Shifts of integration variable within four- and N-dimensional Feynman integrals

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(Received 20 December 1982; revised manuscript received 1 July 1983)

We resolve inconsistencies between integration in four dimensions, where shifts of integration variable may lead to surface terms, and dimensional regularization, where no surface terms accompany such shifts, by showing that surface terms arise only for discrete values of the dimension parameter. General formulas for variable-of-integration shifts within N-dimensional Feynman integrals are presented, and the *VVA* triangle anomaly is interpreted as a manifestation of surface terms occurring in exactly four dimensions.

I. INTRODUCTION

It has long been known that shifts of integration variables within linearly divergent Feynman integrals in four dimensions are accompanied by finite "surface terms"; for example,[1,2]

$$\int \frac{d^4k\ k_\mu}{[(k-p)^2-m^2]^2}-\int \frac{d^4k\,(k+p)_\mu}{[k^2-m^2]^2}=-i\pi^2 p_\mu/2\ . \tag{1.1}$$

It is also well known that naive shifts of integration variable are always permitted within dimensional regularization[3,4]:

$$\int \frac{d^nk\ k_\mu}{[(k-p)^2-m^2]^2}-\int \frac{d^nk\,(k+p)_\mu}{[k^2-m^2]^2}=0\ . \tag{1.2}$$

Equations (1.1) and (1.2) are contradictory, and this contradiction merits further investigation (as opposed to defining or postulating it away) if perturbative field theory is to be truly self-consistent.

The basic point of our paper is that if dimensional regularization is to be a procedure for continuing four-dimensional Feynman integrals to n dimensions, then it should reproduce finite results that occur from standard four-dimensional manipulations when n is taken to be equal to four. In particular, Eq. (1.1) should be reproduced in n dimensions when n is taken to equal four. This criterion is certainly reasonable from a mathematical point of view. We emphasize that the criterion has a physical motivation as well; the triangle anomaly can be made to vanish within dimensional regularization if surface terms are not included, unless *ad hoc* rules for manipulating the matrix γ_5 are constructed.[3] These rules appear to be incompatible with supersymmetry,[5] and alternative rules[6] consistent with supersymmetry (in which $\{\gamma_5,\gamma_\mu\}=0$ for all γ_μ, as in four dimensions) appear to be inconsistent with the Adler-Bardeen theorem, whose perturbation-theory proof in dimensional reduction seems still to require the nonfully anticommuting γ_5 matrix of Ref. 3.[7] Consequently, we believe features of n-dimensional integration need to be reexamined in order to facilitate the development of a regularization procedure fully consistent with supersymmetric theories.

In this paper, we generalize the arguments of Ref. 1 to calculate finite terms accompanying shifts of integration variables in 2ω dimensions. We argue that such terms occur only for discrete values of ω. For example, we find in Sec. II that

$$\int \frac{d^{2\omega}k\ k_\mu}{[(k-p)^2-m^2]^2}-\int \frac{d^{2\omega}k\,(k+p)_\mu}{[k^2-m^2]^2}=-\frac{i\pi^2}{2}p_\mu\delta_{\omega,2} \tag{1.3}$$

for $\omega<\frac{5}{2}$. Hence, Eqs. (1.1) and (1.2) reflect a discontinuity at $\omega=2$. This discontinuity is not merely a curiosity. We demonstrate in Sec. V the explicit connection between shift-of-integration-variable "surface terms" and the *VVA* triangle anomaly, and we argue that the anomaly may be peculiar to four (as opposed to $4+\epsilon$) dimensions if we wish to maintain a sensible and self-consistent Dirac algebra.

In Sec. III, we examine the difference

$$\int \frac{d^{2\omega}k\prod_{j=1}^{2n+1}k_{\mu_j}}{[(k-p)^2-m^2]^r}-\int \frac{d^{2\omega}k\prod_{j=1}^{2n+1}(k+p)_{\mu_j}}{[k^2-m^2]^r}$$

for less-than-quadratically divergent integrals in which neither ω nor r is constrained to be an integer. We find this difference to be zero unless $\omega=r-n$. A general formula [consistent with Eq. (1.3)] for the difference is obtained when this condition is satisfied.

In Sec. IV, we make a similar analysis of the difference

$$\int \frac{d^{2\omega}k\prod_{j=1}^{2n}k_{\mu_j}}{[(k-p)^2-m^2]^r}-\int \frac{d^{2\omega}k\prod_{j=1}^{2n}(k+p)_{\mu_j}}{[k^2-m^2]^r}$$

for less-than-cubically divergent integrals, and obtain a nonzero result only if $\omega=r+1-n$. Once again, a general formula for the difference is obtained when this constraint is satisfied.

In Sec. V, we briefly discuss the significance of these discontinuities in ω. We show that divergences of all three VVA-triangle-graph currents may be expressed as differences of variable-shifted linearly divergent integrals, and we show how the usual chiral anomaly is obtained from finite terms accompanying such a shift when $\omega=2$. We compare this approach to regularization procedures in which the anomaly arises from either having γ_5 commute with (fractionally indexed) γ matrices associated with $2\omega-4$ dimensions,[3] or from abandoning the cyclic trace property of γ matrices.[6] We present a case for retaining conventional γ-matrix properties and for regarding the anomaly (and observed $\pi^0\to\gamma\gamma$ rate) as a manifestation of the discontinuity in Eq. (1.3).

Two lengthy appendices follow Sec. V. In Appendix A, we examine symmetric integration (over k) in 2ω dimensions of numerator factors

$$\prod_{j=1}^{m}(k+pz)_{\mu_j}\ ,$$

where z can be zero, one, or a Feynman parameter. Results of Appendix A are extensively used in Secs. III and IV.

In Appendix B, we "bootstrap" Eq. (1.1) to derive formulas for differences between divergent variable-shifted integrals in (exactly) four dimensions. These results provide a consistency cross check of the general formulas obtained in Secs. II and III. Appendix B is also intended for those readers who would like to pick up where Ref. 1 leaves off without having to fight through the more general arguments of Secs. III and IV, where ω is not constrained to be 2.

II. THE JAUCH-ROHRLICH SURFACE TERM

We begin by considering the difference between

$$I^\mu(\omega,r)\equiv\int d^{2\omega}k\frac{k^\mu}{[(k-p)^2-m^2]^r} \tag{2.1}$$

and

$$J^\mu(\omega,r)\equiv\int d^{2\omega}k\frac{(k+p)^\mu}{[k^2-m^2]^r} \tag{2.2}$$

for three illustrative cases:

Case I: $I^\mu(2,2)-J^\mu(2,2)$

$$=\int d^4k\left[\frac{k^\mu}{[(k-p)^2-m^2]^2}-\frac{(k+p)^\mu}{[k^2-m^2]^2}\right], \tag{2.3a}$$

Case II: $I^\mu(\omega,2)-J^\mu(\omega,2)$

$$=\int d^{2\omega}k\left[\frac{k^\mu}{[(k-p)^2-m^2]^2}-\frac{(k+p)^\mu}{[k^2-m^2]^2}\right], \tag{2.3b}$$

Case III: $I^\mu(\omega,\omega)-J^\mu(\omega,\omega)$

$$=\int d^{2\omega}k\left[\frac{k^\mu}{[(k-p)^2-m^2]^\omega}-\frac{(k+p)^\mu}{[k^2-m^2]^\omega}\right]. \tag{2.3c}$$

Case I corresponds to the case considered by Jauch and Rohrlich[1]; cases II and III continue the dimensionality and the power of the propagator from the $\omega=r=2$ limit of case I. To proceed further we shall need the following relations:

$$a^{-r}-b^{-r}=r\int_0^1 dz\,(b-a)[az+b(1-z)]^{-(r+1)}\ , \tag{2.4}$$

$$\int\frac{d^{2\omega}k\,(k^2)^t}{[k^2+X]^q}=i\frac{\pi^\omega\Gamma(\omega+t)\Gamma(q-\omega-t)}{\Gamma(\omega)\Gamma(q)X^{q-\omega-t}}\ ,\qquad \mathrm{Re}(q-\omega-t)\neq 0\ , \tag{2.5}$$

$$\int d^{2\omega}k\,[f(k)+g(k)]=\int d^{2\omega}k f(k)+\int d^{2\omega}k g(k)\ , \tag{2.6}$$

$$\int d^{2\omega}k f(k,x,y)(x-y)=\begin{cases}0 \text{ if } x=y\\ (x-y)\int d^{2\omega}k f(k,x,y)\ , & \text{if } x\neq y\ ,\end{cases} \tag{2.7}$$

where x and y do not depend on k^μ, and

$$\int d^{2\omega}k\int_a^b dz\, f(k+p,z)=\int d^{2\omega}k\int_a^b dz\, f(k,z) \tag{2.8}$$

for less-than-linearly divergent integrals,[1] i.e., for

$$f(k)\underset{k\to\infty}{\sim}(k^2)^{-r}\text{ when }2\omega-2r<1\ .$$

Equation (2.4) is standard Feynman parametrization. Equation (2.5) follows from the definition of the beta-function; note that Eq. (2.5) is *finite* on both sides and well defined on the right-hand side. If $2\omega\notin Z^+$ (positive integers), then (2.5) is the definition of the left-hand side. It is important that $\mathrm{Re}(q-\omega-t)\neq 0$; otherwise the right-hand side is *not* well defined and (2.5) cannot be employed. Equations (2.6) and (2.7) follow from the linear algebra of integration operations. Equation (2.8) indicates the conditions under which shifts of the integration variable are permitted.[1] Note that in Eq. (2.8) a reversal of k and z integrations is not allowed if the k integral is divergent[1]; such a reversal is permitted only for finite integrals. We also employ the relations

$$\int d^{2\omega}k\; k^\mu k^\nu f(k^2)=\int d^{2\omega}k\frac{k^2 g^{\mu\nu}}{2\omega}f(k^2)\ , \tag{2.9a}$$

$$\int d^{2\omega}k\; k^\mu f(k^2)=0\ . \tag{2.9b}$$

Equation (2.9a) holds for $2\omega\in Z^+$; if $2\omega\notin Z^+$, then (2.9a) defines the left-hand side. Equation (2.96) follows from setting $k_\mu\to -k_\mu$.

We shall find it useful to briefly recapitulate the arguments of Ref. 1 in evaluating case I. Using (2.4) we find that

$$I^\mu(2,2)-J^\mu(2,2)=\int d^4k\int_0^1 dz\left[\frac{-p^\mu}{[k^2-m^2]^2}+\frac{2k^\mu(2p\cdot k-p^2)}{[k^2-2p\cdot kz+p^2z-m^2]^3}\right]. \tag{2.10}$$

Since the second integral is logarithmically divergent, (2.8) may be used (note that k and z integrations are *not* interchanged) in order to obtain

$$I^{\mu}(2,2)-J^{\mu}(2,2) = \int d^4k \int_0^1 dz \left[\frac{-p^{\mu}}{[k^2-m^2]^2} + \frac{4k^{\mu}k^{\nu}p_{\nu}}{[k^2+z(1-z)p^2-m^2]^3} - \frac{2p^{\mu}p^2 z(1-2z)}{[k^2+z(1-z)p^2-m^2]^3} \right] . \quad (2.11)$$

In the last integrand, we may partial integrate over z to find that

$$\int_0^1 \frac{dz\, z(1-2z)}{[k^2+z(1-z)p^2-m^2]^3} = \frac{-1/(2p^2)}{[k^2-m^2]^2} + \frac{1}{2p^2}\int_0^1 \frac{dz}{[k^2+z(1-z)p^2-m^2]^2} . \quad (2.12)$$

Substitution into (2.11) yields

$$I^{\mu}(2,2)-J^{\mu}(2,2) = \int d^4k \int_0^1 dz \left[\frac{4k^{\mu}k^{\nu}p_{\nu}}{[k^2+z(1-z)p^2-m^2]^3} - \frac{[k^2+z(1-z)p^2-m^2]p^{\mu}}{[k^2+z(1-z)p^2-m^2]^3} \right] . \quad (2.13)$$

Using (2.9a), we find $k^{\mu}k^{\nu}\rightarrow g^{\mu\nu}k^2/4$, explicitly canceling the factor k^2p^{μ} in the second integrand of (2.13) [Eq. (2.7)]. The remaining terms lead to finite integrals, so k and z integrations may be interchanged; using (2.5) we obtain

$$I^{\mu}(2,2)-J^{\mu}(2,2) = \int_0^1 dz \int d^4k \frac{-p^{\mu}[p^2z(1-z)-m^2]}{[k^2+z(1-z)p^2-m^2]^3} = -i\pi^2 p^{\mu}/2 . \quad (2.14)$$

Thus we reobtain the Jauch-Rohrlich surface term [Eq. (1.1)].

Consider now case II. All manipulations leading to Eq. (2.11) remain valid provided $2\omega-4<1$, so that (2.8) may be used to obtain

$$I^{\mu}(\omega,2)-J^{\mu}(\omega,2) = \int d^{2\omega}k \int_0^1 dz \left[\frac{-p^{\mu}}{[k^2-m^2]^2} + \frac{4k^{\mu}k^{\nu}p_{\nu}-2p^{\mu}p^2z(1-z)}{[k^2+z(1-z)p^2-m^2]^3} \right] . \quad (2.15)$$

For case II, (2.12) is still valid, so

$$I^{\mu}(\omega,2)-J^{\mu}(\omega,2) = \int d^{2\omega}k \int_0^1 dz \left[\frac{4k^{\mu}k^{\nu}p_{\nu}}{[k^2+z(1-z)p^2-m^2]^3} - \frac{[k^2+z(1-z)p^2-m^2]p^{\mu}}{[k^2+z(1-z)p^2-m^2]^3} \right] . \quad (2.16)$$

Using (2.9a), $k^{\mu}k^{\nu}\rightarrow g^{\mu\nu}k^2/2\omega$, and (2.16) becomes

$$I^{\mu}(\omega,2)-J^{\mu}(\omega,2) = \int d^{2\omega}k \int_0^1 dz \left[\frac{p^{\mu}k^2[(2/\omega)-1]}{[k^2+z(1-z)p^2-m^2]^3} - \frac{p^{\mu}[z(1-z)p^2-m^2]}{[k^2+z(1-z)p^2-m^2]^3} \right] . \quad (2.17)$$

We may interchange k and z integrations provided $2\omega-4<0$. Using (2.5) and (2.8), (2.17) becomes

$$I^{\mu}(\omega,2)-J^{\mu}(\omega,2) = \int_0^1 dz \left[\frac{-i\pi^{\omega}(1-2/\omega)\Gamma(\omega+1)\Gamma(2-\omega)}{2\Gamma(\omega)[p^2z(1-z)-m^2]^{2-\omega}} - \frac{i\pi^{\omega}\Gamma(3-\omega)}{2[p^2z(1-z)-m^2]^{2-\omega}} \right] , \quad (2.18)$$

or

$$I^{\mu}(\omega,2)-J^{\mu}(\omega,2)=0 , \quad \omega<2 . \quad (2.19)$$

Hence

$$I^{\mu}(\omega,2)-J^{\mu}(\omega,2)= \begin{cases} 0 , & \omega<2 , \\ -i\pi^2p^{\mu}/2 , & \omega=2 , \end{cases} \quad (2.20)$$

and so $I^{\mu}(\omega,2)-J^{\mu}(\omega,2)$ is discontinuous at $\omega=2$, consistent with Eq. (1.3).

For case III, the same manipulations that lead to (2.17) now yield

$$I^{\mu}(\omega,\omega)-J^{\mu}(\omega,\omega) = \int d^{2\omega}k \int_0^1 dz \left[\frac{p^{\mu}k^2[(\omega/\omega)-1]}{[k^2+z(1-z)p^2-m^2]^{\omega+1}} - \frac{p^{\mu}[z(1-z)p^2-m^2]}{[k^2+z(1-z)p^2-m^2]^{\omega+1}} \right] = -i\pi^{\omega}p^{\mu}/[\Gamma(\omega+1)] , \quad (2.21)$$

where the first integrand vanishes because of (2.7), and where (2.5) has been used to evaluate the second integrand, which is finite.

We stress that cases I and III do indeed differ from case II, in the limit $\omega\rightarrow2$. This is not surprising; in general, we find that

$$\lim_{\omega\to 2}\left[\int d^{2\omega}k\, f(k,\omega)\right] \neq \lim_{\omega\to 2}\left[\int d^{2\omega}k\, f(k,2)\right], \quad (2.22)$$

reflecting a discontinuity at $\omega=2$. As an example, consider the integral

$$\zeta(\omega,\alpha,\beta)\equiv \int \frac{d^{2\omega}k}{(k^2)^\alpha[(k+p)^2]^\beta}$$

$$=i\pi^\omega \frac{\Gamma(\alpha+\beta-\omega)}{\Gamma(\alpha)\Gamma(\beta)}\beta(\omega-\alpha,\omega-\beta)(p^2)^{\omega-\alpha-\beta}$$

$$(\omega \le \alpha+\beta)\,. \quad (2.23)$$

One can show that

$$\zeta(\omega,1,1)=i\pi^\omega\Gamma(2-\omega)\beta(\omega-1,\omega-1)(p^2)^{\omega-2} \quad (2.24)$$

and

$$\zeta(\omega,1,3-\omega)= i\pi^\omega \frac{\Gamma(4-2\omega)}{\Gamma(3-\omega)}$$

$$\times \beta(\omega-1,2\omega-3)(p^2)^{2\omega-4}\,, \quad (2.25)$$

in which case[8]

$$\lim_{\omega\to 2}[\zeta(\omega,1,3-\omega)]=\tfrac{1}{2}\lim_{\omega\to 2}[\zeta(\omega,1,1)]\,, \quad (2.26)$$

and not

$$\lim_{\omega\to 2}(\omega,1,1)$$

as one might expect.

III. SURFACE TERMS FROM LINEARLY DIVERGENT INTEGRALS

We wish to consider the difference between any pair of Feynman integrals that are related to each other by an additive shift in the variable of integration, and are linearly divergent when integrated over (exactly) four dimensions. To proceed, we must first establish that naive shifts of the integration variable are permitted in any Feynman integral whose degree of divergence is less than one. Consider the difference between two integrals whose degree of divergence is $D_0=2\omega-2r$,

$$I^{(0)}=\int \frac{d^{2\omega}k}{[(k-p)^2-m^2]^r}-\int \frac{d^{2\omega}k}{[k^2-m^2]^r}\,. \quad (3.1)$$

Using the formula[1]

$$a^{-r}-b^{-r}=r(b-a)\int_0^1 dz[az+b(1-z)]^{-(r+1)}\,, \quad (3.2)$$

we find that

$$I^{(0)}=r\int d^{2\omega}k\int_0^1 \frac{dz(2p\cdot k-p^2)}{[(k-pz)^2+p^2z(1-z)-m^2]^{r+1}}\,. \quad (3.3)$$

The degree of divergence of the integral in Eq. (3.3) is one less than that of the integral of Eq. (3.1). If $\omega<r+\frac{1}{2}$, the integral over k in Eq. (3.3) is finite and a naive shift of variable $k-pz\to k$ is allowed,

$$I^{(0)}=r\int d^{2\omega}k\int_0^1 \frac{dz\,p^2(1-2z)}{[k^2+p^2z(1-z)-m^2]^{r+1}}\,. \quad (3.4)$$

The integral over z can be shown to be zero by noting that

$$d[p^2z(1-z)]=p^2(1-2z)dz\,,$$

and that $z(1-z)$ is symmetrically double valued between $z=0$ and $z=1$. Therefore $I^{(0)}=0$ provided $\omega<r+\frac{1}{2}$, in which case naive shifts in the variable of integration for $I^{(0)}$ are allowed provided $D_0<1$.

In fact, this argument can be applied to any less-than-linearly divergent integral; henceforth, we shall always assume that naive shifts of integration variable are permitted in integrals with degree of divergence less than one.

Let us now consider the difference between variable-shifted integrals containing the product of an odd number of numerator momenta:

$$I^{2n+1,r}_{\mu_1\mu_2\cdots\mu_{2n+1}}\equiv \int d^{2\omega}k\left[\prod_{j=1}^{2n+1}k_{\mu_j}\right][(k-p)^2-m^2]^{-r}\,, \quad (3.5)$$

$$J^{2n+1,r}_{\mu_1\mu_2\cdots\mu_{2n+1}}\equiv \int d^{2\omega}k\left[\prod_{j=1}^{2n+1}(k+p)_{\mu_j}\right][k^2-m^2]^{-r}\,. \quad (3.6)$$

If $2\omega+2n+1-2r<1$, these integrals are equal. We wish to consider the difference between these integrals in the regime $2>2\omega+2n+1-2r>1$. Using Eq. (3.2), we find that

$$I^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} = \int d^{2\omega}k\left[\prod_{j=1}^{2n+1}k_{\mu_j}\right][k^2-m^2]^{-r} + r\int d^{2\omega}k\int_0^1 dz\frac{[2k\cdot p-p^2]\prod_{j=1}^{2n+1}k_{\mu_j}}{[(k-pz)^2+p^2z(1-z)-m^2]^{r+1}}\,. \quad (3.7)$$

The first integral in Eq. (3.7) is odd in at least one component of k and must therefore vanish [Eq. (A1a) of Appendix A]. The second integral has degree of divergence $2\omega+2n+2-2(r+1)<1$, in which case the variable shift $k\to k+pz$ is allowed. Consequently, we find that

$$I^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}}=r\int d^{2\omega}k\int_0^1 dz\left[2p^{\mu_{2n+2}}\prod_{j=1}^{2n+2}(k+pz)_{\mu_j}-p^2\prod_{j=1}^{2n+1}(k+pz)_{\mu_j}\right][k^2+p^2z(1-z)-m^2]^{r+1}\,. \quad (3.8)$$

Let us now define an n-indexed object $\sigma^{j_1\cdots j_n}$ which equals unity if all j's are different and zero if any two j's are the same:

$$\sigma^{j_1 j_2 \cdots j_n} = \epsilon^{j_1 j_2 \cdots j_n}(-1)^{\text{sign}(\epsilon)} , \tag{3.9}$$

$$\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_m=1}^{m} \sigma^{j_1 j_2 \cdots j_m} = m! . \tag{3.10}$$

Using the "σ tensor" (which is not, strictly speaking, a tensor), we see that

$$\prod_{j=1}^{m} k_{\mu_j} \equiv k_{\mu_1} k_{\mu_2} \cdots k_{\mu_m}$$
$$= \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_m=1}^{m} (k_{\mu_{j_1}} k_{\mu_{j_2}} \cdots k_{\mu_{j_m}}) \frac{\sigma^{j_1 j_2 \cdots j_m}}{m!} . \tag{3.11}$$

Note that values of μ_{j_i} need not necessarily be different for different values of j_i. For notational convenience we will treat the summation over j's as understood whenever indices j_i are repeated.

In Appendix A, we prove that numerator factors of (3.8) can be expressed in terms of the contraction of the σ tensor onto products of the momenta p_{μ_j} and the metric tensors

$$g_{\mu_{j_{i-1}}\mu_{j_i}} .$$

Substituting Eqs. (A20) and (A15) of Appendix A into Eq. (3.8), we find that

$$I^{2n+1,r}_{\mu_1 \cdots \mu_{2n+1}} = r \int d^{2\omega}k \int_0^1 dz \sum_{t=0}^{n} ((k^2)^t z^{2n+1-2t}[2(\omega+t)p^2 z] + (k^2)^t z^{2n-2t}[(2n-2t+1)k^2] - (k^2)^t z^{2n+1-2t}[(\omega+t)p^2])$$
$$\times [G_{t,2n+1}(p)][k^2+p^2 z(1-z)-m^2]^{-r-1} , \tag{3.12}$$

where [from Eqs. (A11) and (A7) of Appendix A]

$$G_{t,2n+1}(p) = (g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})(p_{\mu_{j_{2t+1}}} \cdots p_{\mu_{j_{2n+1}}})$$
$$\times \sigma^{j_1 \cdots j_{2n+1}} \Gamma(\omega)/[t! 2^{2t} \Gamma(\omega+t+2)(2n+1-2t)!] . \tag{3.13}$$

We choose to regroup the terms of Eq. (3.12) as follows:

$$I^{2n+1,r}_{\mu_1 \cdots \mu_{2n+1}} = \int d^{2\omega}k \sum_{t=0}^{n} (k^2)^t (2n-2t+1) G_{t,2n+1}(p)$$
$$\times \left[r \int_0^1 \frac{dz\, z^{2n-2t} k^2}{[k^2+p^2 z(1-z)-m^2]^{r+1}} + r(\omega+t) \int_0^1 \left[\frac{p^2(1-2z)dz}{[k^2+p^2 z(1-z)-m^2]^{r+1}} \right] \left[\frac{z^{2n-2t+1}}{2n-2t+1} \right] \right] . \tag{3.14}$$

An obvious integration by parts yields

$$I^{2n+1,r}_{\mu_1 \cdots \mu_{2n+1}} = \int d^{2\omega}k \sum_{t=0}^{n} (k^2)^t (2n-2t+1) G_{t,2n+1}(p)$$
$$\times \left[(\omega+t)(2n-2t+1)^{-1}(k^2-m^2)^{-r} + \int_0^1 dz \left[z^{2n-2t} \frac{[k^2(r-\omega-t)-(p^2 z(1-z)-m^2)(\omega+t)]}{[k^2+p^2 z(1-z)-m^2]^{r+1}} \right] \right] . \tag{3.15}$$

In (3.15), we are performing a 2ω-dimensional integral, where the value of ω is fixed. In particular, the coefficient of k^2 in the numerator of the last integral on the right-hand side is $(r-\omega-t)$, where both r and ω have fixed values which are not necessarily integral.

Let us first suppose that there is no integer t for which $r-\omega-t=0$. Then the coefficient of k^2 is nonzero for all t. If we employ the formula[9]

$$\int \frac{d^{2\omega}k\,(k^2)^m}{(k^2+X)^q} = \frac{i\pi^\omega \Gamma(\omega+m)\Gamma(q-\omega-m)}{\Gamma(\omega)\Gamma(q)X^{q-\omega-m}} \tag{3.16}$$

in (3.15), we see that the integrand of the right-hand side integral over z vanishes when integrated over k.

Now let us suppose for our particular choices of r and ω that there exists an integer t such that $r-\omega-t=0$. Since $2\omega+2n+1-2r<2$, the only possible value of t for which r can equal $\omega+t$ is n. Thus, if $r=\omega+n$, the integrand of the integral over z in (3.15) when $t=n$ is just

$$\frac{-[p^2z(1-z)-m^2](\omega+n)}{[k^2+p^2z(1-z)-m^2]^{\omega+n+1}},$$

leading to a surface term when integrated over k:

$$I^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} = \int d^{2\omega}k \sum_{t=0}^{n}(k^2)^t G_{t,2n+1}(p)(\omega+t)/(k^2-m^2)^r - \delta_{r,\omega+n}\left[\frac{i\pi^\omega}{\Gamma(\omega)}G_{n,2n+1}(p)\right]. \quad (3.17)$$

One can attempt to argue away this surface term for the case of $r=\omega+n$ by choosing $\omega=r-n+\epsilon$ and considering the limit $\epsilon\to 0^+$. The relevant term in (3.15) becomes

$$\lim_{\epsilon\to 0^+}\int \frac{d^{2\omega}k\,(k^2)^{t+1}\epsilon}{[k^2+p^2z(1-z)-m^2]^{\omega+t+1+\epsilon}},$$

which leads to a nonzero result proportional to

$$\epsilon\Gamma(\epsilon)=\Gamma(\epsilon+1)\underset{\epsilon\to 0^+}{\longrightarrow} 1$$

that would cancel the surface term. Such an approach *assumes* continuity in ω, inevitably removing any surface terms that may arise at discrete values of ω.[10] We take the point of view here that Eq. (3.15) has a discontinuity when $r=\omega+n$ which the $\epsilon\to 0^+$ limiting procedure necessarily defines away. Indeed, by saying that the contribution of $(r-\omega-t)k^2$ is exactly zero for $r=\omega+n$ when $t=n$, we are assuming that

$$r\int d^{2\omega}k\left[\frac{(k^2)^S}{[(k+p)^2-m^2]^r}-\frac{(k^2)^S}{[(k+p)^2-m^2]^r}\right]=0,$$

consistent with Eq. (2.7), even if

$$\int \frac{d^{2\omega}k\,(k^2)^S}{[(k+p)^2-m^2]^r}$$

diverges for the choices of r, ω, and s considered.[11] Hence, we cannot use the pole to cancel the zero as in case II of the previous section of our paper; to do so would violate Eqs. (2.7) and (2.8), equations which are necessary for us to be able to continue to values of ω outside of the range $0<\omega<\frac{3}{2}$.

We now use Eq. (3.17) to evaluate the difference between (3.5) and (3.6). The first term on the right-hand side of (3.17) is equal to $J^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}}$, a result easily seen by setting $z=1$ in Eq. (A15) and then substituting directly into (3.6). Therefore, we find that

$$I^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}}-J^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} = -\frac{i\pi^\omega G_{n,2n+1}(p)}{\Gamma(\omega)}\delta_{r,\omega+n}, \quad (3.18)$$

where

$$G_{n,2n+1}(p)=\{\Gamma(\omega)/[\Gamma(\omega+n+1)n!2^{2n}]\}\times(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2n-1}}\mu_{j_{2n}}}p_{\mu_{j_{2n+1}}}\sigma^{j_1\cdots j_{2n+1}}). \quad (3.19)$$

Note that in the $(2n+1)!$ terms of (3.19), each of the $(2n+1)$ possible values for p_{μ_j} multiply $(2n-1)!!$ distinct products of n g's, as is discussed in Appendix A. For example, if $n=2$, each j_i goes from 1 to 5, and

$$\begin{aligned} g_{\mu_{j_1}\mu_{j_2}}g_{\mu_{j_3}\mu_{j_4}}p_{\mu_{j_5}}\sigma^{j_1j_2j_3j_4j_5} &= 8p_{\mu_1}(g_{\mu_2\mu_3}g_{\mu_4\mu_5}+g_{\mu_2\mu_4}g_{\mu_3\mu_5}+g_{\mu_2\mu_5}g_{\mu_3\mu_4}) \\ &\quad +8p_{\mu_2}(g_{\mu_1\mu_3}g_{\mu_4\mu_5}+\cdots)+8p_{\mu_3}(g_{\mu_1\mu_2}g_{\mu_4\mu_5}+\cdots) \\ &\quad +8p_{\mu_4}(g_{\mu_1\mu_2}g_{\mu_3\mu_5}+\cdots)+8p_{\mu_5}(g_{\mu_1\mu_2}g_{\mu_3\mu_4}+\cdots). \end{aligned} \quad (3.20)$$

Also note that if $n=0$,

$$G_{0,1}(p)=[\Gamma(\omega)/\Gamma(\omega+1)]p_{\mu_1},$$

consistent with Eq. (1.3).

We stress that no surface term arises unless ω and r differ by an integer. When $\omega=r-n$, the integrals of Eqs. (3.5) and (3.6) are exactly linearly divergent. We see, therefore, that variable shifts do not lead to surface terms within less-than-quadratically divergent integrals with products of an odd number of numerator momenta unless the degree of divergence is *exactly* one.

IV. DIFFERENCES OF MORE-THAN-LINEARLY DIVERGENT VARIABLE-SHIFTED INTEGRALS

Let us now consider the difference between

$$I^{2n,r}_{\mu_1\cdots\mu_{2n}} = \int d^{2\omega}k\left[\prod_{j=1}^{2n}k_{\mu_j}\right][(k-p)^2-m^2]^{-r} \quad (4.1)$$

and

$$J^{2n,r}_{\mu_1\cdots\mu_{2n}} = \int d^{2\omega}k\left[\prod_{j=1}^{2n}(k+p)_{\mu_j}\right][k^2-m^2]^{-r}, \quad (4.2)$$

where $2\omega+2n-2r<3$. We once again use Eq. (3.2) to find that

$$I^{2n,r}_{\mu_1\cdots\mu_{2n}} = \int d^{2\omega}k\left[\prod_{j=1}^{2n}k_{\mu_j}\right][k^2-m^2]^{-r} + r\int d^{2\omega}k\int_0^1\frac{dz[2p\cdot k-p^2]\prod_{j=1}^{2n}k_{\mu_j}}{[(k-pz)^2+p^2z(1-z)-m^2]^{r+1}}. \quad (4.3)$$

We want to make the variable shift $k\to k+pz$ in the second integral of Eq. (4.3). Such a shift results in a surface term when $2\omega+2n+1-2(r+1)=1$; using Eqs. (3.18) and (3.19) we find that

$$I^{2n,r}_{\mu_1 \cdots \mu_{2n}} = \int d^{2\omega}k \left[\prod_{j=1}^{2n} k_{\mu_j} \right] [k^2-m^2]^{-r}$$

$$+2r \int d^{2\omega}k \int_0^1 dz \frac{p^{\mu_{2n+1}} \prod_{j=1}^{2n+1} (k+pz)_{\mu_j}}{[k^2+p^2z(1-z)-m^2]^{r+1}} - rp^2 \int d^{2\omega}k \int_0^1 dz \frac{\prod_{j=1}^{2n} (k+pz)_{\mu_j}}{[k^2+p^2z(1-z)-m^2]^{r+1}}$$

$$+2rp^{\mu_{2n+1}} \int_0^1 dz \{ -i\pi^\omega \delta_{\omega,r+1-n} / [2^{2n} n! \Gamma(\omega+n+1)] \} (g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2n-1}}\mu_{j_{2n}}} p_{\mu_{j_{2n+1}}} z \sigma^{j_1 \cdots j_{2n+1}}) . \tag{4.4}$$

The integrands of the second and third integrals in (4.4) can be evaluated using Eqs. (A16) and (A19). The contraction of $p^{\mu_{2n+1}}$ into terms in the final curly brackets is given by

$$p^{\mu_{2n+1}}(g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2n-1}}\mu_{j_{2n}}} p_{\mu_{j_{2n+1}}} \sigma^{j_1 \cdots j_{2n+1}})$$

$$= p^2(g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2n-1}}\mu_{j_{2n}}} \sigma^{j_1 \cdots j_2}) + 2n (g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2n-3}}\mu_{j_{2n-2}}} p_{\mu_{j_{2n-1}}} p_{\mu_{j_{2n}}} \sigma^{j_1 \cdots j_{2n}}) , \tag{4.5}$$

a result most easily understood when one considers that $(2n+1)!$ permutations of indices on the left-hand side of (4.5) may be partitioned into either (1) a choice of

$$p_{\mu_{j_{2n+1}}} = p_{\mu_{2n+1}} ,$$

in which case there are $(2n)!$ permutations of the remaining indices running between μ_1 and μ_{2n}, or (2) the $[(2n+1)!-(2n)!] = 2n(2n)!$ permutations of $2n+1$ indices constrained such that $p_{\mu_{j_{2n+1}}}$ does *not* equal $p_{\mu_{2n+1}}$ [for both cases $(2n)!$ corresponds to the contraction of the $2n$ indices $\mu_{j_1} \cdots \mu_{j_{2n}}$ into $\sigma^{j_1 \cdots j_{2n}}$].

Upon substitution of Eqs. (A16), (A19), and (4.5) into Eq. (4.4), we find that

$$I^{2n,r}_{\mu_1 \cdots \mu_{2n}} = \int d^{2\omega}k \left[\prod_{j=1}^{2n} k_{\mu_j} \right] [k^2-m^2]^{-r}$$

$$+ r \sum_{t=0}^{n-1} \int d^{2\omega}k \int_0^1 dz \frac{z^{2n-2t-1}(k^2)^{t+1}(2n-2t) G_{t,2n}(p)}{[k^2+p^2z(1-z)-m^2]^{r+1}}$$

$$- r \sum_{t=0}^{n} \int d^{2\omega}k \int_0^1 dz \frac{z^{2n-2t}(k^2)^t(\omega+t)p^2(1-2z) G_{t,2n}(p)}{[k^2+p^2z(1-z)-m^2]^{r+1}}$$

$$- \frac{i\pi^\omega r \delta_{\omega,r+1-n}}{2^{2n} n! \Gamma(\omega+n+1)} \sigma^{j_1 \cdots j_{2n}} \{ g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2n-3}}\mu_{j_{2n-2}}} [p^2 g_{\mu_{j_{2n-1}}\mu_{j_{2n}}} + 2np_{\mu_{j_{2n-1}}} p_{\mu_{j_{2n}}}] \} . \tag{4.6}$$

The third integral in (4.6) can be integrated by parts over z by identifying

$$du = p^2(1-2z)dz / [k^2+p^2z(1-z)-m^2]^{r+1} .$$

We see that when $t=n$, the third integral vanishes, and (after a little algebra) we find that

$$I^{2n,r}_{\mu_1 \cdots \mu_{2n}} = \int d^{2\omega}k \left[\prod_{j=1}^{2n} k_{\mu_j} \right] [k^2-m^2]^{-r} + \sum_{t=0}^{n-1} \int d^{2\omega}k \int_0^1 dz \frac{z^{2n-2t-1}(2n-2t)(r-\omega-t)(k^2)^{t+1} G_{t,2n}(p)}{[k^2+p^2z(1-z)-m^2]^{r+1}}$$

$$- \sum_{t=0}^{n-1} \int d^{2\omega}k \int_0^1 dz \frac{z^{2n-2t-1}(2n-2t)(\omega+t)(k^2)^t [p^2z(1-z)-m^2]}{[k^2+p^2z(1-z)-m^2]^{r+1}} G_{t,2n}(p)$$

$$+ \sum_{t=0}^{n-1} \int d^{2\omega}k (\omega+t)(k^2)^t G_{t,2n}(p) [k^2-m^2]^{-r}$$

$$- \frac{i\pi^\omega r \delta_{\omega,r+1-n}}{2^{2n} n! \Gamma(\omega+n+1)} \{ g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2n-3}}\mu_{j_{2n-2}}} [p^2 g_{\mu_{j_{2n-1}}\mu_{j_{2n}}} + 2np_{\mu_{j_{2n-1}}} p_{\mu_{j_{2n}}}] \sigma^{j_1 \cdots j_{2n}} \} . \tag{4.7}$$

If $r \neq \omega + t$, the second and third integrals cancel [Eq. (3.16)] as before,

$$(r-\omega-t)\int d^{2\omega}k(k^2)^{t+1}[k^2+p^2z(1-z)-m^2]^{-r-1}$$
$$=(\omega+t)[p^2z(1-z)-m^2]\int d^{2\omega}k(k^2)^t[k^2+p^2z(1-z)-m^2]^{-r-1}$$
$$=i\pi^\omega\Gamma(\omega+t+1)\Gamma(r-\omega-t+1)[p^2z(1-z)-m^2]^{\omega+t-r}/\{\Gamma(\omega)\Gamma(r+1)\}\ . \quad (4.8)$$

If $r=\omega+t$, our restriction that $2\omega+2n-2r<3$ is satisfied only if t equals $n-1$, its largest allowed value. Thus, we find that

$$I^{2n,r}_{\mu_1\cdots\mu_{2n}}=\int d^{2\omega}k\left[\prod_{j=1}^{2n}k_{\mu_j}\right][k^2-m^2]^{-r}+\int d^{2\omega}k\sum_{t=0}^{n-1}(\omega+t)(k^2)^tG_{t,2n}(p)[k^2-m^2]^{-r}$$
$$-\delta_{r,\omega+n-1}\left[\frac{2i\pi^\omega}{\Gamma(\omega)}G_{n-1,2n}+\frac{i\pi^\omega}{2^{2n}\Gamma(n+1)\Gamma(\omega+n+1)}\right.$$
$$\left.\times[g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2n-3}}\mu_{j_{2n-2}}}(p^2g_{\mu_{j_{2n-1}}\mu_{j_{2n}}}+2np_{\mu_{j_{2n-1}}}p_{\mu_{j_{2n}}})\sigma^{j_1\cdots j_{2n}}]\right]\ . \quad (4.9)$$

The first two integrals on the right-hand side of Eq. (4.9) are equal to $J^{2n,r}_{\mu_1\cdots\mu_{2n}}$ as is seen from substituting Eq. (A16) (with $z=1$), (A1b) and (A11) (with $t=n,m=2n$) into Eq. (4.9). If we then use Eq. (A11) to evaluate $G_{n-1,2n}$, we find that

$$I^{2n,r}_{\mu_1\cdots\mu_{2n}}-J^{2n,r}_{\mu_1\cdots\mu_{2n}}$$
$$=-\delta_{r,\omega+n-1}\left[\frac{i\pi^\omega\sigma^{j_1\cdots j_{2n}}}{2^{2n}\Gamma(n+1)\Gamma(\omega+n+1)}\{g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2n-3}}\mu_{j_{2n-2}}}\right.$$
$$\left.\times[(\omega+n-1)p^2g_{\mu_{j_{2n-1}}\mu_{j_{2n}}}+2n(2\omega+2n-1)p_{\mu_{j_{2n-1}}}p_{\mu_{j_{2n}}}]\}\right]\ . \quad (4.10)$$

As an example, note that for four indices there are 24 permutations, facilitating determination of

$$\sigma^{j_1j_2j_3j_4}(g_{\mu_{j_1}\mu_{j_2}}g_{\mu_{j_3}\mu_{j_4}})=8[g_{\mu_1\mu_2}g_{\mu_3\mu_4}+g_{\mu_1\mu_3}g_{\mu_2\mu_4}+g_{\mu_1\mu_4}g_{\mu_2\mu_3}]\ , \quad (4.11a)$$

$$\sigma^{j_1j_2j_3j_4}(g_{\mu_{j_1}\mu_{j_2}}p_{\mu_{j_3}}p_{\mu_{j_4}})=4[g_{\mu_1\mu_2}p_{\mu_3}p_{\mu_4}+g_{\mu_1\mu_3}p_{\mu_2}p_{\mu_4}+g_{\mu_1\mu_4}p_{\mu_2}p_{\mu_3}+g_{\mu_2\mu_3}p_{\mu_1}p_{\mu_4}+g_{\mu_2\mu_4}p_{\mu_1}p_{\mu_3}+g_{\mu_3\mu_4}p_{\mu_1}p_{\mu_2}]\ . \quad (4.11b)$$

Once again, we see from Eq. (4.10) that naive shifts of variables are permitted for products of an even number of numerator momenta unless ω is exactly equal to $r+1-n$, corresponding to a degree of divergence *exactly* equal to 2.

V. DISCUSSION

In Secs. III and IV, we have shown that naive variable shifts are permitted in divergent integrals with nonintegral degrees of divergence, but that additional finite terms accompany such shifts when the degree of divergence is 1 or 2. In other words, the function $F(\omega)$ defined by

$$F^{m,r}_{\mu_1\cdots\mu_m}(\omega)\equiv I^{m,r}_{\mu_1\cdots\mu_m}(\omega)-J^{m,r}_{\mu_1\cdots\mu_m}(\omega) \quad (5.1)$$

is not a continuous function of the dimensionality ω. This result has its most obvious ramifications in calculations of the *VVA* triangle anomaly.

Consider $S_{\mu\rho\sigma}$, the sum of the triangle graphs in Fig. 1 (in which interior momenta are parametrized in the most general possible manner). In four dimensions, divergences of all three triangle-graph currents are proportional to differences of variable-shifted linearly divergent integrals (note that $p_2'-p_2=p_1'-p_1$):

$$k^\sigma S_{\mu\rho\sigma}=\frac{-ie^2}{(2\pi)^4}\int d^4r\{\mathrm{Tr}[\gamma_\rho(\not r+\not p_1+\not q)^{-1}\gamma_\mu\gamma_5(\not r+\not p_1)^{-1}]-\mathrm{Tr}[\gamma_\rho(\not r+\not p_2'+\not q)^{-1}\gamma_\mu\gamma_5(\not r+\not p_2')^{-1}]\}$$
$$+\frac{ie^2}{(2\pi)^4}\int d^4r\{\mathrm{Tr}[\gamma_\rho(\not r+\not p_2'+\not q)^{-1}\gamma_\mu\gamma_5(\not r+\not p_1')^{-1}]-\mathrm{Tr}[\gamma_\rho(\not r+\not p_2+\not q)^{-1}\gamma_\mu\gamma_5(\not r+\not p_1)^{-1}]\}\ , \quad (5.2)$$

$$-(q+k)^{\rho}S_{\mu\rho\sigma}=\frac{-ie^2}{(2\pi)^4}\int d^4r\{\mathrm{Tr}[(\not r+\not p_2+\not q)^{-1}\gamma_\sigma(\not r+\not p_1+\not q)^{-1}\gamma_\mu\gamma_5]-\mathrm{Tr}[(\not r+\not p_2')^{-1}\gamma_\sigma(\not r+\not p_1')^{-1}\gamma_\mu\gamma_5]\}$$
$$+\frac{ie^2}{(2\pi)^4}\int d^4r\{\mathrm{Tr}[(\not r+\not p_1)^{-1}\gamma_\sigma(\not r+\not p_1+\not q)^{-1}\gamma_\mu\gamma_5]-\mathrm{Tr}[(\not r+\not p_2')^{-1}\gamma_\sigma(\not r+\not p_2'+\not q)^{-1}\gamma_\mu\gamma_5]\}\ , \quad (5.3)$$

$$-q^{\mu}S_{\mu\rho\sigma}=\frac{ie^2}{(2\pi)^4}\int d^4r\{\mathrm{Tr}[\gamma_\rho(\not r+\not p_2+\not q)^{-1}\gamma_\sigma\gamma_5(\not r+\not p_1)^{-1}]-\mathrm{Tr}[\gamma_\rho(\not r+\not p_2'+\not q)^{-1}\gamma_\sigma\gamma_5(\not r+\not p_1')^{-1}]\}$$
$$+\frac{ie^2}{(2\pi)^4}\int d^4r\{\mathrm{Tr}[\gamma_\rho(\not r+\not p_2+\not q)^{-1}\gamma_\sigma(\not r+\not p_1+\not q)^{-1}\gamma_5]-\mathrm{Tr}[\gamma_\rho(\not r+\not p_2')^{-1}\gamma_\sigma(\not r+\not p_1')^{-1}\gamma_5]\}\ . \quad (5.4)$$

Each difference of traces within curly brackets yields a finite surface term contributing to anomalous divergences of triange-graph currents. If we define

$$p_2'-p_2(=p_1'-p_1)$$

as an arbitrary linear combination of the external momenta q and k such that

$$(p_2'-p_2)_\mu=Aq_\mu+Bk_\mu\ , \quad (5.5)$$

we repeatedly apply Eq. (1.3) and find that

$$k^{\sigma}S_{\mu\rho\sigma}=(-ie^2/8\pi^2)(1+A)\epsilon_{\mu\tau\rho\eta}q^{\tau}k^{\eta}\ , \quad (5.6)$$

$$-(q+k)^{\rho}S_{\mu\rho\sigma}=(ie^2/8\pi^2)(2-A+B)\epsilon_{\mu\tau\sigma\eta}q^{\tau}k^{\eta}\ , \quad (5.7)$$

$$-q^{\mu}S_{\mu\rho\sigma}=(ie^2/8\pi^2)(1-B)\epsilon_{\rho\tau\sigma\eta}q^{\tau}k^{\eta}\ . \quad (5.8)$$

These three equations in two unknowns forbid simultaneous conservation of vector and axial-vector currents. Imposition of vector-current conservation upon Eqs. (5.6) and (5.7) yields $A=-1, B=-3$; this value for B yields the usual chiral anomaly in Eq. (5.8).

We stress that this result is obtained directly from surface terms associated with the difference of variable-shifted linearly divergent integrals in exactly four dimensions.[12] If naive variable shifts were permitted in Eqs. (5.1), (5.2), and (5.3), no anomalous divergence would occur,[13] despite the fact that the anomaly corresponds to the physical field-theoretical amplitude for $\pi^0\to\gamma\gamma$.

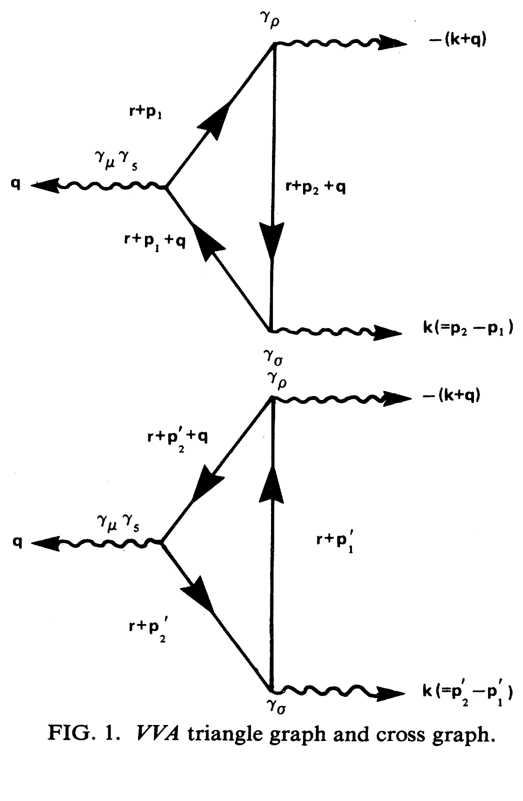

FIG. 1. *VVA* triangle graph and cross graph.

One can, of course, obtain the triangle anomaly in $n>4$ dimensions (where naive variable shifts are permitted) at the price of defining a γ_5 which commutes with γ_n for $n>4$ (Ref. 3). In essence, an exotic definition of γ_5 (not to mention nonintegral γ-matrix indices) compensates for removal of the discontinuity in dimensionality that occurs in Eq. (3.18).

Such a γ_5 does not seem to be appropriate for supersymmetric theories; $\{\gamma_5,\gamma_\mu\}=0$ and $\mathrm{Tr}(1)=4$ are necessary conditions to preserve supersymmetry in the Wess-Zumino model.[5,6] Dimensional reduction, an alternative regularization procedure in which naive variable shifts are also permitted, manages to uphold these conditions by partitioning four-dimensional space into n and $4-n$ dimensions and by using the usual four-dimensional γ matrices.[14] Unfortunately, the calculation of the triangle anomaly under dimensional reduction is somewhat ambiguous. Nicolai and Townsend[6] show that to recover the usual triangle anomaly, one must selectively abandon cyclicity of Dirac-matrix traces. For example, we retain having $\mathrm{Tr}\gamma_\mu\gamma_\nu=\mathrm{Tr}\gamma_\nu\gamma_\mu$, but we also require that

$$\mathrm{Tr}[\gamma_5\not b\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\not b]\neq -b^2\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma]\ ,$$

contradicting the equality that follows from cyclic rearrangement.

Under these circumstances, we find it reasonable to question the exclusion of variable-shift surface terms at integral values of dimensionality. In particular, if γ matrices retain their usual properties, the triangle anomaly appears to be peculiar to four dimensions, a manifestation of the discontinuity of Eq. (3.18) when $\omega-r$ is integral.[15,16]

Consequences of retaining finite terms associated with variable shifts in divergent integrals have also been examined in the renormalization of a spontaneously broken gauge theory in which fermions are absent.[17] The inclusion of such terms to full two-loop order has been shown to lead only to absorbable divergences (even though the absence of fermions precludes any γ-matrix peculiarities that may compensate for surface terms). Finite terms associated with variable shifts in divergent integrals have also been examined in two-dimensional QED[18]; results obtained are consistent with the theory first-discussed obtained by Schwinger.[19]

ACKNOWLEDGMENTS

The authors are grateful to G. Leibbrandt and A. R. Swift for useful correspondence. Financial support by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

APPENDIX A: SYMMETRIC INTEGRATION IN 2ω DIMENSIONS

Consider the k^2-symmetric integration of a product of numerator momenta k_{μ_j} in 2ω dimensions:

$$\int d^{2\omega}k\, f(k^2) \prod_{j=1}^{2r+1} k_{\mu_j} = 0 \,, \tag{A1a}$$

$$\int d^{2\omega}k\, f(k^2) \prod_{j=1}^{2r} k_{\mu_j} = \int d^{2\omega}k\, f(k^2)(k^2)^r \left[g_{\mu_{j_1}\mu_{j_2}} g_{\mu_{j_3}\mu_{j_4}} \cdots g_{\mu_{j_{2r-1}}\mu_{j_{2r}}} \frac{N_r \sigma^{j_1 \cdots j_{2r}}}{r!2^r} \right] . \tag{A1b}$$

(The "σ tensor" is defined in the text.) In general, there are $(2r)!$ terms on the right-hand side of Eq. (A1b), corresponding to $(2r)!$ permutations of the $2r$ indices $\mu_1, \mu_2, \ldots, \mu_{2r}$ obtained when the product of g's is contracted into σ. Of these $(2r)!$ terms, only $(2r)!/(r!2^r) = (2r-1)!!$ are distinct, as there are $r!$ ways to arrange a product of r g's, and for each one of these g's, there are two equivalent choices of indices ($g_{\alpha\beta} = g_{\beta\alpha}$). For example, when $r=2$,

$$g_{\mu_{j_1}\mu_{j_2}} g_{\mu_{j_3}\mu_{j_4}} \sigma^{j_1 j_2 j_3 j_4} = 8(g_{\mu_1\mu_2} g_{\mu_3\mu_4} + g_{\mu_1\mu_3} g_{\mu_2\mu_4} + g_{\mu_1\mu_4} g_{\mu_2\mu_3}) \,, \tag{A2}$$

corresponding to $3(=3!!)$ distinct terms within $4!$ permutations of four indices.

We wish first to derive the value of N_r in terms of r and ω. To do this, consider

$$\int d^{2\omega}k\, \phi(k^2) \prod_{j=1}^{2r+2} k_{\mu_j} = \int d^{2\omega}k\, \phi(k^2)(k^2)^{r+1} \left[g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2r+1}}\mu_{j_{2r+2}}} \frac{N_{r+1} \sigma^{j_1 \cdots j_{2r+2}}}{(r+1)!2^{r+1}} \right] . \tag{A3}$$

Let us call $\mu_{2r+1} \equiv \rho$ and $\mu_{2r+2} \equiv \sigma$. Consider contractions of $g^{\rho\sigma}$ on both sides of Eq. (A3); using Eq. (A1), we find that contraction on the left-hand side yields

$$\int d^{2\omega}k\, \phi(k^2) k^2 \prod_{j=1}^{2r} k_{\mu_j} = \int d^{2\omega}k\, \phi(k^2)(k^2)^{r+1} \left[g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2r-1}}\mu_{j_{2r}}} \frac{N_r \sigma^{j_1 \cdots j_{2r}}}{r!2^r} \right] . \tag{A4}$$

Contraction of $g^{\rho\sigma}$ into the right-hand side of Eq. (A3) will be proportional to the same product of r g's and the σ tensor as in Eq. (A4). To find the constant of proportionality, consider the $(2r+2)!$ permutations of indices on the right-hand side of Eq. (A3). The number of permutations containing an explicit factor of $g_{\rho\sigma}$ is $(r+1)\times(2)\times(2r)!$, corresponding to $r+1$ choices for either $g_{\rho\sigma}$ or $g_{\sigma\rho}$ in the product of $r+1$ g's, and $(2r)!$ permutations of the remaining indices $\mu_1 - \mu_{2r}$ excluding ρ and σ. Moreover, the number of permutations *not* containing an explicit factor of $g_{\rho\sigma}$ (i.e., ρ and σ indices occur in different g's, as in $g_{\rho\mu_1} g_{\sigma\mu_2}$) is

$$(2r+2)! - (r+1)(2)(2r)! = [2(r+1)(2r)](2r)! \,,$$

where once again the factor of $(2r)!$ corresponds to the number of ways of arranging the $2r$ indices $\mu_1 - \mu_{2r}$. Consequently, contraction of $g^{\rho\sigma}$ into the right-hand side of Eq. (A3) yields

$$\begin{aligned} g^{\rho\sigma} \int d^{2\omega}k\, \phi(k^2)(k^2)^{r+1} &\left[g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2r+1}}\mu_{j_{2r+2}}} \frac{N_{r+1} \sigma^{j_1 \cdots j_{2r+2}}}{(r+1)!2^{r+1}} \right] \\ &= \int d^{2\omega}k\, \phi(k^2)(k^2)^{r+1} \{ g^{\rho\sigma} g_{\rho\sigma} [2(r+1)] + [2(r+1)(2r)] \} \left[g_{\mu_{j_1}\mu_{j_2}} \cdots g_{\mu_{j_{2r-1}}\mu_{j_{2r}}} \frac{N_{r+1} \sigma^{j_1 \cdots j_{2r}}}{(r+1)!2^{r+1}} \right] . \end{aligned} \tag{A5}$$

Since $g^{\rho\sigma} g_{\rho\sigma} = 2\omega$, comparison of Eqs. (A5) and (A4) shows that

$$(2\omega + 2r) N_{r+1} = N_r \,. \tag{A6}$$

Since $N_1 = 1/2\omega$, we see from induction that

$$N_r = 2^{-r} \Gamma(\omega)/\Gamma(\omega + r) \,. \tag{A7}$$

We now wish to consider the integration of a product of variable-shifted numerator momenta over a function of k^2 in 2ω dimensions:

$$I_{2n+1}(q) \equiv \int d^{2\omega}k\, f(k^2) \prod_{j=1}^{2n+1} (k+q)_{\mu_j} \,. \tag{A8}$$

Using the σ tensor defined in the text, we see that

$$\prod_{j=1}^{2n+1}(k+q)_{\mu_j}=\sum_{s=0}^{2n+1}(k_{\mu_{j_1}}\cdots k_{\mu_{j_s}})(q_{\mu_{j_{s+1}}}\cdots q_{\mu_{j_{2n+1}}})\frac{\sigma^{j_1\cdots j_{2n+1}}}{s!(2n+1-s)!}\ . \tag{A9}$$

[Summation over $j_1,j_2\cdots j_{2n+1}$ over integers $1\le j_i\le 2n+1$ is understood. Note that the "out-of-order" products $(k_{\mu_{j_1}}\cdots k_{\mu_{j_0}})$ and $(q_{\mu_{j_{2n+2}}}\cdots q_{\mu_{j_{2n+1}}})$ corresponding to $s=0$ and $s=2n+1$, respectively, are understood to be unity.]

If we substitute Eq. (A9) into (A8) and make use of Eqs. (A1), we find that

$$\begin{aligned}I_{2n+1}(q)&=\int d^{2\omega}k\,f(k^2)\sum_{t=0}^{n}(k_{\mu_{j_1}}\cdots k_{\mu_{j_{2t}}})(q_{\mu_{j_{2t+1}}}\cdots q_{\mu_{j_{2n+1}}})(\sigma^{j_1\cdots j_{2n+1}})/[(2t)!(2n-2t+1)!]\\&=\int d^{2\omega}k\,f(k^2)\sum_{t=0}^{n}\Bigg[(2t)!(g_{\mu_{j_1}\mu_{j_2}}g_{\mu_{j_3}\mu_{j_4}}\cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})\\&\qquad\times\left[\frac{(k^2)^tN_t}{t!2^t}\right](q_{\mu_{j_{2t+1}}}\cdots q_{\mu_{j_{2n+1}}})\left[\frac{\sigma^{j_1\cdots j_{2n+1}}}{(2t)!(2n-2t+1)!}\right]\Bigg]\ .\end{aligned} \tag{A10}$$

[Note that the factor $(k_{\mu_{j_1}}\cdots k_{\mu_{j_{2t}}})$ corresponds to $(2t)!$ products of $2t$ k's.]

We define

$$G_{t,m}(q)=2(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})(q_{\mu_{j_{2t+1}}}\cdots q_{\mu_{j_m}})\left[\frac{\sigma^{j_1\cdots j_m}N_{t+1}}{t!2^t[m-2t]!}\right] \tag{A11}$$

and use Eq. (A6) to find that

$$I_{2n+1}(q)=\int d^{2\omega}k\,f(k^2)\sum_{t=0}^{n}(k^2)^t(\omega+t)G_{t,2n+1}(q)\ . \tag{A12}$$

This result corresponds to making the following substitution in the integrand of Eq. (A8):

$$\prod_{j=1}^{2n+1}(k+q)_{\mu_j}\to\sum_{t=0}^{n}(k^2)^t(\omega+t)G_{t,2n+1}(q)\ . \tag{A13}$$

An identical argument yields the result

$$\prod_{j=1}^{2n}(k+q)_{\mu_j}\to\sum_{t=0}^{n}(k^2)^t(\omega+t)G_{t,2n}(q)\ . \tag{A14}$$

When $q=pz$, we see that $G_{t,m}(q)=z^{m-2t}G_{t,m}(p)$, leading to the following formulas for products of variable-shifted momenta within k^2-symmetric integrands of 2ω-dimensional integrals:

$$\prod_{j=1}^{2n+1}(k+pz)_{\mu_j}\to\sum_{t=0}^{n}(k^2)^t(\omega+t)z^{2n+1-2t}G_{t,2n+1}(p)\ , \tag{A15}$$

$$\prod_{j=1}^{2n}(k+pz)_{\mu_j}\to\sum_{t=0}^{n}(k^2)^t(\omega+t)z^{2n-2t}G_{t,2n}(p)\ . \tag{A16}$$

Let us now consider the numerator factors

$$p^{\mu_{2n+1}}\prod_{j=1}^{2n+1}(k+pz)_{\mu_j}\ ,$$

which occur in

$$p^{\mu_{2n+1}}I_{2n+1}(pz)$$

[Eq. (A8)]. From Eq. (A10), we see that

$$\begin{aligned}p^{\mu_{2n+1}}I_{2n+1}(pz)=\int d^{2\omega}k\,f(k^2)\sum_{t=0}^{n}&\left[\frac{(k^2)^tN_tz^{2n-2t+1}}{t!2^t(2n-2t+1)!}\right]\\&\times p^{\mu_{2n+1}}[(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})(p_{\mu_{j_{2t+1}}}\cdots p_{\mu_{j_{2n+1}}})\sigma^{j_1\cdots j_{2n+1}}]\ .\end{aligned} \tag{A17}$$

There are $[(2n+1)-2t]$ p indices; if any one of them is μ_{2n+1}, contraction with $p^{\mu_{2n+1}}$ will reduce the number of p indices by 1. Similarly, there are $2t$ g indices such that if any one of them is μ_{2n+1}, contraction with $p^{\mu_{2n+1}}$ replaces $g_{\mu_{j_i}\mu_{j_{2n+1}}}$ with $p_{\mu_{j_i}}$, thereby *increasing* the number of p indices by 1.

Hence, we find that

$$\begin{aligned}&p^{\mu_{2n+1}}[(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})(p_{\mu_{j_{2t+1}}}\cdots p_{\mu_{j_{2n+1}}})\sigma^{j_1\cdots j_{2n+1}}]\\&\quad=p^2[(2n+1)-2t][(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})(p_{\mu_{j_{2t+1}}}\cdots p_{\mu_{j_{2n}}})\sigma^{j_1\cdots j_{2n}}]\\&\quad\quad+2t[(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-3}}\mu_{j_{2t-2}}})(p_{\mu_{j_{2t-1}}}\cdots p_{\mu_{j_{2n}}})\sigma^{j_1\cdots j_{2n}}]\ .\end{aligned} \tag{A18}$$

Substitution of Eqs. (A18), (A11), and (A6) into Eq. (A17) shows that

$$p^{\mu_{2n+1}}\prod_{j=1}^{2n+1}(k+pz)_{\mu_j}$$

$$\rightarrow \sum_{t=0}^{n} z^{2n-2t}(k^2)^t[(\omega+t)p^2z]G_{t,2n}(p) + \sum_{t=1}^{n}\frac{z^{2n+1-2t}(k^2)^tN_t}{(t-1)!2^{t-1}(2n+1-2t)!}(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-3}}\mu_{j_{2t-2}}}p_{\mu_{j_{2t-1}}}\cdots p_{\mu_{j_{2n}}}\sigma^{j_1\cdots j_{2n}})$$
$$= \sum_{t=0}^{n} z^{2n-2t}(k^2)^t[(\omega+t)p^2z]G_{t,2n}(p) + \sum_{t=0}^{n-1}\frac{z^{2n-2t-1}(k^2)^{t+1}N_{t+1}}{t!2^t(2n-2t-1)!}[(g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2t-1}}\mu_{j_{2t}}})(p\ldots)\sigma^{\cdots}]$$
$$= \sum_{t=0}^{n} z^{2n-2t}(k^2)^t[(\omega+t)p^2z]G_{t,2n}(p) + \sum_{t=0}^{n-1} z^{2n-2t-1}(k^2)^t[(n-t)k^2]G_{t,2n}(p)\ . \qquad (A19)$$

Similarly, we find that

$$p^{\mu_{2n+2}}\prod_{j=1}^{2n+2}(k+pz)_{\mu_j} \rightarrow \sum_{t=0}^{n} z^{2n-2t+1}(k^2)^t[(\omega+t)p^2z]G_{t,2n+1}(p) + \sum_{t=0}^{n} z^{2n-2t}(k^2)^t[(2n-2t+1)k^2/2]G_{t,2n+1}(p)\ . \qquad (A20)$$

APPENDIX B: VARIABLE-OF-INTEGRATION SHIFTS WITHIN FOUR-DIMENSIONAL FEYNMAN INTEGRALS

In this appendix, we use Eq. (1.1) in order to derive the following formulas for integration-variable shifts in four dimensions:

$$\int \frac{d^4k\, k_\mu k_\nu k_\lambda}{[(k-p)^2-m^2]^3} - \int \frac{d^4k\,(k+p)_\mu(k+p)_\nu(k+p)_\lambda}{[k^2-m^2]^3} = -\frac{i\pi^2}{12}[p_\mu g_{\nu\lambda}+p_\nu g_{\mu\lambda}+p_\lambda g_{\mu\nu}]\ , \qquad (B1)$$

$$\int \frac{d^4k\, k_\mu k_\nu}{[(k-p)^2-m^2]^2} - \int \frac{d^4k\,(k+p)_\mu(k+p)_\nu}{[k^2-m^2]^2} = -\frac{i\pi^2}{6}[p^2g_{\mu\nu}+5p_u p_\nu]\ , \qquad (B2)$$

$$\int \frac{d^4k\, k^2}{[(k-p)^2-m^2]^2} - \int \frac{d^4k\,(k+p)^2}{[k^2-m^2]^2} = -\frac{3i\pi^2}{2}p^2\ , \qquad (B3)$$

$$\int \frac{d^4k}{[(k-p)^2-m^2]} - \int \frac{d^4k}{[k^2-m^2]} = -\frac{i\pi^2}{2}p^2\ . \qquad (B4)$$

These results are consistent with Eqs. (3.18) and (4.10).

Consider first the integral

$$I_{\mu\nu\lambda} \equiv \int \frac{d^4k\ k_\mu k_\nu k_\lambda}{[(k-p)^2-m^2]^3}\ . \qquad (B5)$$

Following Ref. 1, we use the identity

$$\frac{1}{[(k-p)^2-m^2]^n} - \frac{1}{(k^2-m^2)^n} = -n(p^2-2p\cdot k) \times \int_0^1 \frac{dz}{[(k-pz)^2-(p^2z^2-p^2z+m^2)]^{n+1}} \qquad (B6)$$

in order to write

$$I_{\mu\nu\lambda} = \int \frac{d^4k\ k_\mu k_\nu k_\lambda}{(k^2-m^2)^3} - 3p^2\int d^4k \int_0^1 \frac{dz\, k_\mu k_\nu k_\lambda}{[(k-pz)^2-(p^2z^2-p^2z+m^2)]^4} + 6p^\tau \int d^4k \int_0^1 \frac{dz\, k_\mu k_\nu k_\lambda k_\tau}{[(k-pz)^2-(p^2z^2-p^2z+m^2)]^4}\ . \qquad (B7)$$

The final two integrals in Eq. (B7) are at most logarithmically divergent in k. Consequently, the variable k may be replaced in these integrals with $k+pz$, and

$$I_{\mu\nu\lambda} = \int \frac{d^4k\ k_\mu k_\nu k_\lambda}{(k^2-m^2)^3} - 3p^2\int d^4k \int_0^1 dz \frac{(k+pz)_\mu(k+pz)_\nu(k+pz)_\lambda}{[k^2-(p^2z^2-p^2z+m^2)]^4} + 6p^\tau\int d^4k\int_0^1 dz\frac{(k+pz)_\mu(k+pz)_\nu(k+pz)_\lambda(k+pz)_\tau}{[k^2-(p^2z^2-p^2z+m^2)]^4}$$
$$= \int \frac{d^4k\ k_\mu k_\nu k_\lambda}{(k^2-m^2)^3}$$

$$-3p_\mu p_\nu p_\lambda \left[p^2 \int d^4k \int_0^1 \frac{dz\, z^3(1-2z)}{[k^2-(p^2z^2-p^2z+m^2)]^4} - \frac{3}{2} \int d^4k \int_0^1 \frac{dz\, z^2k^2}{[k^2-(p^2z^2-p^2z+m^2)]^4} \right]$$
$$+(p_\mu g_{\nu\lambda}+p_\nu g_{\mu\lambda}+p_\lambda g_{\mu\nu}) \left[\frac{-3p^2}{4} \int d^4k \int_0^1 \frac{dz\, k^2 z(1-2z)}{[k^2-(p^2z^2-p^2z+m^2)]^4} + \frac{1}{4} \int d^4k \int_0^1 \frac{dz\, k^4}{[k^2-(p^2z^2-p^2z+m^2)]^4} \right] . \tag{B8}$$

The coefficient integrals multiplying $p_\mu p_\nu p_\lambda$ are finite and can be easily evaluated,

$$I_{\mu\nu\lambda} = \int \frac{d^4k\; k_\mu k_\nu k_\lambda}{(k^2-m^2)^3} - \frac{i\pi^2}{2m^2} p_\mu p_\nu p_\lambda + (p_\mu g_{\nu\lambda}+p_\nu g_{\mu\lambda}+p_\lambda g_{\mu\nu})\{ \cdots \} . \tag{B9}$$

Now consider the integral

$$J_{\mu\nu\lambda} \equiv \int \frac{d^4k\,(k+p)_\mu (k+p)_\nu (k+p)_\lambda}{(k^2-m^2)^3} , \tag{B10}$$

corresponding to a naive shift of integration variable in $I_{\mu\nu\lambda}$. It is straightforward to obtain the coefficient of $p_\mu p_\nu p_\lambda$ in $J_{\mu\nu\lambda}$. This coefficient is precisely the same as that obtained for $I_{\mu\nu\lambda}$ on the right-hand side of Eq. (B9). Consequently, we find that any surface term in $I_{\mu\nu\lambda}$ can only be proportional to $(p_\mu g_{\nu\lambda}+p_\nu g_{\mu\lambda}+p_\lambda g_{\mu\nu})$:

$$I_{\mu\nu\lambda} - J_{\mu\nu\lambda} = A(p^2,m^2)(p_\mu g_{\nu\lambda}+p_\nu g_{\mu\lambda}+p_\lambda g_{\mu\nu}) . \tag{B11}$$

The coefficient A can be determined by contracting two indices in Eq. (B10), using Eqs. (B5) and (B10) for I and J:

$$6A(p^2,m^2)p_\mu = g_{\nu\lambda}(I_{\mu\nu\lambda} - J_{\mu\nu\lambda}) = \int \frac{d^4k\; k_\mu k^2}{[(k-p)^2-m^2]^3} - \int \frac{d^4k\,(k+p)_\mu (k+p)^2}{(k^2-m^2)^3} . \tag{B12}$$

Now we see that

$$\int \frac{d^4k\; k_\mu k^2}{[(k-p)^2-m^2]^3} = \int \frac{d^4k\{[(k-p)^2-m^2]+m^2+2p\cdot k-p^2]\}k_\mu}{[(k-p)^2-m^2]^3}$$
$$= \int \frac{d^4k\; k_\mu}{[(k-p)^2-m^2]^2} + m^2 \int \frac{d^4k\; k_\mu}{[(k-p)^2-m^2]^3} + \int \frac{d^4k\,(2p\cdot k-p^2)k_\mu}{[(k-p)^2-m^2]^3} . \tag{B13}$$

Only the first integral on the far right-hand side of Eq. (B13) is more-than-logarithmically divergent. The surface term for that integral is given by Eq. (1.1). Naive shifts of integration variable are permitted for the remaining integrals on the far right-hand side of Eq. (B13), leading to the following result:

$$\int \frac{d^4k\; k_\mu k^2}{[(k-p)^2-m^2]^3} = \left[\int \frac{d^4k\,(k+p)_\mu}{(k^2-m^2)^2} - \frac{i\pi^2}{2} p_\mu \right] + m^2 \int \frac{d^4k\,(k+p)_\mu}{(k^2-m^2)^3} + \int \frac{d^4k\,(p^2+2p\cdot k)(k+p)_\mu}{(k^2-m^2)^3}$$
$$= \int d^4k \frac{(k+p)_\mu (k+p)^2}{(k^2-m^2)^3} - \frac{i\pi^2}{2} p_\mu . \tag{B14}$$

Substitution of Eq. (B14) into Eq. (B12) determines A:

$$6A(p^2,m^2)p_\mu = -\frac{i\pi^2}{2} p_\mu , \tag{B15}$$

in which case $A(p^2,m^2) = -i\pi^2/12$. Thus, Eq. (B1) is obtained by substitution of this value for A into Eq. (B11).

Equation (B2) can be derived using the identity of Eq. (B6) and, subsequently, Eq. (B1). First note that

$$I_{\mu\nu} \equiv \int \frac{d^4k\; k_\mu k_\nu}{[(k-p)^2-m^2]^2} = \int \frac{d^4k\; k_\mu k_\nu}{[k^2-m^2]^2} + 4p^\tau \int d^4k \int_0^1 dz \frac{k_\mu k_\nu k_\tau}{[(k-pz)^2-(p^2z^2-p^2z+m^2)]^3}$$
$$-2p^2 \int d^4k \int_0^1 dz \frac{k_\mu k_\nu}{[(k-pz)^2-(p^2z^2-p^2z+m^2)]^3} . \tag{B16}$$

The integral multiplying p^2 in Eq. (B16) is logarithmically divergent, in which case the shift of integration variable $k \to k+pz$ is permitted. The integral multiplying p_τ is of the same form as the first integral in Eq. (B1). Hence, we can use Eq. (B1) in order to find that

$$I_{\mu\nu}=\int \frac{d^4k\, k_\mu k_\nu}{(k^2-m^2)^3}+4p^\tau \int_0^1 dz \left[\frac{-i\pi^2}{12}(p_\mu z g_{\nu\tau}+p_\nu z g_{\mu\tau}+p_\tau z g_{\mu\nu})\right]$$

$$+4p^\tau \int d^4k \int_0^1 dz \frac{(k+pz)_\mu (k+pz)_\nu (k+pz)_\tau}{[k^2-(p^2z^2-p^2z+m^2)]^3}-2p^2 \int d^4k \int_0^1 dz \frac{(k+pz)_\mu (k+pz)_\nu}{[k^2-(p^2z^2-p^2z+m^2)]^3}$$

$$=\int \frac{d^4k\, k_\mu k_\nu}{(k^2-m^2)^3}-\frac{i\pi^2}{6}[p^2 g_{\mu\nu}+2p_\mu p_\nu]-2p^2\int d^4k \int_0^1 dz \frac{(1-2z)k_\mu k_\nu}{\{k^2-[p^2(z^2-z)+m^2]\}^2}$$

$$-2p^2 p_\mu p_\nu \int d^4k \int_0^1 dz \frac{z^2(1-2z)}{\{k^2-[p^2(z^2-z)+m^2]\}^3}+2p_\mu p_\nu \int d^4k \int_0^1 dz \frac{zk^2}{\{k^2-[p^2(z^2-z)+m^2]\}^3} . \tag{B17}$$

The first of the last three integrals in Eq. (B17) vanishes, since

$$\int_0^1 dz\, p^2(1-2z) f(p^2(z^2-z)) = \int_0^0 du\, f(u) = 0 . \tag{B18}$$

The second of the three integrals can be integrated by parts over z,

$$-2p_\mu p_\nu \int d^4k \int_0^1 dz \frac{p^2(1-2z)z^2}{\{k^2-[p^2(z^2-z)+m^2]\}^3}=p_\mu p_\nu \int \frac{d^4k}{(k^2-m^2)^2}-2p_\mu p_\nu \int d^4k \int_0^1 dz \frac{z}{\{k^2-[p^2(z^2-z)+m^2]\}^2} . \tag{B19}$$

The last integral of Eq. (B19) can be combined with the last integral of (B17) as follows:

$$-2p_\mu p_\nu \int d^4k \int_0^1 dz\, z \left[\frac{1}{\{k^2-[p^2(z^2-z)+m^2]\}^2}-\frac{k^2}{\{k^2-[p^2(z^2-z)+m^2]\}^3}\right]$$

$$=2p_\mu p_\nu \int_0^1 dz \int d^4k \frac{z(p^2z^2-p^2z+m^2)}{[k^2-(p^2z^2-p^2z+m^2)]^3}=p_\mu p_\nu \left[-\frac{i\pi^2}{2}\right] . \tag{B20}$$

[We are allowed to interchange the order of integration in Eq. (B20) because the integral over k is finite.] Substituting all of these results into Eq. (B17), we find that

$$I_{\mu\nu}=\int \frac{d^4k\,(k_\mu k_\nu+p_\mu p_\nu)}{(k^2-m^2)^2}-\frac{i\pi^2}{6}[p^2 g_{\mu\nu}+5p_\mu p_\nu] , \tag{B21}$$

which is a rearrangement of Eq. (B2).

Equation (B3) can be found trivially by contracting the indices of Eq. (B2). Equation (B4) can be found through judicious application of Eqs. (1.1) and (B3) as follows:

$$\int \frac{d^4k}{[(k-p)^2-m^2]}=\int \frac{d^4k\,[k^2-2p\cdot k+(p^2-m^2)]}{[(k-p)^2-m^2]^2}$$

$$=\left[\int \frac{d^4k\,(k+p)^2}{(k^2-m^2)^2}-\frac{3i\pi^2}{2}p^2\right]-2p^\tau\left[\int \frac{d^4k\,(k+p)_\tau}{(k^2-m^2)^2}-\frac{i\pi^2}{2}p_\tau\right]+(p^2-m^2)\left[\int \frac{d^4k}{(k^2-m^2)^2}\right]$$

$$=\int \frac{d^4k}{k^2-m^2}-\frac{i\pi^2}{2}p^2 . \tag{B22}$$

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[10]We reiterate that such a procedure eliminates precisely the surface term necessary for a consistent perturbative calculation of the anomaly when ω is explicitly equal to two (and when the Dirac algebra is explicitly four dimensional).
[11]This last assumption is liberally employed in Ref. 3 as well; 't Hooft and Veltman differ from us only in that they explicitly drop any surface terms (in their "partial-p" continuation operation) by *defining* all four-dimensional Feynman integrals $I(4)$ in terms of a limit approached by noninteger-dimensional Feynman integrals $I(n)$:

$$I(4) \equiv \lim_{n\to 4} I(n) \ .$$

This approach necessitates a nonfully-anticommuting γ_5 matrix, as discussed in Sec. V.
[12]A similar calculation of the chiral anomaly from shift-of-integration-variable surface terms is presented by J. C. Taylor, *Gauge Theories of Weak Interactions* (Cambridge University Press, Cambridge, 1976), p. 106.
[13]This point is also made by S. Adler, in *Lectures on Quantum Field Theory,* edited by S. Deser, M. T. Grisaru, and H. Pendleton (MIT, Cambridge, Mass., 1970), p. 1.
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[15]It is possible to calculate $S_{\mu\rho\sigma}$ directly in four dimensions without explicit use of surface terms [L. Rosenberg, Phys. Rev. 129, 2786 (1963)]. Similarly, $S_{\mu\rho\sigma}$ may be calculated in n dimensions without explicit use of γ-matrix peculiarities [S. Gottlieb and J. T. Donohue, Phys. Rev. D 20, 3378 (1979)]. The point we are making here is that a perfectly legitimate calculational route exists that will yield $q^\mu S_{\mu\rho\sigma}=0$ unless variable-shift surface terms, nonfully-anticommuting γ_5's, or noncyclic γ-matrix traces are explicitly incorporated into the calculation.
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