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Equivalence theorem and gauge theory at finite temperature

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The Feynman rules of non-Abelian gauge theory at finite temperature are studied in a real-time formulation of quantum field theory at finite temperature. In order to have a simple prescription to obtain the Feynman rules at finite temperatures, an equivalence theorem which is a straightforward generalization of the same theorem at zero temperature to finite temperature will be presented in perturbation theory.

I. INTRODUCTION

Recently several authors have tried to formulate gauge theory at finite temperature[1] and found that the statistics of Faddeev-Popov ghost fields do not obey the Fermi statistics although they are anticommuting fields.[2] In the operator formalism, it was shown that this is a result of the condition that only the physical states contribute to the trace calculation in the statistical average of a physical operator.[3]

In quantum-field-theoretical analysis at finite temperature, the temperature Green's-function formalism of Matsubara has been used extensively. In this formalism the nature of the statistics manifests itself through the periodic boundary condition in imaginary time.[4] The choice of the boundary condition for unphysical fields is determined by the requirement that the Lagrangian is invariant under the gauge transformation of gauge functions with bosonic boundary condition in imaginary time. This is a new requirement at finite temperature; it does not appear when the temperature is zero, because the time domain at zero temperature extends to infinity. In fact the Ward-Takahashi relations usually become complicated in the Matsubara formalism because the symmetry transformations are associated with the imaginary time (τ) which is confined to the finite region ($0 \leq \tau \leq \beta$) and therefore the boundary condition at the end points of the imaginary-time domain must be carefully chosen. One of the most prominent examples of this kind is given by supersymmetry theory.[5] Recall that in the usual quantum field theory at zero temperature the gauge function is required to damp reasonably fast in the infinite-time limit. Such a simple boundary condition cannot be used for finite temperature in the Matsubara formalism. In the analysis of Ref. 3 where the operator formalism is used, the physical-state condition played a crucial role in determining the statistical weight of the ghost fields. However, such an analysis requires that the physical-state condition at finite temperature must be fully formulated before one calculates the statistical average. This is usually quite an involved problem. All of these problems disappear when we can extend the usual quantum field theory with real time to consider the case at finite temperature. In this paper we present such an analysis by using the recent formulation for real-time quantum field theory at finite temperature.

Since the gauge invariance means the physical equivalence among different choices of gauge, we first study the equivalence theorem in perturbation for our study of gauge theory at finite temperature. Two field theories are said to be equivalent when both theories predict the same results for physical amplitudes. Here physical amplitudes mean matrix elements of the physical observables.

In the framework of quantum field theory at zero temperature, many equivalent theorems have been known in the past. A pseudoscalar and a pseudovector coupling are equivalent.[6] A massive Proca field is equivalent to a massive vector field with a Stueckelberg field.[7] The study of chiral symmetry disclosed the fact that nonlinear Lagrangians which are related to each other through a redefinition of the field lead to the same physical results.[8] In the gauge theory, physical matrix elements are equivalent independently of the gauge conditions. The theories with auxiliary fields frequently permit a variety of choices of the auxiliary fields without violating the physical equivalence.

In this paper we discuss how such equivalent theorems can be extended to the finite-temperature domain and then apply those results to the non-Abelian gauge theories at finite temperature. Our analysis relies on perturbation theory and thermofield dynamics (TFD) which has recently been developed as a quantum field theory at finite temperature.

In thermofield dynamics,[9] each dynamical degree of freedom is doubled. A Hilbert space is constructed as a Fock space on a thermal ground state by operating creation operators of quasiparticle modes. A formalism which comprises both the Matsubara formalism and TFD was presented in Ref. 10. There it was shown that the Feynman-type perturbative expansion in TFD is the same as a Feynman-type perturbative expansion in which the time-ordered products are replaced by the path-ordered products. Here the path-ordered products are the products ordered along a suitably chosen path on the complex time plane (the conditions for the choice of this path are presented in Ref. 10, and, therefore, are not repeated here). This path-ordered product formalism is related to the perturbative expansions in the Matsubara formalism through an analytic continuation. In this way it was proven that to *all orders* the finite-temperature real-time Green's function is the same as the one calculated via the imaginary-time expressions. The equivalence between TFD and the so-called axiomatic quantum statistical mechanics was proven in an excellent paper by Ojima.[11] One of the merits of TFD is that the operator formalism and Feynman diagram method at zero temperature are naturally extended to those at finite temperature without having to consider imaginary time. Therefore the relationship between the zero-temperature and finite-temperature situations becomes transparent and most of the relations based on symmetry requirements preserve the same forms even when at finite temperature.[12] We will use this close similarity with the zero-temperature field-theoretical structure to extend the equivalence theorems to finite temperature.

After we review TFD in the next section, we will show in Sec. III the following theorems by use of perturbation theory.

Theorem I: When the Hilbert space has no unphysical subspace which is unobservable, an equivalence theorem at zero temperature is preserved at finite temperature.

Theorem II: Even when the Hilbert space contains some unphysical part which is not observable, we can always choose the unphysical part in such a way that an equivalence theorem at zero temperature is preserved at finite temperature. This specifies the choice of statistics of unphysical fields. In the proofs, we extensively use the fact that the topological structure of the Feynman diagrams in the thermofield dynamics is exactly the same as the one at zero temperature. This means that every combinatorics rule in the Feynman diagrams in TFD is exactly the same as the one at zero temperature. Therefore, the proofs do not need any change even when a certain symmetric property of the Lagrangian conditions the combinatorics in the Feynman diagrams. In Sec. IV, we apply theorem *II* to obtain Feynman rules at finite temperature in non-Abelian gauge theories. We consider the covariant gauge as an example. Section V is devoted to concluding remarks.

II. THERMOFIELD DYNAMICS

In thermofield dynamics,[9] the statistical average of an operator A is given by the vacuum expectation value with the temperature-dependent vacuum $|0(\beta)\rangle$:

$$tr[e^{-\beta H}A]/\mathrm{tr}\,[e^{-\beta H}]=\langle 0(\beta)|A\,|0(\beta)\rangle\ . \tag{2.1}$$

To construct such a field theory requires a doubling of the field degrees of freedom. Let us suppose that the Lagrangian of a system $\mathscr{L}(x)$ is a function of a boson field $\phi(x)$ and a Dirac field $\psi(x)$. We assume that the Hamiltonian formalism exists for this $\mathscr{L}(x)$. The doubling of the field operators are performed by the tilde operation rules. The tilde operation is indicated either by the tilde symbol on operators or by the symbol "til" in front of operators: til$[O]\equiv\widetilde{O}$. The tilde operation rules are defined as

$$\text{til}\ [O_1O_2]=\widetilde{O}_1\widetilde{O}_2\ , \tag{2.2a}$$

$$\text{til}\ [C_1O_1+C_2O_2]=C_1^*\widetilde{O}_1+C_2^*\widetilde{O}_2\ , \tag{2.2b}$$

$$\widetilde{\widetilde{O}}=\eta O,\quad \begin{matrix}\eta=+1 \text{ for a bosonlike operator,}\\ \eta=-1 \text{ for a fermionlike operator .}\end{matrix} \tag{2.2c}$$

Following these tilde operation rules, we introduce the tilde fields $\widetilde{\phi}(x)$ and $\widetilde{\psi}(x)$. The thermal doublets $\phi^\alpha(x)$ and $\psi^\alpha(x)$ $(\alpha=1,2)$ are formed as

$$\begin{bmatrix}\phi^1(x)\\ \phi^2(x)\end{bmatrix}=\begin{bmatrix}\phi(x)\\ \widetilde{\phi}^{\dagger t}(x)\end{bmatrix},\quad \begin{bmatrix}\psi^1(x)\\ \psi^2(x)\end{bmatrix}=\begin{bmatrix}\psi(x)\\ C\widetilde{\overline{\psi}}^{\,t}(x)\end{bmatrix}, \tag{2.3}$$

where t indicates transpose of vectors when ϕ and ψ form column vectors. $\overline{\psi}$ is defined by $\overline{\psi}=\psi^\dagger\gamma^0$ and C is the charge conjugation matrix. The doublets (2.3) are constructed in such a manner that the first and the second components have the same transformation properties. The tilde operation applied to $\mathscr{L}(x)$ gives $\widetilde{\mathscr{L}}(x)$. The total Lagrangian is then given by

$$\hat{\mathscr{L}}(x)=\mathscr{L}(x)-\widetilde{\mathscr{L}}(x)\ . \tag{2.4}$$

By use of the thermal doublet notation ϕ^α and ψ^α with $\alpha=1$ being the first component and $\alpha=2$ being the second component in (2.3), we have

$$\hat{\mathscr{L}}(x)=\sum_\alpha \epsilon_\alpha\mathscr{L}_\alpha(x) \tag{2.5}$$

with

$$\mathscr{L}_\alpha(x)=P_\alpha\mathscr{L}(\phi^\alpha,\psi^\alpha)\ , \tag{2.6}$$

$$\epsilon_\alpha=\pm 1 \text{ for } \alpha=\{^1_2\ , \tag{2.7}$$

where the ordering operator P_α is defined for arbitrary operators $A,B,\ldots,C$:

$$P_\alpha(A^\alpha B^\alpha\cdots C^\alpha)=\begin{cases}A^1B^1\cdots C^1 \text{ for } \alpha=1,\\ C^2\cdots B^2A^2 \text{ for } \alpha=2\ .\end{cases} \tag{2.8}$$

The $\hat{\mathscr{L}}(x)$ determines the dynamics of fields.

The Feynman rules are constructed by dividing $\hat{\mathscr{L}}(x)$ into an unperturbative part and an interaction part:

$$\hat{\mathscr{L}}(x)=\hat{\mathscr{L}}_0(x)+\hat{\mathscr{L}}_I(x)\ . \tag{2.9}$$

Let us suppose that $\hat{\mathscr{L}}_0(x)$ is given by

$$\hat{\mathscr{L}}_0(x)=\sum_{\alpha=1}^{2}\{\epsilon_\alpha[\partial_\mu\phi^{\dagger\alpha}(x)\partial^\mu\phi^\alpha(x)-\mu^2\phi^{\dagger\alpha}(x)\phi^\alpha(x)] +\overline{\psi}^\alpha(x)[i\gamma^\mu\partial_\mu-m^2]\psi^\alpha(x)\}\ . \tag{2.10}$$

In general, the boson fields are obtained through the Bogoliubov transformation

$$\phi^\alpha(x)=U_B\left[\left|-i\frac{\partial}{\partial t}\right|\right]^{\alpha\gamma}\phi^\gamma_\beta(x)\ , \tag{2.11a}$$

while fermion fields are obtained through the transformation

$$\psi^\alpha(x)=U_F\left[\left|-i\frac{\partial}{\partial t}\right|\right]^{\alpha\gamma}\psi^\gamma_\beta(x)\ , \tag{2.11b}$$

where

$$U_B(\omega)=\frac{1}{(e^{\beta\omega}-1)^{1/2}}\begin{bmatrix} e^{\beta\omega/2} & 1 \\ 1 & e^{\beta\omega/2}\end{bmatrix}, \quad (2.12a)$$

$$U_F(\omega)=\frac{1}{(e^{\beta\omega}+1)^{1/2}}\begin{bmatrix} e^{\beta\omega/2} & 1 \\ -1 & e^{\beta\omega/2}\end{bmatrix}. \quad (2.12b)$$

The fields ϕ_β^α and ψ_β^α ($\alpha=1,2$) in (2.11) are diagonal with respect to the thermal ground states in the interaction picture:

$$\langle 0,\beta \mid T\phi_\beta^\alpha(x)\phi_\beta^{\dagger\gamma}(y)\mid 0,\beta\rangle = i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\Delta_0^{\alpha\gamma}(k)\ , \quad (2.13a)$$

$$\langle 0,\beta \mid T\psi_\beta^\alpha(x)\bar{\psi}_\beta^\gamma(x)\mid 0,\beta\rangle = i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}S_0^{\alpha\gamma}(k)\ , \quad (2.13b)$$

with

$$\Delta_0(k)=\tau(k^2-\mu^2+i\epsilon\tau)^{-1}\ , \quad (2.14a)$$

$$S_0(k)=(\gamma k+m)(k^2-m^2+i\epsilon\tau)^{-1}\ , \quad (2.14b)$$

$$\tau=\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}. \quad (2.15)$$

Then the boson propagators are given by

$$\langle 0,\beta \mid T\phi^\alpha(x)\phi^{\dagger\gamma}(y)\mid 0,\beta\rangle = i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\Delta^{\alpha\gamma}(k)\ , \quad (2.16)$$

where

$$\Delta(k)=U_B(\,|\,k_0\,|\,)\Delta_0(k)U_B(\,|\,k_0\,|\,) \quad (2.17a)$$

$$=\Delta_0(k)+\Delta_\beta(k)\ , \quad (2.17b)$$

with

$$\Delta_\beta^{\alpha\gamma}(k)=-2\pi i\delta(k^2-\mu^2)[f_B(\,|\,k_0\,|\,)]^{\alpha\gamma}\ , \quad (2.18)$$

$$[f_B(\omega)]=\frac{1}{e^{\beta\omega}-1}\begin{bmatrix} 1 & e^{\beta\omega/2} \\ e^{\beta\omega/2} & 1\end{bmatrix}. \quad (2.19)$$

The fermion propagators are given by

$$\langle 0,\beta \mid T\psi^\alpha(x)\bar{\psi}^\gamma(y)\mid 0,\beta\rangle = i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}S^{\alpha\gamma}(k)\ , \quad (2.20)$$

where

$$S(k)=U_F(\,|\,k_0\,|\,)S_0(k)U_F(\,|\,k_0\,|\,)^\dagger \quad (2.21a)$$

$$=S_0(k)+S_\beta(k)\ , \quad (2.21b)$$

with

$$S_\beta^{\alpha\gamma}(k)=+2\pi i\delta(k^2-m^2)(\gamma k+m)[f_F(\,|\,k_0\,|\,)]^{\alpha\gamma}\ , \quad (2.22)$$

$$[f_F(\omega)]=\frac{1}{e^{\beta\omega}+1}\begin{bmatrix} 1 & e^{\beta\omega/2} \\ e^{\beta\omega/2} & -1\end{bmatrix}. \quad (2.23)$$

The merit of the expressions (2.17b) and (2.21b) lies in the fact that the β dependence besides the β dependence of ω appears only through Δ_β and S_β; Δ_0 and S_0 have the same structure as the zero-temperature propagators.

The Feynman diagrams consist of the vertices given by $\hat{\mathscr{L}}_I(x)$ and the propagators (the internal lines) given by Eqs. (2.17) and (2.21). When one considers the renormalization, the fields in $\hat{\mathscr{L}}_0$ should be the renormalized fields and $\hat{\mathscr{L}}_I$ should contain the appropriate counterterms. When the masses in $\hat{\mathscr{L}}_0$ are chosen to be the ones defined for zero temperature, the renormalization constants should also be those defined at zero temperature. It has been shown that such counterterms are sufficient to renormalize the finite-temperature theory.[13]

When the zero-temperature masses are used, all of the temperature effects in TFD arise from the propagators Δ_β or S_β which are proportional to the δ functions restricting the four-momenta to the mass shells. In the Matsubara formalism for temperature Green's functions, a real-time propagator can be obtained by means of an analytic continuation.[14] The result corresponds to the 1-1 component of the above propagator in (2.17) or (2.21). However, one cannot perform the perturbative calculation with the Feynman diagrams consisting of such an analytic continued propagator only, since one then immediately finds that the Kubo-Martin-Schwinger (KMS) condition is upset by the higher-order corrections.[15] The doubling of the components is required in order that the analytic continuation in perturbative calculation is usable.[10]

Because of the structure of $\hat{\mathscr{L}}(x)$ in (2.4), the tilde fields and nontilde fields mix only through the thermal weights U_B and U_F. Therefore at zero temperature, the tilde and nontilde fields completely decouple; the upper components give usual amplitudes of zero temperature and lower components give their duplicate.

Another important aspect of the Feynman rules in TFD is that the topological structures of the Feynman diagrams are the same as those at zero temperature. In fact, it is easy to see from the structure of $\hat{\mathscr{L}}_I(x)$ in (2.5) that Feynman amplitudes at finite temperature are obtained from those at zero temperature by replacing $\Delta_0(k)$ by $\Delta^{\alpha\gamma}(k)$, $S_0(k)$ by $S^{\alpha\gamma}(k)$, and vertices $g(k_1,\ldots,k_n)$ by $g^\alpha(k_1,\ldots,k_n)$, where $g^\alpha(k_1,\ldots,k_n)$ is given by

$$g^\alpha(k_1,\ldots,k_n)=g(k_1,\ldots,k_n)\ \ (\text{for } \alpha=1)$$
$$=-(-)^N g^*(k_1,\ldots,k_n)\ \ (\text{for } \alpha=2) \quad (2.24)$$

with N being the number of fermion pairs associated with the vertex g. The weight number of each diagram in TFD also agrees with the one at zero temperature. These properties of the Feynman diagrams together with the fact that each internal line in the Feynman diagram of TFD consists of the temperature-independent part (Δ_0 or S_0) and the temperature-dependent one (Δ_β or S_β) with k on the mass shell constitute the basis of our proofs for the equivalent theorems.

III. EQUIVALENCE THEOREM AT FINITE TEMPERATURE

In this section we prove the equivalence theorems at finite temperature by means of the Feynman diagram method. Let us suppose that there exist two Lagrangians $\mathscr{L}_A(x)$ and $\mathscr{L}_B(x)$ to describe the same phenomena. When the physical amplitudes F_A calculated from $\mathscr{L}_A$ and F_B calculated from $\mathscr{L}_B$ are equal, we say that there exists an equivalence theorem. Here physical amplitudes mean matrix elements of the physical observables.

Assuming that an equivalence theorem holds at zero temperature, then the following theorem can be proven in perturbation theory at finite temperature.

Theorem I. When all the internal lines in the Feynman diagrams given by $\mathscr{L}_A$ and $\mathscr{L}_B$ are the free propagators of physically observable particles (so that there is no unphysical internal line), the equivalence theorem holds true also at finite temperature.

Proof of Theorem I. At zero temperature, physical amplitudes are constructed by Feynman rules as functions of propagators Δ_{ab} and vertices $g_{ab\cdots c}$, where Δ includes fermion and boson propagators. The indices $a,b,\ldots,c$ indicate possible internal degrees of freedom and polarizations. Since it is assumed that physical amplitudes at $T=0$ are equal, one has

$$F_A[\Delta_A,g_A]=F_B[\Delta_B,g_B]\ . \tag{3.1}$$

It can happen that forms of vertices g_A and g_B and propagators Δ_A and Δ_B are different. However, since there is no unphysical internal line, propagators in (3.1) are the physical internal lines. Then we can identify the same physical internal line (say, ith line) in F_A and F_B as $\Delta^{ab}_{(i)A}$ and $\Delta^{ab}_{(i)B}$, the pole structures of which are the same. When we confine a four-momentum of the ith internal lines on the mass shell in (3.1), we find that the integrands, $F_A^{(i)}$ and $F_B^{(i)}$, are physical amplitudes:

$$\begin{aligned}&\int d^4k_iF_A^{(i)}(\Delta_A,g_A;k_i)^{ab}\mathscr{P}^{ab}_{(i)A}(k_i)\delta(k_i{}^2-m_i{}^2)\\&\quad=\int d^4k_iF_B^{(i)}(\Delta_B,g_B;k_i)^{ab}\mathscr{P}^{ab}_{(i)B}(k_i)\delta(k_i{}^2-m_i{}^2)\ ,\end{aligned} \tag{3.2}$$

where $\mathscr{P}^{ab}_A$ and $\mathscr{P}^{ab}_B$ are projection operators associated with the spin and other internal degrees of freedom.

At finite temperature, the Feynman rules are modified by replacing Δ by $\Delta^{\alpha\gamma}$ and g by g^{α}. The topological structures and the weight numbers of the diagrams are not modified. Note that $\Delta^{\alpha\gamma}$ is given by $\Delta^{\alpha\gamma}=\Delta_0^{\alpha\gamma}+\Delta_\beta^{\alpha\gamma}$ as in (2.17) or (2.21). Since the mixing of the $\alpha=1$ and 2 components occur only through Δ_β, we can expand the finite-temperature amplitudes $\hat F$ functionally in terms of Δ_β:

$$\begin{aligned}\hat F&=\hat F[\Delta_0+\Delta_\beta,g]\\&=\sum_n\int d^4q_1\cdots d^4q_nF_n(\Delta_0,g;q_1,\ldots,q_n)_{i_1\cdots i_n}^{\alpha_1\gamma_1\cdots\alpha_n\gamma_n}\Delta_{\beta i_1}^{\alpha_1\gamma_1}(q_1)\cdots\Delta_{\beta i_n}^{\alpha_n\gamma_n}(q_n)\ ,\end{aligned} \tag{3.3}$$

in which the summation $\sum_n$ is interpreted to include all possible combinations of various kinds of propagators specified by internal degrees of freedom i_k $(1\le k\le n)$. Since $\Delta_\beta(q)$ restricts the integration to the mass shell [see (2.18) and (2.22)] with appropriate projection operator $\mathscr{P}^{ab}_A$ or $\mathscr{P}^{ab}_B$, the coefficients F_n in (3.3) consist of combinations of the physical amplitudes. Since Δ_0 is the same propagator as the one at zero temperature, the coefficients in (3.3) are given by amplitudes at zero temperature and are identical in theories of $\mathscr{L}_A$ and $\mathscr{L}_B$. Therefore, we have

$$\hat F_A[\Delta_A,g_A]=\hat F_B[\Delta_B,g_B] \tag{3.4}$$

at finite temperature. (Q.E.D.)

In the above discussion we considered the matrix elements of an operator among the physical particle states. When we consider the S matrix, we must replace in the above argument every external propagator by the free particle wave function with the zero-temperature renormalized mass. Note that the wave-function renormalization constant is also the one determined at zero temperature.

When there exist certain unphysical fields in a theory, it sometimes happens that the structure of the propagators of those fields at finite temperature are not obvious. (For example, we have no criterion to determine the statistics of double poles.) When a canonical formalism exists including unphysical fields and conditions for physical observables are specified, the treatment of unphysical fields are well defined in the operator formalism of TFD as was illustrated by the gauge theory in Ref. 11. Without going into the detail of these physical conditions of the Fock space, we can prove the following theorem by means of the Feynman rules.

Theorem II: When $\mathscr{L}_A$ and/or $\mathscr{L}_B$ include certain unphysical fields and are physically equivalent to each other at zero temperature, one can preserve the equivalence theorem at finite temperature by choosing propagators (i.e., the internal lines) of unphysical fields suitably.

Proof of Theorem II. We first point out that the doubling of freedom of fields is applied also to unphysical fields. Our question now is to ask how propagators of unphysical fields should be determined at finite temperature in order that the equivalence theorem is satisfied. The equivalence theorem at zero temperature means that at zero temperature the physical amplitude F_A is equal to F_B. Let us consider all Feynman diagrams calculated to the same order in perturbation theory. Pick up from one diagram an internal line corresponding to a field φ^i and from the other diagram an internal line corresponding to a field φ^j. When these two internal lines have a common pole in the momentum plane and the same values for all the other internal degrees of freedom, we say that these two fields belong to a two-member set. One can enlarge this set in such a way that any two-member set of a multimember set $(\varphi^a,\varphi^b,\ldots,\varphi^c)$ has a common element belonging to another two-member set of this multimember set. For example, in the R_ξ gauge[16] of the spontaneous-breakdown gauge theories, a massive gauge field and an unphysical Higgs scalar field form a two-member set and an unphysical Higgs scalar field and a ghost field form another two-member set then those three are elements of a multimember set. On the other hand, the auxiliary field which appears in the Gross-Neveu model[17] does not have a pole structure. Such an auxiliary field does not belong to any multimember set and does not change the form of its propagator even at finite temperature, as will be shown later. There are two kinds of multimember sets; class I, which includes at least one physical field (i.e., fields with a physical pole) and class II, which does not include any physical field. The equivalence theorem at $T=0$ implies that, corresponding to each multimember set of class I for $\mathscr{L}_B$, there exists a multimember set of class I for $\mathscr{L}_A$; they give rise to the same physical result and they contain the same physical fields. The equivalence theorem at $T=0$ implies also that the pole contribution to a physical amplitude from the members of the multimember set of class II compensates among themselves. When the four-momentum of one internal line belonging to the same class is put on the mass shell in the Feynman diagrams, we have the integration over the on-shell states

$$\sum_i \int d^4k\, F^{(i)}(\Delta,g;k_i)^{ab}\mathscr{P}^{ab}_{(i)}(k_i)\delta(k_i{}^2-m_i{}^2)\ , \qquad (3.5)$$

where the summation $\sum_i$ is over the same class and $\mathscr{P}_i^{ab}(k)$ is the projection operator associated with the spin and other internal degrees of freedom. The physical equivalence at $T=0$ means that only the physical projection remains after the summation. Therefore when the on-shell amplitudes with the projection operators belonging to the same class are summed over as in the zero-temperature case, only a physical amplitude remains. Similar to the proof in theorem I, we separate all propagators as $\Delta_0+\Delta_\beta$ in which the temperature dependence is included only in Δ_β, then we expand the Feynman amplitude as a functional of Δ_β. It is now clear that we have an equivalence theorem at finite temperature when every member of a set obeys the same statistics. The contribution from class I leaves only the physical amplitudes at $T=0$ and the contributions from the fields in class II cancel. Thus the statistics of an unphysical field which belongs to a set of class I is the same as the statistics of a physical field in the same set, while the statistics of an unphysical field which belongs a set of class II is arbitrary. (Q.E.D.)

The above theorem leads to a simple rule for construction of Feynman propagators at finite temperature when those at zero temperature are known. Namely, when a zero-temperature propagator is given by

$$\Delta^{ab}(k)=\mathscr{P}^{ab}(k,k^2+i\epsilon)/(k^2-m^2+i\epsilon)\ , \qquad (3.6)$$

its finite-temperature extension is given, depending on whether this field shares the same set with physical bosons or fermions, by

$$\Delta^{ab}(k)=U_B(|k_0|)\{\tau\mathscr{P}^{ab}(k,k^2+i\epsilon\tau)/(k^2-m^2+i\epsilon\tau)\}U_B(|k_0|) \qquad (3.7a)$$

or

$$\Delta^{ab}(k)=U_F(|k_0|)\{\mathscr{P}^{ab}(k,k^2+i\epsilon\tau)/(k^2-m^2+i\epsilon\tau)\}U_F(|k_0|)^\dagger\ , \qquad (3.7b)$$

respectively.

In the non-Abelian gauge theories, we have the equivalence theorem between all the various choices of gauge conditions. When this equivalence theorem is considered as a first principle at zero temperature, we can determine by means of theorem II the statistics of unphysical fields which appear due to the choice of the gauge condition. The immediate consequence of theorem II is that the Faddeev-Popov ghost fields should obey Bose statistics, since they belong to the same class as the gauge fields.

IV. FEYNMAN RULES AT FINITE TEMPERATURE IN THE COVARIANT GAUGE

We are now ready to derive the Feynman rules for non-Abelian gauge theories through the following procedure. First, the Lagrangian is constructed in the manner of Faddeev for a given gauge condition and the doubling of field operators are performed according to the tilde operation rule. Second, the propagators at finite temperature $\Delta^{\alpha\gamma}=\Delta_0^{\alpha\gamma}+\Delta_\beta^{\alpha\gamma}$ are obtained by the following rules. The temperature-independent parts $\Delta_0^{\alpha\gamma}$ are given by duplicating the zero temperature ones; one for the nontilde fields and another for the tilde fields. The temperature dependent parts $\Delta_\beta^{\alpha\gamma}$ consist of the δ functions which confine the four-momentum k_μ to the mass shell, the projection operators for a given gauge, and the temperature-dependent weight functions. When the gauge condition introduces the ghost fields such as the Faddeev-Popov ghost, the method for construction of the projection operator in the temperature-dependent part is supplied by theorem II in the last section. This will be illustrated in the following discussion.

As an example, let us consider the following Lagrangian with the covariant gauge:

$$\mathscr{L}(x)=-\tfrac{1}{4}F_{\mu\nu}(x)\cdot F^{\mu\nu}(x)+\partial_\mu\bar{C}(x)\cdot D^\mu(A)C(x) -\frac{1}{2\alpha}[\partial_\mu A^\mu(x)]^2\ . \qquad (4.1)$$

Here

$$F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x) -gA_\mu(x)\times A_\nu(x)\ , \qquad (4.2)$$

$$D_\mu(A)C(x)=\partial_\mu C(x)-gA_\mu(x)\times C(x)\ , \qquad (4.3)$$

in which abbreviations

$$A\cdot B=A^aB^a,\quad (A\times B)^a=f^{abc}A^bB^c \qquad (4.4)$$

are used for vectors on the gauge group. The set $\{f^{abc}\}$ denotes that of the structure constants of the group. The fields $C(x)$ and $\bar{C}(x)$ are the Fadeev-Popov ghost fields. The thermal Lagrangian is given by

$$\hat{\mathscr{L}}(x) = -\tfrac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu} + \partial_\mu \bar{C}\cdot D^\mu(A)C - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \tfrac{1}{4}\tilde{F}_{\mu\nu}\cdot\tilde{F}^{\mu\nu} - \partial_\mu\tilde{\bar{C}}\cdot D^\mu(\tilde{A})\tilde{C} + \frac{1}{2\alpha}(\partial_\mu\tilde{A}^\mu)^2 . \quad (4.5)$$

We form the thermal doublet as

$$A_\mu^\alpha(x) = \begin{cases} A_\mu(x) & (\alpha=1) \\ \tilde{A}_\mu(x) & (\alpha=2) , \end{cases} \quad (4.6)$$

$$C^\alpha(x) = \begin{cases} C(x) & (\alpha=1) \\ \tilde{C}(x) & (\alpha=2) , \end{cases}$$

$$\bar{C}^\alpha(x) = \begin{cases} \bar{C}(x) & (\alpha=1) \\ \tilde{\bar{C}}(x) & (\alpha=2) . \end{cases} \quad (4.7)$$

As was pointed out in Ref. 18, C and $\bar{C}$ should not be Hermitian conjugates of each other when the Hermiticity of $\mathscr{L}(x)$ is required. Rather they should satisfy

$$C^\dagger(x) = C(x), \quad \bar{C}^\dagger(x) = -\bar{C}(x) , \quad (4.8)$$

although $C(x)$ and $\bar{C}(x)$ are mutually canonical conjugate:

$$\left\{ \frac{\partial}{\partial t} C(\vec{x},t), \bar{C}(\vec{y},t) \right\} = -i\delta(\vec{x}-\vec{y}) . \quad (4.9)$$

Therefore there exists a certain ambiguity in the formation of the thermal doublet of Faddeev-Popov ghost fields. The choice (4.7) gives

$$\left\{ \frac{\partial}{\partial t} C^\alpha(\vec{x},t), \bar{C}^\gamma(\vec{y},t) \right\} = -i\tau^{\alpha\gamma}\delta(\vec{x}-\vec{y}) , \quad (4.10)$$

while the choice

$$\bar{C}^\alpha(x) = \begin{cases} \bar{C}(x) & (\alpha=1) \\ -\tilde{\bar{C}}(x) & (\alpha=2) , \end{cases} \quad (4.11)$$

leads to

$$\left\{ \frac{\partial}{\partial t} C^\alpha(\vec{x},t), \bar{C}^\gamma(\vec{y},t) \right\} = -i\delta^{\alpha\gamma}\delta(\vec{x}-\vec{y}) . \quad (4.12)$$

The thermal Bogoliubov transformation (2.11) keeping (4.10) is bosonlike, but it is fermionlike in the case (4.12). Theorem II in Sec. III indicates that we should choose the bosonlike property which leads to the choice (4.7).

Now the thermal Lagrangian is given by

$$\hat{\mathscr{L}}(x) = \hat{\mathscr{L}}_0(x) + \hat{\mathscr{L}}_I(x) , \quad (4.13)$$

with

$$\hat{\mathscr{L}}_0(x) = \sum_\alpha \epsilon_\alpha \left[-\tfrac{1}{4}(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)^2 - \frac{1}{2\alpha}(\partial^\mu A_\mu^\alpha)^2 + \partial_\mu \bar{C}^\alpha \cdot \partial^\mu C^\alpha \right] , \quad (4.14)$$

$$\hat{\mathscr{L}}_I(x) = \sum_\alpha \epsilon_\alpha [\tfrac{1}{2}g(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)\cdot(A^{\mu\alpha}\times A^{\nu\alpha}) - \tfrac{1}{4}g^2(A_\mu^\alpha \times A_\nu^\alpha)\cdot(A^{\mu\alpha}\times A^{\nu\alpha}) - g\partial_\mu\bar{C}^\alpha\cdot(A^{\mu\alpha}\times C^\alpha)] . \quad (4.15)$$

The Lagrangian (4.13) leads to the following Feynman rules.

(a) Gauge-boson propagators

$$\langle 0,\beta | T A_\mu^{a\alpha}(x) A_\nu^{b\gamma}(y) | 0,\beta \rangle = i\delta^{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta_{\mu\nu}^{\alpha\gamma}(k,\alpha) , \quad (4.16)$$

where

$$\Delta_{\mu\nu}(k,\alpha) = U_B(|k_0|)\Delta_{0\mu\nu}(k,\alpha)U_B(|k_0|) \quad (4.17a)$$

$$= \Delta_{0\mu\nu}(k,\alpha) + \Delta_{\beta\mu\nu}(k,\alpha) , \quad (4.17b)$$

$$\Delta_{0\mu\nu}(k,\alpha) = \tau \frac{-g_{\mu\nu} + (1-\alpha)k_\mu k_\nu/(k^2+i\epsilon\tau)}{k^2+i\epsilon\tau} , \quad (4.18)$$

$$\Delta_{\beta\mu\nu}(k,\alpha) = -2\pi i\delta(k^2)\left[-g_{\mu\nu} + (1-\alpha)k_\mu k_\nu Pr\frac{2}{k^2} \right] \frac{1}{e^{\beta|k_0|}-1} \begin{bmatrix} e^{\beta|k_0|/2} & 1 \\ 1 & e^{\beta|k_0|/2} \end{bmatrix} \text{(Ref. 19)} . \quad (4.19)$$

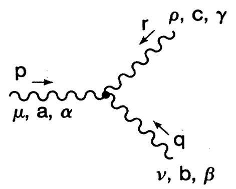
(b) Ghost propagators

$$\langle 0,\beta | T C^{a\alpha}(x) \bar{C}^{b\gamma}(y) | 0,\beta \rangle = i\delta^{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta^{\alpha\gamma}(k) \quad (4.20)$$

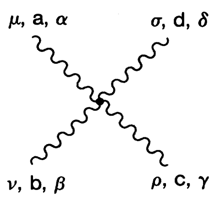
with

$$\Delta(k)=U_B(|k_0|)\Delta_0(k)U_B(|k_0|)$$
$$=\Delta_0(k)+\Delta_\beta(k)\ , \tag{4.21}$$
$$\Delta_0(k)=\tau[k^2+i\epsilon\tau]^{-1}\ , \tag{4.22}$$
$$\Delta_\beta(k)=-2\pi i\delta(k^2)\frac{1}{e^{\beta|k_0|}-1}\begin{bmatrix} 1 & e^{\beta|k_0|/2} \\ e^{\beta|k_0|/2} & 1 \end{bmatrix}\ . \tag{4.23}$$

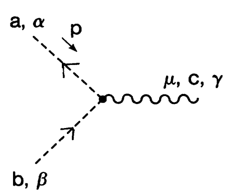
(c) Vertices



$$=-ig\epsilon_\alpha\delta^{\alpha\beta\gamma}f^{abc}[(q-r)_\mu g_{\nu\rho}+(r-p)_\nu g_{\rho\mu}+(p-q)_\rho g_{\mu\nu}]\ , \tag{4.24}$$



$$=-\epsilon_\alpha\delta^{\alpha\beta\gamma\delta}g^2[f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})+f^{ace}f^{bde}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho})+f_{ade}f_{bce}(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\rho}g_{\nu\sigma})]\ , \tag{4.25}$$



$$=-ig\epsilon_\alpha\delta^{\alpha\beta\gamma}f^{abc}p_\mu\ . \tag{4.26}$$

Here we use the notation $\delta^{\alpha\beta\gamma\cdots}=1$ when $\alpha=\beta=\gamma=\cdots$ and $\delta^{\alpha\beta\gamma\cdots}=0$ otherwise. These are the Feynman rules at the finite temperature of the non-Abelian gauge theory with the covariant guage. Explicit demonstrations of the gauge invariance at the finite-temperature calculation will be presented elsewhere.

V. CONCLUDING REMARKS

We showed by the use of perturbation theory that, when certain equivalence theorems hold at zero temperature, the same equivalence theorems hold true also at finite temperature. The proof is greatly simplified with the help of thermofield dynamics. The structure of the free field causal propagator at finite temperature in the thermofield dynamics plays a key role in the proof; this propagator can be written as a sum of a temperature-independent part and a temperature-dependent one, the four-momentum of the latter part being confined to the mass shell. This particular structure of the Feynman internal line at finite temperature simplifies the analysis in perturbation theory of the temperature dependence of quantum field systems when thermofield dynamics is employed. The proof of the equivalence theorem is an example of this kind of analysis.

When a theory contains some auxiliary unphysical fields, we can determine their propagators and statistics by requiring that the equivalence theorem which holds at zero temperature should also be true at finite temperature. This has many applications. In this paper, we applied this method to gauge theories with covariant gauge and determined the propagators and statistics of the Faddeev-Popov ghost fields. The equivalence theorem discussed in this paper illustrates the general fact that most of the relations which originate from the symmetric nature of the Lagrangian preserve the same form at finite temperature when thermofield dynamics is employed.

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[19]In Eq. (4.19), we use the following definition:

$$-2\pi i\delta(k^2)F(k)Pr\frac{2}{k^2}=\lim_{\epsilon\to 0}\left[\frac{1}{(k^2+i\epsilon)^2}-\frac{1}{(k^2-i\epsilon)^2}\right]F(k)$$

for an arbitrary function $F(k)$. There is some arbitrariness in the term proportional to $k_\mu k_\nu$ since such a term does not contribute at $k^2=0$ due to the Ward-Takahashi relation for the longitudinal part of the gauge bosons. We have chosen $\Delta_{\beta\mu\nu}(k,\alpha)$ in order that $\Delta_{\mu\nu}(k,\alpha)$ reduces to a compact form as in (4.17a).