[1,2]:

$$\sum_{n=3}^{\infty}\frac{1}{(n-1)!}\widetilde{T}_{i_1\cdots i_n}[\widetilde{A}_0,\widetilde{A}_0]\Delta\widetilde{A}^{i_1}_{0}\cdots\Delta\widetilde{A}^{i_n}_{0}=(u^{i}_{A}a_A+u^{i}_{A}{}^{*}a_A{}^{*})\vec{S}^{0}{}_{,ij}\Delta\widetilde{A}^{j}_{0} \tag{70}$$

or

$$\sum_{n=3}^{\infty}\frac{1}{(n-1)!}T_{i_1\cdots i_n}[\widetilde{A}_0]\Delta\widetilde{A}^{i_1}_{00}\cdots\Delta\widetilde{A}^{i_n}_{00}=(u^{i}{}_{A}a_A+u^{i}_{A}{}^{*}a^{*}_{A})\vec{S}^{0}{}_{,ij}\Delta\widetilde{A}^{j}_{00}\,, \tag{71}$$

where

$$\Delta\widetilde{A}^{i}_{0}=\widetilde{\Gamma}^{ij}\overleftarrow{S}^{0}{}_{,jk}(u^{k}{}_{A}a_A+u^{k}{}_{A}{}^{*}a_A{}^{*})$$

and

$$\Delta\widetilde{A}^{i}_{00}=G^{ij}\overleftarrow{S}^{0}{}_{,jk}(u^{k}{}_{A}a_A+u^{k}{}_{A}{}^{*}a_A{}^{*})\,.$$

The $u^{i}{}_{A}$ and $u^{i}{}_{A}{}^{*}$ are normalized positive-frequency wave functions which satisfy

$$\vec{S}^{0}{}_{,ij}u^{j}{}_{A}=0\,, \tag{72}$$

where $S^{0}{}_{,ij}$ is $S_{,ij}$ evaluated at the classical vacuum and the arrow denotes the direction in which $S^{0}{}_{,ij}$ acts as a differential operator. The $T_n[\widetilde{A}_0]$ are the sums of tree functions which result from iteration of (65) for $\Delta\widetilde{A}$ in terms of $\Delta\widetilde{A}_{00}$. It is also easy to show[2,8] that (71) is invariant under the gauge transformations

$$\begin{aligned}\delta\Delta\widetilde{A}^{i}_{00}&=Q^{i}{}_{\alpha}[\widetilde{A}_0]\delta\xi^{\alpha}\,,\\ \delta\Delta\widetilde{A}^{i}&=Q^{i}{}_{\alpha}[\widetilde{A}_0+\Delta\widetilde{A}]\delta\xi^{\alpha}\,,\end{aligned} \tag{73}$$

where

$$\delta\xi^{\alpha}\equiv-\hat{\mathfrak{G}}^{\alpha}{}_{\beta}[\widetilde{A}_0,0]Q^{\beta}{}_{j}[\widetilde{A}_0]Q_{\eta}{}^{j}[\widetilde{A}_0+\Delta\widetilde{A}]\delta\xi^{\eta}\,. \tag{74}$$

Since $\Delta\widetilde{A}_0$ equals $\Delta\widetilde{A}_{00}$ modulo a gauge transformation and since (71) is invariant under gauge transformation it follows that both (71) and (70) yield the same S-matrix elements. DeWitt has shown that $\Delta\widetilde{A}_0\approx\Delta\widetilde{A}_{00}$ for the case where $\widetilde{A}_0$ is the vacuum. Here, we have extended his proof of S-matrix equivalence to the general case where $\widetilde{A}_0$ need only be a solution to the field equations. The tree amplitudes derived using (71) will be composed of combinations of G^{ij} and the vertices $\Gamma_{,i_1 \cdots i_n}$, $n \geq 3$. As mentioned earlier in Sec. III, these vertices are not the same as the usual theory; however, the Ward identities (53) provide an extremely simple and useful check in actual calculations.

VI. DISCUSSION AND SUMMARY

In the preceding sections we have shown that the various methods of 't Hooft, DeWitt, Boulware, and Abbott for constructing a GIEA are all equivalent and have discussed in some detail the construction of what may be considered the prototypical GIEA. We did so by first carefully examining the gauge transformation and invariance properties of the usual effective action in Sec. II. When the background-field gauge condition $P=\hat{P}[\varphi]$ is used we showed that $\hat{W}$ and $\hat{\Gamma}$ obey the simple invariance relations (30) and (35). We next defined a new generating functional $\tilde{W} \equiv \hat{W} + J_i \varphi^i$ and showed that the new effective action $\tilde{\Gamma}[\varphi,\tilde{A}]=\hat{\Gamma}[\varphi,\hat{\phi}]$ can be considered to be the usual effective action with an unusual gauge-fixing condition. If the external source J is chosen to be a functional of the background field $\varphi, J=J[\varphi]$, in such a way that $\hat{\phi}=0$ or equivalently we choose $\varphi=\tilde{A}$, we found that the GIEA $\Gamma[\varphi]\equiv\tilde{\Gamma}[\varphi,\varphi]$ is manifestly gauge invariant and that all solutions to the GIEA field equation $\Gamma_{,i}=0$ are physical.

In Sec. IV we showed how to prove renormalizability of the theory based upon $\Gamma[\varphi]$. Since subdiagrams (i.e., subdivergences which give renormalization parts) of Γ are not generated by functional derivatives of Γ we found it necessary to proceed in a somewhat different manner than usual. We first showed, using the renormalization theory of Caswell and Kennedy that $\tilde{\Gamma}[\varphi,\tilde{A}]$ can be made finite by adding a possibly infinite set of purely local Hermitian counterterms to the action. We next argued that since $\Gamma[\varphi]$ is gauge invariant so must all divergences be gauge invariant. For a renormalizable theory there are only a finite number of possible counterterms and for pure Yang-Mills theory there is only a multiple of the action. We then showed that Z must be independent of the gauge condition, i.e., $Z\neq Z(\alpha)$, and discussed how one may compute Z very simply via computation of $\Gamma_{,ij}$. Finally, in Sec. IV we discussed how to compute S-matrix elements using $\Gamma[\varphi]$ and, in particular, proved that the method is valid for all background fields which solve the GIEA field equations thereby proving a conjecture of DeWitt.

It is important to realize that there does not exist any simple method as in Eqs. (10) and (13) for computing the chronological average of an arbitrary operator using the GIEA. This is because the various $\partial^n\hat{\Gamma}/\partial\hat{\phi}^n$ and $\hat{\Gamma}^{ij}$ are not related to the $\Gamma_{,i_1 \cdots i_m}$ in any easily calculable manner as is clear from Eq. (49). In other words $\partial^3\hat{\Gamma}/\partial\hat{\phi}^3 \neq \Gamma_{,3}$ but is instead some complicated combination of the GIEA vertices and propagator.[22] This is not really too much of a disadvantage, however, since one is usually only interested in calculating expectation values of gauge-invariant operators (physical observables). The case of S-matrix elements was given in Sec. V while, for an arbitrary gauge-invariant operator $\mathscr{O}_{\rm GI}$ we should add a source term to the usual action[3]: $X\mathscr{O}_{\rm GI}$. It is then clear that functional derivatives of the new GIEA $\Gamma_{\mathscr{O}_{\rm GI}}[\varphi,X]$ with respect to X will give us the necessary expectation values. Since $\mathscr{O}_{\rm GI}$ is gauge invariant it is easy to show that the proofs of gauge invariance and P independence for $\Gamma_{\mathscr{O}_{\rm GI}}$ still hold. It should prove much easier to use this approach, rather than the usual method with $\hat{\Gamma}$, for calculating the renormalization of gauge-invariant operators, anomalous dimension, etc.[3] The extension of the present approach, if possible, to supersymmetric theories and applications to cosmology and to low-energy effective field theory also deserves further study.

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[12]One can also introduce a gauge index raising and lowering metric $\eta_{\alpha\beta}$; however, we choose $\eta_{\alpha\beta}=\delta_{\alpha\beta}$. It can easily be reintroduced and variations with respect to η also taken to discuss η dependence. See also Ref. 5.

[13]The fields φ and $\hat{\phi}$ must satisfy the asymptotic Feynman boundary conditions implicit in the definition of the functional integral (1).

[14]The conditions $Q^{i}{}_{\alpha,i}=0$ and $c^{\alpha}{}_{\gamma\alpha}=0$ are also needed in the derivation. See B. S. DeWitt, in *Recent Developments in Gravitation, Cargese 1978,* edited by M. Levy and S. Deser (Plenum, New York, 1979).

[15]This is, in fact, a *condition,* not an identity as stated in Ref. 7.

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[17]In Ref. 6, the implication seems to be that if $\hat{\phi}=0$ then $P_{\alpha i}\hat{\phi}=0$; however since P is usually a differential operator there is no reason for this to be true.

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[21]$\Delta\widetilde{A}_{00}$ and $\Delta\widetilde{A}_{0}$ will have different renormalization constants due to the different manners in which they are renormalized.

[22]Equation (69) places conditions upon how the various tree functions are related; however, the implications of these constraints is unclear.