Functional-integral approach to Parisi-Wu stochastic quantization: Scalar theory

E. Gozzi
Physics Department, City College of C.U.N.Y., New York, New York 10031
(Received 23 June 1983)

The 5th-time stochastic-quantization approach to field theory, recently proposed by Parisi and Wu, is put in a path-integral form. The procedure of taking the limit $\tau \rightarrow \infty$ is analyzed and based on new grounds through the introduction of the vacuum-vacuum generating functional. Different aspects of the interplay between forward and backward Fokker-Planck dynamics are studied in detail in connection with the supersymmetry recently discovered in Gaussian stochastic processes.

INTRODUCTION

Recently, Parisi and Wu[1] proposed a new and interesting method for the quantization of physical systems. The idea was to introduce a 5th time τ and postulate a stochastic Langevin dynamics for the system. Those authors[1] showed that, at least at the perturbative level, the usual quantum theory is reproduced by the equilibrium limit $\tau \rightarrow \infty$ of that dynamics. The first advantage of this method is the possibility to quantize gauge theories without fixing the gauge, and much work[1,2] has already been done in this direction. Another recent application is the use of the Langevin equation for the computer simulation of lattice-field-theory models[3]: simulation that should have better properties than the usual Monte Carlo one. The third application,[4] and we hope not the last, is a better and deeper understanding of the so-called quenched reduced models.

In view of all these connections and hoping for more to come, we give in this paper a functional-integral reformulation of this new method of quantization. The hope is to be able to use all the techniques developed in recent years in path integration and bring out the rich content, still undiscovered, in this approach of Parisi and Wu.

The paper is organized as follows: in Sec. I we briefly review the work of Parisi and Wu. In Sec. II, starting from the Langevin equation, we derive the corresponding generating functional. In Sec. III we impose on the Langevin equation to describe a stationary process, and the procedure of taking the limit $\tau \rightarrow \infty$ is put on a new basis through the introduction of the vacuum-vacuum generating functional. In Sec. IV we make contact with the recently discovered hidden supersymmetry in stochastic Gaussian processes, and analyze in detail the nice interplay of forward and backward Fokker-Planck dynamics present in the supersymmetric form of the generating functional.

In this work we limit ourselves to scalar theories without any internal symmetry.

I. REVIEW OF PARISI-WU STOCHASTIC QUANTIZATION

We all know that the "quantum" correlation functions for a Euclidean system, described by an action $S[\phi]$, are given by

$$\langle 0 | T\phi(x_1)\phi(x_2)\cdots\phi(x_l) | 0\rangle = \frac{\int \mathscr{D}\phi[\phi(x_1)\cdots\phi(x_l)]e^{-S[\phi]}}{\int \mathscr{D}\phi e^{-S[\phi]}} . \tag{1}$$

Parisi and Wu[1] proposed the following alternative method to get the quantum averages:

(i) Introduce a 5th time τ, in addition to the usual four-space-time x^μ, and postulate the following Langevin equation for the dynamics of the field ϕ in this extra time τ:

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S[\phi]}{\delta\phi} + \eta(x,\tau) . \tag{2}$$

η is a Gaussian random variable,

$$\begin{aligned} &\langle \eta(x,\tau)\rangle_\eta = 0 , \\ &\langle \eta(x,\tau)\eta(x'\tau')\rangle_\eta = 2\delta(x-x')\delta(\tau-\tau') , \\ &\langle \eta\cdots\eta\rangle_\eta = 0 . \end{aligned} \tag{3}$$

The angular brackets denote connected average with respect to the random variable η.

(ii) Evaluate the stochastic average of fields ϕ_η satisfying Eq. (2), that means

$$\langle \phi_\eta(x_1\tau_1)\phi_\eta(x_2\tau_2)\cdots\phi_\eta(x_l\tau_l)\rangle_\eta . \tag{4}$$

(iii) Put $\tau_1=\tau_2=\cdots=\tau_l$ in (4) and take the limit $\tau_1 \rightarrow \infty$.

It is possible to prove,[1] at least perturbatively, that

$$\lim_{\tau_1\rightarrow\infty} \langle \phi_\eta(x_1\tau_1)\phi_\eta(x_2\tau_1)\cdots\phi_\eta(x_l\tau_1)\rangle_\eta = \frac{\int \mathscr{D}\phi[\phi(x_1)\cdots\phi(x_l)]e^{-S[\phi]}}{\int \mathscr{D}\phi e^{-S[\phi]}} . \tag{5}$$

To understand this relation we have to introduce the notion of probability $P(\phi,\tau)$, that is, the probability of having the system in the configuration ϕ at time τ. There exists for $P(\phi,\tau)$ an equation that describes its evolution in the time τ. It is called the Fokker-Planck (FP) equation and it has been derived many times in the literature[5]:

$$\frac{\partial P}{\partial \tau}=\frac{\partial^2 P}{\partial \phi^2}+\frac{\partial}{\partial \phi}\left[P\frac{\partial S}{\partial \phi}\right] . \tag{6}$$

It is possible to recast this equation in a Schrödinger-type form:

$$\frac{\partial \Psi}{\partial \tau}=-2\hat{H}^{\mathrm{FP}}\Psi , \tag{7}$$

where

$$\Psi \equiv P(\phi,\tau)e^{S[\phi]/2}$$

and

$$\hat{H}^{\mathrm{FP}}\equiv -\frac{1}{2}\frac{\delta^2}{\delta\phi^2}+\frac{1}{8}\left[\frac{\partial S}{\partial \phi}\right]^2-\frac{1}{4}\frac{\partial^2 S}{\partial \phi^2} .$$

Because of this form, we call $\hat{H}^{\mathrm{FP}}$ the Fokker-Planck Hamiltonian. It is a positive semi-definite operator $\hat{H}^{\mathrm{FP}}\Psi_n=E_n\Psi_n$, $E_n\geq 0$ whose ground state $E_0=0$ is $\Psi_0=e^{-S[\phi]/2}$. The solution of (7) is

$$\Psi(\phi,\tau)=\sum_{\eta} c_n\Psi_n e^{-2E_n\tau} ,$$

where c_n are normalizing constants. The probability can be written as

$$P(\phi,\tau)=e^{-S[\phi]/2}\sum_n c_n\Psi_n e^{-2E_n\tau}$$

In the limit $\tau\to\infty$ the only term that does not disappear in this expression is Ψ_0, so we have

$$\lim_{\tau\to\infty}P(\phi,\tau)=c_0e^{-S(\phi)/2}e^{-S(\phi)/2}=c_0e^{-S(\phi)} . \tag{8}$$

This is the reason why (5) holds.

II. FUNCTIONAL-INTEGRAL APPROACH TO STOCHASTIC QUANTIZATION

In this section we would like to reformulate the Parisi-Wu method in a path-integral form.

We want to build a generating functional (that we will call $Z^{\mathrm{FP}}[J]$ for the Fokker-Planck generating functional) from which the correlations (4) can be derived in the usual fashion:

$$\langle \phi_\eta(x_1\tau_1)\cdots\phi_\eta(x_l\tau_l)\rangle_\eta=\left.\frac{\delta^l Z^{\mathrm{FP}}[J]}{\delta J(x_1\tau_0)\cdots\delta J(x_l\tau_l)}\right|_{J=0} .$$

This can be easily done retracing steps (i), (ii), and (iii) of Sec. I. $Z^{\mathrm{FP}}[J]$ becomes

$$Z^{\mathrm{FP}}[J]=\mathcal{N}\int\tilde{\mathcal{D}}\phi\tilde{\mathcal{D}}\eta P(\phi(0))\delta(\phi-\phi_\eta)\exp\left[-\int_0^\tau J\phi\, d\tau'\right]\exp\left[-\int_0^\tau\frac{\eta^2}{4}d\tau'\right] . \tag{9}$$

ϕ_η that appears in (9) is the solution of the Langevin equation (2), solved with some initial probability $P(\phi(0))$; $\mathcal{N}$ is a normalizing constant and $\tilde{\mathcal{D}}\phi=\lim_{N\to\infty}\prod_{i=0}^{N}\mathcal{D}\phi_{\tau_i}$ where ϕ_{τ_i} are the field configurations at the time τ_i, having sliced the interval 0 to τ in N infinitesimal parts ϵ with $\tau_i=i\epsilon$. This measure is a product of the usual four-dimensional path-integral measures. The $\delta(\phi-\phi_\eta)$ in (9) is a "formal" expression that we can write as

$$\delta(\phi-\phi_\eta)=\delta\left[\dot{\phi}+\frac{\partial S}{\partial\phi}-\eta\right]\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right| , \tag{10}$$

where $||\delta\eta/\delta\phi||$ is the Jacobian of the transformation $\eta\to\phi$, that is,

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right|=\det\left[\left[\partial_\tau+\frac{\partial^2 S}{\partial\phi(\tau)\delta\phi(\tau')}\right]\delta(\tau-\tau')\right] . \tag{11}$$

With well-known manipulations we can write this as

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right|=\exp\left[\mathrm{tr}\ln\left[\partial_\tau+\frac{\partial^2 S}{\partial\phi(\tau)\delta\phi(\tau')}\right]\delta(\tau-\tau')\right]$$
$$=\exp\left[\mathrm{tr}\ln\partial_\tau\left[\delta(\tau-\tau')+\partial_{\tau'}{}^{-1}\frac{\partial^2 S}{\partial\phi(\tau)\delta\phi(\tau')}\right]\right] ,$$

where $(\partial_{\tau'})^{-1}$ is just to indicate the Green's function $G(\tau-\tau')$ that satisfies

$$\partial_\tau G(\tau-\tau')=\delta(\tau-\tau') . \tag{12}$$

The solutions are $G(\tau-\tau')=\theta(\tau-\tau')$ if we choose propagation forward in time, or $G(\tau-\tau')=-\theta(\tau'-\tau)$ for propagation backward in time. It is also possible to choose $G(\tau-\tau')=\frac{1}{2}[\theta(\tau-\tau')-\theta(\tau'-\tau)]$ but we will concentrate on the first two. In the first case, propagation forward in time (that is, the one chosen by the Parisi and Wu), we get

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right|=\exp\left\{\mathrm{tr}\left[\ln\partial_\tau+\ln\left[\delta(\tau-\tau')+\theta(\tau-\tau')\frac{\partial^2 S}{\partial\phi(\tau)\delta\phi(\tau')}\right]\right]\right\}$$
$$=\exp(\mathrm{tr}\ln\partial_\tau)\exp\left[\mathrm{tr}\ln\left[\delta(\tau-\tau')+\theta(\tau-\tau')\frac{\delta^2 S}{\delta\phi(\tau)\delta\phi(\tau')}\right]\right] . \tag{13}$$

The term $\exp(\operatorname{tr}\ln\partial_\tau)$ can be dropped, as it cancels with the same term in the denominator of (9), once we normalize $\hat{Z}[J]=Z[J]/Z[0]$. So in (13) we are left with

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right|=\exp\left[\operatorname{tr}\ln\left[\delta(\tau-\tau')+\theta(\tau-\tau')\frac{\delta^2 S}{\delta\phi(\tau)\delta\phi(\tau')}\right]\right].$$

Doing the usual expansion for the ln, we obtain

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right|=\exp\left[\operatorname{tr}\left[\theta(\tau-\tau')\frac{\delta^2 S}{\delta\phi(\tau)\delta\phi(\tau')}+\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\partial^2 S}{\delta\phi(\tau)\delta\phi(\tau')}\frac{\partial^2 S}{\delta\phi(\tau')\delta\phi(\tau'')}+\cdots\right]\right]$$

$$=\exp\left[\int d\tau\,\theta(0)\frac{\delta^2 S}{\delta\phi^2(\tau)}+\int d\tau' d\tau\,\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\partial^2 S}{\partial\phi(\tau)\delta\phi(\tau')}\frac{\delta^2 S}{\delta\phi(\tau')\delta\phi(\tau)}+\cdots\right].$$

The second term in this expression is zero because $\theta(\tau-\tau')\theta(\tau'-\tau)=0$ and the same for all the subsequent terms. The only one left is the first term and choosing $\theta(0)=\frac{1}{2}$ (Ref. 6) we get

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right|=\exp\left[\frac{1}{2}\int_0^\tau d\tau'\frac{\partial^2 S}{\partial\phi^2}\right]. \tag{14}$$

Inserting (14) and (10) back into (9) and performing the η integration, we have

$$Z^{\text{FP}}[J]=\int\tilde{\mathscr{D}}\phi P(\phi(0))\exp\left\{-\int_0^\tau\left[\frac{1}{4}\left[\dot{\phi}+\frac{\partial S}{\partial\phi}\right]^2-\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]d\tau'-\int_0^\tau J\phi\,d\tau'\right\}. \tag{15}$$

If we want also to specify that we are interested only in the correlations at the same 5th time τ_1, we have just to choose $J(x,\tau')$ of the form $J(x,\tau')=\tilde{J}(x)\delta(\tau'-\tau_1)$, $\tau_1<\tau$.

Expression (15) then becomes

$$Z^{\text{FP}}[J]=\int\tilde{\mathscr{D}}\phi P(\phi(0))\exp\left\{-\int_0^\tau\left[\frac{1}{4}\left[\dot{\phi}+\frac{\partial S}{\partial\phi}\right]^2-\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]d\tau'-\tilde{J}(x\tau_1)\phi(x\tau_1)\right\}.$$

In all this we have to remember, of course, that once we send $\tau_1\to\infty$ we have also to extend the interval of integration from $\int_0^\tau$ to $\int_0^\infty$. In (15) we are neglecting the normalizing constant $\mathscr{N}$ and all the usual four-space integration. It is easy, anyhow, to reinstate them when necessary. The Lagrangian in the exponent of (15), which we call the FP Lagrangian, does not seem to have any relation to the Hamiltonian in (7). It is easy, anyhow, to see the connection: let us first, in the action of (15), perform the integration of the term $\int_0^\tau\frac{1}{2}\dot{\phi}(\partial S/\partial\phi)d\tau'=\frac{1}{2}[S(\phi(\tau))-S(\phi(0))]$ (Ref. 7) and second, let us rescale the time $\tau'\to\tau'/2$, so that we get

$$Z^{\text{FP}}[J]=\int\mathscr{D}\phi(0)P(\phi(0))e^{+S(\phi(0))/2}\mathscr{D}(\phi(2\tau))e^{-S(\phi(2\tau))/2}\tilde{\mathscr{D}}''\phi\exp\left[-\int_0^{2\tau}\mathscr{L}^{\text{FP}}d\tau'-\int_0^{2\tau}J\phi\,d\tau'\right], \tag{16}$$

where

$$\tilde{\mathscr{D}}''\phi=\lim_{N\to\infty}\prod_{i=1}^{N-1}\mathscr{D}\phi_{\tau_i}$$

and

$$\int_0^{2\tau}\mathscr{L}^{\text{FP}}d\tau'=\int_0^{2\tau}\left[\dot{\phi}^2/2+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2-\frac{1}{4}\frac{\partial^2 S}{\partial\phi^2}\right]d\tau'.$$

Now it is clear why we called $\mathscr{L}^{\text{FP}}$ the Fokker-Planck Lagrangian. It is a sort of "Euclidean" Lagrangian for the Hamiltonian $\hat{H}^{\text{FP}}$ in (7), and for this reason we also call Z^{FP} the generating functional of the Fokker-Planck dynamics.

If in (13) we had made the choice $G(\tau-\tau')=-\theta(\tau'-\tau)$, then the action in (17) would have been

$$\int_0^{2\tau}\mathscr{L}^{\text{FP}}d\tau'=\int_0^{2\tau}\left[\frac{1}{2}\dot{\phi}^2+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2+\frac{1}{4}\left[\frac{\partial^2 S}{\partial\phi^2}\right]\right]d\tau'.$$

The only difference from (17) is in the sign of the third term. The corresponding Hamiltonian has been known in the literature for a long time as the Kolmogoroff-Fokker-Planck backward Hamiltonian:

$$\hat{H}^{\text{FP}}_{\text{backward}}=-\frac{1}{2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2+\frac{1}{4}\frac{\partial^2 S}{\partial\phi^2}. \tag{17a}$$

Going back to the derivation of (15) we want to stress what has been done: We have integrated away the η and replaced the role it plays in the Langevin equation with a sort of "effective" action $\mathscr{L}^{\text{FP}}$. This Lagrangian might look very complicated but it contains only the field ϕ as a dynamical variable (Fokker-Planck dynamics). In the Langevin equation the dynamical variables were both ϕ and η on the same ground and they were interlocked in a dynamics (Langevin dynamics) that only apparently looked simpler. With the generating functional (15) we can, of course, develop perturbation theory using the Feynman rules dictated by $\mathscr{L}^{FP}$. This perturbation theory is the parallel of the one[1] that has been developed starting from the Langevin equation. Differently from that we do not, anyhow, have to integrate over the η at the level of Feynman graphs, as this is already done at the level of Z^{FP}. The number of graphs is very large in both approaches: In our case the high number comes from the extra vertices contained in $\frac{1}{8}(\partial S/\partial\phi)^2-\frac{1}{4}\partial^2 S/\partial\phi^2$.

Before concluding this section we want to make a remark concerning the Jacobian (11). All the steps from (9) to (16), that we have done to derive the generating function Z^{FP}, are possible only if the Jacobian is not identically zero. If this happens the Z^{FP} itself is zero. The same Langevin equation, starting point of the stochastic quantization, loses all its meaning. In fact, $\|\delta\eta/\delta\phi\|=0$ means that there is no field associated, through (2), to a particular η. In technical language this can be expressed by saying that the winding number of the transformation $\eta\rightarrow\phi$ is zero. This number called Δ has been studied in great detail in Ref. 9 in connection with supersymmetry and it is known as the Witten index. Our conclusion is that, in case $\Delta=0$, the stochastic quantization does not hold any more. In this case, anyhow, the same traditional method of quantization [given by (1)] does not hold. In fact, it has been shown in Ref. 9 that $\Delta=0$ implies non-normalizability for $e^{-S/2}$; that means the "quantum" probability e^{-S} cannot be used in (1) any more.

III. VACUUM-VACUUM GENERATING FUNCTIONAL

Of the random process (3), we have used, up to now, only the property that it is Gaussian. The action (15) that we have obtained is a consequence of this. Besides this property there is another very interesting one: the stochastic process (3) is stationary. By stationary[10] we mean a process whose "momenta" $c(\tau_1\tau_2\cdots\tau_l)$,

$$c(\tau_1\cdots\tau_l)\equiv\langle\eta(\tau_1)\cdots\eta(\tau_l)\rangle_\eta\ ,$$

are functions only of the differences $(\tau_i-\tau_J)$. The process (3) has exactly this feature. A question that arises naturally is if also the correlation functions

$$\langle\phi_\eta(x_1\tau_1)\cdots\phi_\eta(x_l\tau_l)\rangle_\eta \tag{17b}$$

manifest this property. The answer is generally no. In fact the averages that we perform are not only in η, but also on the initial configuration $\phi(0)$ for which we give the $P(\phi(0))$. It is the form of this $P(\phi(0))$ that determines if (17b) is a function only of $(\tau_i-\tau_J)$. The choice of Parisi and Wu was $P(\phi(0))=\delta(\phi(0)-\phi_1)$ (with ϕ_1 a definite configuration) and the perturbative calculation done by them (for details see Ref. 1) showed that their choice of $P(\phi(0))$ does not make (17b) stationary. To find out which is the right one, let us start supposing (17b) is stationary:

$$\langle\phi_\eta(x_1\tau_1)\cdots\phi_\eta(x_l\tau_l)\rangle_{\eta,P(\phi(0))}=\delta(\tau_1-\tau_2,\tau_2-\tau_3,\ldots,\tau_i-\tau_{i-1})\ , \tag{18}$$

where we use the notation $\langle\ \ \rangle_{\eta,P(\phi(0))}$ to remind us of the average over both η and $\phi(0)$. From (18) we see that, if we rescale all the τ_i of a quantity T, nothing changes on the right-hand side, so

$$\langle\phi_\eta(x_1\tau_1)\cdots\phi_\eta(x_l\tau_l)\rangle_{\eta,P(\phi(0))}=\langle\phi_\eta(x_1,\tau_1+T)\cdots\phi_\eta(x_l,\tau_l+T)\rangle_{\eta,P(\phi(0))}\ .$$

Let us first put on both sides $\tau_1=\tau_2\cdots=\tau_l$,

$$\langle\phi_\eta(x_1\tau_1)\phi_\eta(x_2\tau_1)\cdots\phi_\eta(x_l,\tau_1)\rangle_{\eta,P}=\langle\phi_\eta(x_1,\tau_1+T)\phi_\eta(x_2,\tau_1+T)\cdots\phi_\eta(x_l,\tau_1+T)\rangle_{\eta,P}$$

and second let us take the limit of $T\rightarrow\infty$,

$$\lim_{T\rightarrow\infty}\langle\phi_\eta(x_1\tau_1)\phi_\eta(x_2\tau_1)\cdots\phi_\eta(x_l\tau_1)\rangle_{\eta,P}=\lim_{T\rightarrow\infty}\langle\phi_\eta(x_1,\tau_1+T)\phi_\eta(x_2,\tau_1+T)\cdots\phi_\eta(x_l,\tau_1+T)\rangle_{\eta,P}\ .$$

The left-hand side does not depend on T, while the right-hand side of (5) is the "quantum" correlation function [see (5)], so we have

$$\langle\phi_\eta(x_1\tau_1)\phi_\eta(x_2\tau_1)\cdots\phi_\eta(x_l\tau_1)\rangle_{\eta,P}=\frac{\int\mathscr{D}\phi[\phi(x_1)\cdots\phi(x_l)]e^{-S(\phi)}}{\int\mathscr{D}\phi e^{-S(\phi)}}\ . \tag{19}$$

As the left-hand side is stationary, we can rescale all the fields backward of τ_1; that means

$$\langle\phi_\eta(x_10)\phi_\eta(x_20)\cdots\phi_\eta(x_l0)\rangle_{\eta,P}=\langle\phi_\eta(x_1\tau_1)\phi_\eta(x_2\tau_1)\cdots\phi_\eta(x_l\tau_1)\rangle_{\eta,P}$$

and using (19) we conclude

$$\langle\phi_\eta(x_10)\phi_\eta(x_20)\cdots\phi_\eta(x_l0)\rangle_{\eta,P}=\frac{\int\mathscr{D}\phi[\phi(x_1)\cdots\phi(x_l)]e^{-S(\phi)}}{\int\mathscr{D}\phi e^{-S(\phi)}}\ . \tag{20}$$

From this expression we can explicitly derive the form of $P(\phi(0))$: the left-hand side of (20) is at $\tau=0$, so we do not have any random effect caused by η (η has not been switched on yet), the only average is with respect to $P(\phi(0))$; that means the left-hand side of (20) is

$$\int\mathscr{D}\phi(0)P(\phi(0))[\phi(x_1,0)\cdots\phi(x_l,0)]\ .$$

Comparing with its right-hand side we get

$$P(\phi(0))=\frac{e^{-S(\phi(0))}}{\int\mathscr{D}\phi(0)e^{-S(\phi(0))}}\ . \tag{21}$$

From (19) we can derive a second conclusion: for that particular form of $P(\phi(0))$, for which the correlation functions are stationary, we do not need to take the limit $\tau_1\rightarrow\infty$. At every finite τ_1, we have that the stochastic average is already the quantum one. The physical meaning of this is very clear: From the beginning we put the system in the equilibrium distribution $P(\phi)=e^{-S(\phi)}$ and the presence of the Langevin dynamics $\dot{\phi}=-\partial S/\partial\phi+\eta$ does not modify this. On the contrary, in the case of the choice[1] $P(\phi(0))=\delta(\phi(0)-\phi_1)$ we started with every field in configuration ϕ_1 and then the Langevin dynamics was able to spread them to the equilibrium form at time $\tau_1\rightarrow\infty$.

Inserting (21) back into (16), the new generating functional looks like

$$Z^{FP}_{\substack{\text{vacuum}\\ \text{vacuum}}}[J]=\int \mathscr{D}\phi(0)e^{-S(\phi(0))/2}\mathscr{D}\phi(\tau)e^{-S(\phi(\tau))/2}\mathscr{D}''\phi\exp\left[-\int_0^\tau \mathscr{L}^{FP}d\tau'\right]. \tag{22}$$

We like to call this the *"vacuum-vacuum generating functional."* The reason is clear if we remember that the ground-state (vacuum) of $\hat{H}^{FP}$ is $\Psi_0=e^{-S(\phi)/2}$.

Another way (less transparent) to get stationary correlation functions is to start from (15) and take the limit of integration from $\int_0^\tau$ to $\int_{-\infty}^\tau$. What happens is that the Fokker-Planck dynamics builds up a probability between $-\infty$ and 0 equal to $e^{-S(\phi)}$; that means equal to the one we inserted at $\tau=0$ by (21).

Before concluding, a word of caution is needed: Stochastic quantization does not compel us to choose (21). Any normalizable form of $P(\phi(0))$ is acceptable: the result, the limit of $\tau_1\rightarrow\infty$, is independent of $P(\phi(0))$.

The particular choice (21) has the advantage that it avoids step (iii) of the Parisi-Wu prescription.

Somehow this $Z^{FP}_{\substack{\text{vacuum}\\ \text{vacuum}}}$ is another method for representing the traditional quantum generating functional $Z=\int\mathscr{D}\phi e^{-S(\phi)}$, and in Ref. 11 its connection to the new functional method recently proposed by De Alfaro, Fubini, and Furlan[12] has been shown, using nonperturbative techniques.

IV. HIDDEN SUPERSYMMETRY

A. General notation

In this section we want to study another form for Z^{FP}. The expression for the Jacobian that we derived in (14) is not the only manner in which to write it. Another way to do so is by using anticommuting variables $\psi,\bar{\psi}$:

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right| =\int\mathscr{D}\psi\mathscr{D}\bar{\psi}\exp\left\{-\int_0^\tau\left[\bar{\psi}\left[\frac{\partial}{\partial\tau'}+\frac{\partial^2 S}{\partial\phi^2}\right]\psi\right]d\tau'\right\}. \tag{23}$$

With this form for the Jacobian and rescaling the time $\tau'\rightarrow\tau'/2$ the generating functional Z^{FP} becomes

$$Z^{FP}_{SS}=\int\mathscr{D}\phi\mathscr{D}\bar{\psi}P(\phi(0))\exp\left\{-\int_0^{2\tau}\left[\frac{1}{2}\left[\dot{\phi}+\frac{1}{2}\frac{\partial S}{\partial\phi}\right]^2+\bar{\psi}\left[\partial_{\tau'}+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi\right]d\tau'-\int_0^{2\tau}J\phi\,d\tau'\right\}. \tag{24}$$

The reason for the notation SS is that, with a proper choice of $P(\phi(0))$ and of boundary conditions for $\psi,\bar{\psi}$, this system reveals a hidden supersymmetry recently discussed in Ref. 13. Let us choose $P(\phi(0))=\delta(\phi(0)-\phi(2\tau))$. Then Z^{FP}_{SS} can be written as

$$Z^{FP}_{SS}=\int{}'\mathscr{D}\phi{}'\mathscr{D}\psi{}'\mathscr{D}\bar{\psi}\exp\left\{-\int_0^{2\tau}\left[\frac{1}{2}\dot{\phi}^2+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2+\bar{\psi}\left[\partial_{\tau'}+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi\right]d\tau'\right\} \tag{25}$$

with

$${}'\mathscr{D}\phi=\lim_{N\rightarrow\infty}\prod_{i=1}^N\mathscr{D}\phi_{\tau_i}.$$

The Lagrangian that appears in Z^{FP}_{SS} is

$$\mathscr{L}^{FP}_{SS}=\tfrac{1}{2}\dot{\phi}^2+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2+\bar{\psi}\left[\partial_\tau+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi. \tag{26}$$

The corresponding "Euclidean" Hamiltonian

$$H^{FP}_{SS}=-\frac{1}{2}\frac{\delta^2}{\delta\phi^2}+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2+\frac{1}{4}[\bar{\psi},\psi]\frac{\partial^2 S}{\partial\phi^2} \tag{27}$$

is well known in the literature and it has been studied in great detail in Ref. 14. It manifests a sort of *nonrelativistic supersymmetry* whose conserved charges are

$$Q_\psi=\left[\pi_\phi+\frac{1}{2}\frac{\partial S}{\partial\phi}\right]\psi,\qquad \pi_\phi=\frac{\delta}{\delta\phi}, \tag{28}$$
$$\bar{Q}_{\bar{\psi}}=\bar{\psi}\left[\pi_\phi+\frac{1}{2}\frac{\partial S}{\partial\phi}\right].$$

The symmetry transformations generated by them are

$$\delta\phi=-\epsilon_\psi\psi-\epsilon_{\bar{\psi}}-\psi,$$
$$\delta\psi=\epsilon_\psi\left[\pi_\phi+\frac{1}{2}\frac{\partial S}{\partial\phi}\right], \tag{29}$$
$$\delta\bar{\psi}=\epsilon_{\bar{\psi}}\left[\pi_\phi-\frac{1}{2}\frac{\partial S}{\partial\phi}\right],$$

where ϵ_ψ and $\epsilon_{\bar{\psi}}$ are infinitesimal anticommuting parameters. $\hat{H}^{FP}_{SS}$ itself can be written, in perfect supersymmetric fashion, as

$$\hat{H}^{FP}_{SS}=\tfrac{1}{2}\{\bar{Q}_{\bar{\psi}},Q_\psi\}.$$

We can even bring the notation a step further with the use of superfields.[15] Let us *first* rewrite (24) with an auxiliary field ω (that in statistical mechanics is known as a response field)

$$Z_{SS}^{FP}[J]=\int {}'\mathscr{D}\phi\,{}'\mathscr{D}\omega\,{}'\mathscr{D}\psi\,{}'\mathscr{D}\bar{\psi}\exp\left\{+\int_0^{2\tau}\left[\omega^2+\omega\left[\dot{\phi}+\frac{1}{2}\frac{\partial S}{\partial\phi}\right]-\bar{\psi}\left[\frac{\partial}{\partial\tau}+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi\right]d\tau'\right\}. \quad (30)$$

Second let us introduce the superfield Φ,

$$\Phi=\phi(x)+\bar{\theta}\psi+\theta\bar{\psi}+\theta\bar{\theta}\omega\ ,$$

where $\theta,\bar{\theta}$ are elements of a Grassmann algebra. Then the action in (30) can be written, in a very compact form, as

$$A[\Phi]\equiv\int[\tfrac{1}{2}(D_\theta\Phi)(D_{\bar{\theta}}\Phi)-S(\Phi)]d\tau\, d\theta\, d\bar{\theta}\ . \quad (31)$$

S is the usual action of the system (1) from which we started, but whose argument in (31) is the superfield Φ and $D_\theta=\partial_\theta-\bar{\theta}\partial_\tau$ is the so-called covariant derivative.[14] As we can see S plays the role of a sort of "potential" and in supersymmetric jargon its proper name is superpotential. The "space" over which the Lagrangian in (31) is integrated is $\tau,\theta,\bar{\theta}$, and it is the superspace[14] of our system. The symmetry (29), under which $\mathscr{L}_{SS}^{FP}$ is invariant, can be seen as a transformation on the fields $\phi,\psi,\bar{\psi}$ induced by the following "supertranslation" in superspace:

$$\delta\theta=\epsilon_\psi\ ,$$

$$\delta\bar{\theta}=\epsilon_{\bar{\psi}}\ ,$$

$$\delta\tau=-(\bar{\theta}\epsilon_\psi-\epsilon_{\bar{\psi}}\theta)\ .$$

B. Forward and backward Fokker-Planck dynamics

The manner in which we wrote $||\delta\eta/\delta\phi||$ in (23) deserves a deeper analysis. Being $||\delta\eta/\delta\phi||$ nothing else than a determinant (11), we can think of evaluating it as the product of its eigenvalues, that means

$$\left|\left|\frac{\delta\eta}{\delta\phi}\right|\right| =\int\mathscr{D}\psi\mathscr{D}\bar{\psi}\exp\left[-\int_0^{2\tau}\bar{\psi}\left[\frac{\partial}{\partial\tau'}+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi d\tau'\right] =\prod_{n=-\infty}^{+\infty}\alpha_n\ , \quad (32)$$

where α_n are the eigenvalues of the equation

$$\left[\partial_\tau+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\Psi_n=\alpha_n\Psi_n\ . \quad (33)$$

The solutions of (33) are

$$\Psi_n=K\exp\left[\int_0^\tau d\tau'\left[\alpha_n-\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\right]$$

where K is a normalizing constant) (from now on we will call the extreme of integration τ and not 2τ, for convenience).

If we impose, following Ref. 16, antiperiodic boundary conditions,

$$\Psi_n(\tau)=-\Psi_n(0)\ ,$$

we have

$$\alpha_n=\frac{i(2n+1)\pi}{\tau}+\frac{1}{\tau}\int_0^\tau d\tau'\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\ .$$

Unfortunately we cannot make this choice in our case. Our system is supersymmetric and the choice of periodic boundary conditions on the bosonic variable induces the same boundary conditions on the $\psi,\bar{\psi}$. If we had chosen antiperiodic ones, we would have broken supersymmetry explicitly (for more details see Ref. 9).

Taking $\Psi_n(\tau)=\Psi_n(0)$ we have

$$\alpha_n=\frac{2ni\pi}{\tau}+\frac{1}{\tau}\int_0^\tau d\tau'\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2} \quad (34)$$

and, substituting this in (32), we get

$$\prod_{n=-\infty}^{+\infty}\alpha_n=\prod_{n=-\infty}^{+\infty}\left[\frac{i2n\pi}{\tau}+\frac{1}{\tau}\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}d\tau'\right]$$

$$=\left[\prod_{n=-\infty}^{+\infty}\left[\frac{i2n\pi}{\tau}\right]\right]\left[\frac{1}{\tau}\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\prod_{n=-\infty}^{+\infty}{}'\left[1+\frac{\int_0^\tau\frac{1}{2}2(\partial^2 S/\partial\phi^2)d\tau'}{2\pi ni}\right]$$

$$=c'\left[\frac{1}{\tau}\int_0^\tau\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}d\tau'\right]\prod_{n=-\infty}^{\infty}{}'\left[1+\frac{\int_0^\tau\frac{1}{2}(\partial^2 S/\partial\phi^2)d\tau'}{2\pi ni}\right]$$

(Π' is to indicate that we exclude the term $n=0$ from the product)

$$=c'\sinh\tfrac{1}{4}\int_0^\tau d\tau'\frac{\partial^2 S}{\partial\phi^2}$$

$$=\frac{c'}{2}\left[\exp\left[\tfrac{1}{4}\int_0^\tau\frac{\partial^2 S}{\partial\phi^2}d\tau'\right]-\exp\left[-\tfrac{1}{4}\int_0^\tau\frac{\partial^2 S}{\partial\phi^2}d\tau'\right]\right] \quad (35)$$

(c' is a constant). Inserting this expression into (25), we get

$$Z^{\rm FP}_{\rm SS}=\frac{c'}{2}\left[\int{}'\mathscr{D}\phi\exp\left\{-\int_0^\tau\left[\dot\phi^2/2+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2-\frac{1}{4}\frac{\partial^2 S}{\partial\phi^2}\right]d\tau'\right\}\right.$$
$$\left.-\int{}'\mathscr{D}\phi\exp\left\{-\int_0^\tau\left[\dot\phi^2/2+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2+\frac{1}{4}\frac{\partial^2 S}{\partial\phi^2}\right]d\tau'\right\}\right] \tag{36}$$

We see that the first term on the right-hand side is the usual $Z^{\rm FP}$ studied in Sec. II, while the second term is the generating functional corresponding to the backward Kolmogoroff-Fokker-Planck Lagrangian presented in (17).

We can indicate this in a compact way as

$$Z^{\rm FP}_{\rm SS}[J]=\frac{c'}{2}(Z^{\rm FP}_{\rm forward}[J]-Z^{\rm FP}_{\rm backward}[J])\ , \tag{37}$$

where the notation is self-explanatory.

If we had chosen antiperiodic boundary conditions for ψ and $\bar\psi$, we would have gotten

$$Z^{\rm FP}_{\rm SS}[J]=\frac{c}{2}(Z^{\rm FP}_{\rm forward}[J]+Z^{\rm FP}_{\rm backward}[J])\ .$$

In this case the right-hand side, besides the notation, would not have been supersymmetric.

This nice interplay of forward and backward FP dynamics is also evident at the level at $\hat H^{\rm FP}_{\rm SS}$ (27). If we in fact represent ψ and $\bar\psi$ as 2×2 matrices (see Ref. 14),

$$\psi=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad \bar\psi=\begin{bmatrix}0&0\\1&0\end{bmatrix},$$
$$\psi^2=\bar\psi^2=0,\quad \{\psi,\bar\psi\}=\mathbb{1}\ ,$$

$\hat H^{\rm FP}_{\rm SS}$ can be written as

$$\hat H^{\rm FP}_{\rm SS}=-\frac{1}{2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{8}\left[\frac{\partial S}{\partial\phi}\right]^2-\frac{1}{4}\sigma_3\left[\frac{\partial^2 S}{\partial\phi^2}\right]$$
$$=\begin{bmatrix}\hat H^{\rm FP}_{\rm forward}&\\&\hat H^{\rm FP}_{\rm backward}\end{bmatrix}, \tag{38}$$

where $\hat H^{\rm FP}_{\rm forward}$ is the expression in (7) while $\hat H^{\rm HP}_{\rm backward}$ is the one in (17a). Both of these Hamiltonians are positive semi-definite: in fact, $\hat H^{\rm HP}_{\rm forward}=\frac{1}{2}QQ^\dagger$ and $\hat H^{\rm FP}_{\rm backward}=\frac{1}{2}Q^\dagger Q$ with $Q=\partial/\partial\phi-1/2\,\partial S/\partial\phi$. As we said in Sec. I, the ground state of $\hat H^{\rm FP}_{\rm forward}$ is at $E_0=0$ and is $\Psi_0^{\rm forward}=e^{-S/2}$. Also $\hat H^{\rm FP}_{\rm backward}$ has a state at $E_0=0$ and is $\Psi_0^{\rm backward}=e^{+S/2}$, but we have to be cautious about this state. If we assume, in fact, that the $\Psi_0^{\rm forward}=e^{-S/2}$ is normalizable [and we have to assume that for the traditional quantization (1) to hold], then $\Psi_0^{\rm backward}=e^{+S/2}$ is not normalizable and cannot be accepted as part of the spectrum of $\hat H^{\rm FP}_{\rm backward}$. This means that there is no physical state for $\hat H^{\rm FP}_{\rm backward}$ at $E_0=0$: All its states are at $E_n>0$.

These conditions can also be expressed using the language of supersymmetry: there is only one ground state for $\hat H^{\rm FP}_{\rm SS}$, that is

$$\begin{bmatrix}e^{-S/2}\\0\end{bmatrix};$$

the other one,

$$\begin{bmatrix}0\\e^{+S/2}\end{bmatrix},$$

cannot be accepted. This is equivalent to saying that supersymmetry is unbroken (see Ref. 14).

We can thus conclude that the request that the traditional quantization holds implies for the hidden supersymmetry of $Z^{\rm FP}_{\rm SS}$ be unbroken.

The presence of both dynamics in $Z^{\rm FP}_{\rm SS}$ is very amusing but it may bother the careful reader who knows that the prescription of Parisi and Wu[1] is to choose the forward one. It should be remembered, anyhow, that in the stochastic quantization we have to take the limit of integration τ in (36) to infinity. In this limit only the $Z^{\rm FP}_{\rm forward}$ is left in (37). In fact,

$$\lim_{\tau\to\infty}Z^{\rm FP}_{\rm SS}[J]=\lim_{\tau\to\infty}(Z^{\rm FP}_{\rm forward}-Z^{\rm FP}_{\rm backward})$$
$$=\lim_{\tau\to\infty}({\rm tr}e^{-\hat H^{\rm FP}_{\rm forward}\tau}-{\rm tr}e^{-\hat H^{\rm FP}_{\rm backward}\tau})$$
$$=\lim_{\tau\to\infty}\left[\sum_n e^{-E^n_{\rm forward}\tau}-\sum_n e^{-E^n_{\rm backward}\tau}\right]$$
$$=\lim_{\tau\to\infty}Z^{\rm FP}_{\rm forward}[J]\ . \tag{39}$$

(All the $E^n_{\rm backward}$ are positive so the last term goes to zero.) (We have neglected the constant c' because we can get rid of it by properly normalizing $Z^{\rm FP}_{\rm SS}$.)

The presence of this hidden supersymmetry can be further exploited deriving, for example, the corresponding Ward identities. This has been done in Ref. 13, but it does not throw any new light on the problem. These Ward identities only express the fact that correlations involving the ϕ and $\psi,\bar\psi$ fields can be reexpressed as correlations involving only ϕ fields. This is clear already in (37) where we succeed in integrating the $\psi,\bar\psi$ away, leaving only ϕ fields.

We want to derive some different identities here that also stem from the hidden supersymmetry.

Let us define the following generating functional:

$$Z^{[\rm SS]}_\alpha=\int e^{-(1+\alpha)S^{[\rm SS]}}{}'\mathscr{D}\phi{}'\mathscr{D}\psi{}'\mathscr{D}\bar\psi\ ,$$

where $S^{[\rm SS]}$ is a supersymmetric action and α is a parameter. It has been shown in Ref. 17 that, if supersymmetry is unbroken, then

$$\frac{\partial Z^{[\rm SS]}_\alpha}{\partial\alpha}=0\ ,$$

that means $Z^{[\rm SS]}_\alpha$ is independent of α. Let us write this down for our $Z^{\rm FP}_{\rm SS}$ in (24):

$$Z^{\mathrm{FP}}_{(\alpha)\mathrm{SS}}=\int {}'\mathscr{D}\phi\,'\mathscr{D}\psi\,'\mathscr{D}\bar{\psi}\exp\left[-(1+\alpha)S_B^{\mathrm{FP}}-(1+\alpha)\int_0^{\tau}\bar{\psi}\left[\partial_{\tau'}+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi\,d\tau'\right], \tag{40}$$

where S_B^{FP} is the bosonic part of the FP action (26),

$$S_B^{\mathrm{FP}}=\int_0^{\tau}\left[\dot{\phi}^2/2+\frac{1}{8}\frac{\partial S}{\partial\phi}\right]d\tau' .$$

If we make the following rescaling in the fermionic part of the action $\psi\to\sqrt{1+\alpha}\,\psi\equiv\psi'$, $\bar{\psi}\to\sqrt{1+\alpha}\,\bar{\psi}\equiv\bar{\psi}'$ we get

$$Z^{\mathrm{FP}}_{(\alpha)\mathrm{SS}}=\int\exp\left[-(1+\alpha)S_B^{\mathrm{FP}}-\int_0^{\tau}\bar{\psi}'\left[\partial_{\tau'}+\frac{1}{2}\frac{\partial^2 S}{\partial\phi^2}\right]\psi'\right]\frac{'\mathscr{D}\phi\,'\mathscr{D}\psi'\,'\mathscr{D}\bar{\psi}'}{1+\alpha} .$$

Performing the integration in $\psi',\bar{\psi}'$, we obtain

$$Z^{\mathrm{FP}}_{(\alpha)\mathrm{SS}}[J]=\frac{\int {}'\tilde{\mathscr{D}}\phi\, e^{-(1+\alpha)S_B^{\mathrm{FP}}+S_2}-\int {}'\tilde{\mathscr{D}}\phi\, e^{-(1+\alpha S_B^{\mathrm{FP}}-S_2}}{2(1+\alpha)}$$

with

$$S_2=\tfrac{1}{4}\int_0^{\tau}\frac{\partial^2 S}{\partial\phi^2}d\tau' .$$

This Z_α^{FP} is independent of α, so we can set its derivative equal to zero. What we obtain is the following relation:

$$\int(1+S_B^{\mathrm{FP}})\exp\left[-\int_0^{\tau}\mathscr{L}^{\mathrm{FP}}_{\mathrm{forward}}d\tau'\right]{}'\tilde{\mathscr{D}}\phi$$
$$=\int {}'\tilde{\mathscr{D}}\phi(1+S_B^{\mathrm{FP}})\exp\left[-\int_0^{\tau}\mathscr{L}^{\mathrm{FP}}_{\mathrm{backward}}d\tau'\right] .$$

This identity has never been derived before for stochastic processes and it expresses a sort of "time symmetry" between forward and backward Fokker-Planck dynamics. This time symmetry is a reflection of the supersymmetry of the sytem.

CONCLUSION

In this paper we have put the basis of a functional-integral approach to stochastic quantization. We have done this, not to merely develop once again perturbation theory, but with the goal of having a new tool to study the rich nonperturbative content of field theory. The traditional generating functional has proved, in the last 20 years, to be a very powerful instrument. We hope that our $Z^{\mathrm{FP}}[J]$ can at least complement this.

Note added. After this work was completed, I was informed of past and recent works on the functional approach to stochastic problems: F. Langouche *et al.*, Physica 95A, 252 (1979); Y. Nakano, University of Alberta report, 1982 (unpublished); M. Namiki *et al.*, Prog. Theor. Phys. (to be published); C. M. Bender *et al.*, Nucl. Phys. B219, 61 (1983).

ACKNOWLEDGMENTS

A preliminary report of this work was mentioned by Professor B. Sakita at the "Symposium on High Energy Physics," Tokyo, Sept. 1982. I thank him for many discussions and my friend V. Sarzi for convincing me to write up these results. This work was supported in part by the National Science Foundation Grant No. 82-15364 and CUNY-PSC-BHE Faculty Research award.

[1]G. Parisi and Wu-yong-shi, Sci. Sin. 24, 484 (1981).
[2]D. Zwanziger, Nucl. Phys. B192, 259 (1981); B193, 163 (1981); G. Marchesini, *ibid.* B191, 214 (1981).
[3]G. Parisi, Nucl. Phys. B180, 378 (1981); B205, 337 (1982).
[4]J. Alfaro and B. Sakita, Phys. Lett. 121B, 339 (1983); J. Greensite and M. B. Halpern, Nucl. Phys. B211, 343 (1983); A. Guha and S. C. Lee, Phys. Rev. D 27, 2412 (1983).
[5]M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 323 (1945).
[6]This choice corresponds to a midpoint prescription for the path integral in (9).
[7]This integration can be done in the usual way, even inside the path integral if we choose the midpoint prescription (see Ref. 8 and 6).
[8]L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1980), p. 29.
[9]S. Ceccotti and L. Girardello, Phys. Lett. 110B, 39 (1982).
[10]Yu. A. Rozanov, *Stationary Random Processes* (Holden-Day, San Francisco, 1967).
[11]E.Gozzi, Phys. Lett. B (to be published).
[12]V. De Alfaro, S. Fubini, and G. Furlan, Phys. Lett. 105B,
[13]G. Parisi and N. Sourlas, Nucl. Phys. B206, 321 (1982); S.

Ceccotti and L. Girardello, Harvard Report No. 82/4 HTUP, 1982 (unpublished).

[14]E. Witten, Nucl. Phys. B188, 513 (1981); P. Salomonson and J. W. Van Holten, *ibid.* B196, 509 (1982).

[15]A. Salam and J. Strathdee, Fortschr. Phys. 25, 58 (1977).

[16]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 12, 2443 (1975) (Appendix).

[17]H. Nicolai, Nucl. Phys. B176, 419 (1980). 462 (1981).