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Origin of the quantum observable operator algebra in the frame of stochastic mechanics

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In previous work the approach to stochastic quantization, originally proposed by Nelson, has been formulated in the frame of the stochastic variational principles of control theory. Then the Hamilton-Jacobi-Madelung equation is interpreted as the programming equation of the controlled problem, to be associated with the hydrodynamical continuity equation. Here we point out explicitly the canonical Hamiltonian structure of these equations, by introducing a suitable symplectic structure on the underlying phase space in various representations. One possible representation leads to the Schrödinger equation, which, together with its complex conjugate, can be recognized as a particular form of the Hamilton canonical equations in this frame. Then a suitably selected time-invariant subalgebra of the classical hydrodynamical algebra, closed under Poisson bracket pairing, is shown to be connected to the standard quantum observable operator algebra. In this correspondence Poisson brackets for hydrodynamical observables become averages of quantum observables in the given state. From this point of view stochastic quantization can be interpreted as giving an explanation for the standard quantization procedure of replacing the classical particle (or field) observables with operators, according to the scheme $p \to (h/i)\partial/\partial x$, $l \to (h/i)\partial/\partial\phi$, etc. This discussion shows also the relevance of the canonical symplectic structure of the quantum state space, a feature which seems to have been overlooked in the axiomatic approaches to quantum mechanics.

I. INTRODUCTION

In a recent paper,[1] it was shown that it is possible to recast the approach to stochastic quantization, originally proposed by Nelson[2] and further developed by other authors,[3] in the frame of a stochastic variational principle of the type introduced in control theory. The resulting programming equation, for a suitable choice of the stochastic action, is identical to the Hamilton-Jacobi equation of classical mechanics with corrections coming from the random noise and giving rise, for a proper choice of the diffusion coefficient, to the terms present in the Madelung equation[4] of quantum mechanics. It is very well known (see, for example, Ref. 5 and references quoted there) that the Madelung equation and the continuity equation can be interpreted as canonical Hamilton equations for a suitable symplectic system with phase space described by the density and the phase action fields. The main purpose of this paper is to investigate the structure of this symplectic system in order to point out the origin of the quantum observable operator algebra from the more general hydrodynamical algebra. Therefore, starting from the observation that the wave function ψ can be introduced through a particular canonical transformation on phase space, we verify that the Schrödinger equation and its complex conjugate are the Hamilton equations for the symplectic structure coming from a canonical Hamiltonian, which is bilinear in the ψ representation and invariant under rephasing of the wave function as a consequence of the conservation of the total probability. This explains the linearity of the Schrödinger equation and shows that the superposition principle is, in some sense, a consequence of the stochastic variational principle and the particular form of the action. In fact, different choices of the stochastic action may give rise also to nonlinear Schrödinger equations.[5] In the linear case it is very natural to introduce the subalgebra $\mathscr{a}$ of variables on phase space bilinear and phase invariant in the representation ψ. Clearly any invariant phase-space variable can be approximated through polynomials of elements in $\mathscr{a}$. Moreover, $\mathscr{a}$ is closed under Poisson-bracket pairing and contains the Hamiltonian and the generators of all geometric transformations on the particle configuration space. From this point of view $\mathscr{a}$ can be interpreted as a germ for the algebra of all invariant phase-space observables. The invariance of the algebra $\mathscr{a}$ under rephasing of the wave function is equivalent to the fact that all canonical transformations having the generators in $\mathscr{a}$ preserve the total probability in the configuration space of the controlled process.

Then it is amusing to see that the algebra $\mathscr{a}$ is in correspondence with the operator algebra on the Hilbert space generated by the wave function, while the value of the Poisson brackets for two elements in $\mathscr{a}$ is expressed in terms of the average of the commutator of the corresponding operators. In particular, the geometric transformations on the particle configuration space give rise to elements of $\mathscr{a}$ associated with the usual quantum-mechanical expressions for momentum, angular momentum, etc.

Therefore one can see that, in some sense, the superposition principle, the quantum operator observable algebra, and the relevance of the linear representations of group in quantum mechanics are all consequences of the stochastic variational principle and the particular form of the action, which has been chosen in Ref. 1.

The organization of the paper is as follows. In Sec. II, we briefly recall the basic aspects of the variational principle in the frame of control theory as explained in Ref. 1. In Sec. III, we introduce the symplectic structure in phase space and develop the canonical theory in various representations. Section IV contains the main core of our results. We introduce the subalgebra $\mathscr{a}$, in connection with rephasing invariance, and show its relations with the quantum observable operator algebra. Section V is dedicated to some examples dealing with geometric transformations on the configuration space of the controlled system and their related symplectic and quantum generators.

Finally, Sec. VI deals with conclusions and outlooks for possible extension of the results presented here.

II. STOCHASTIC VARIATIONAL PRINCIPLE IN CONTROL THEORY

We refer to Ref. 1 for a complete treatment. Here, for the sake of completeness, we recall some of the main features.

Let us consider a dynamical system with configuration space R^n. In general, we could consider the more general case of a Riemannian manifold (see, for example, Ref. 6), but it is sufficient to use R^n to show the main structural aspects of our results. While we have in mind in general the field case, where $x \in R^n$ is the configuration in some partial-wave representation for a strongly cut-off field theory ($n \to \infty$ at the end), for the sake of simplicity we refer to R^n as a "particle" configuration space. We consider a class of controlled stochastic Markov processes $q(t)$ on R^n satisfying the Ito stochastic equation

$$dq(t) = v_+(q(t),t)dt + dw(t) \;, \tag{1}$$

where $v_+(\cdot,t)$ is some given control field and the Gaussian random noise $dw(t)$ is normalized by the expectations

$$E(dw(t)dw(t)) = (\hbar/m)I\,dt \quad (dt > 0) \;. \tag{2}$$

In (2), I is the identity matrix in R^n and the diffusion coefficient has been written in the form $(\hbar/m)$ for future convenience. We assume that the process has initial density $\rho_0(\cdot)$ at time t_0. Then the density $\rho(\cdot,t)$ evolves according to

$$\partial_t\rho = -\nabla(v_+\rho) + (\hbar/2m)\Delta\rho \;, \tag{3}$$

equivalent to

$$\partial_t\rho = -\nabla(\rho v) \tag{4}$$

if the current density v is introduced according to the standard definitions[2]

$$v(x,t) = (v_+ + v_-)/2 \;, \qquad u(x,t) = (v_+ - v_-)/2 = (\hbar/2m)\nabla\rho/\rho \;. \tag{5}$$

In order to give a dynamical specification for the control $v_+(\cdot,t)$, let us introduce the stochastic action A in terms of the Lagrangian $\mathscr{L}$,

$$A = \int_{t_0}^{t_1} E(\mathscr{L}[q(t),t])dt \;, \qquad \mathscr{L}(x,t) = \tfrac{1}{2}m(v_+\cdot v_-)(x,t) - V(x) \;, \tag{6}$$

where $V(\cdot)$ is some potential (more general cases can be easily introduced).

Then let us consider the particular specifications of v_+ for which the action A is stationary ($\delta A = 0$) under small variations $v_+ \to v_+ + \delta v_+$ of the control subject to the constraint $\delta\rho(\cdot,t_1) = 0$. Then the results of Ref. 1 show that some $S(\cdot,t)$ must necessarily exist such that

$$v(x,t) = \nabla S/m \;, \tag{7}$$

$$\partial_t S + (\nabla S)^2/2m + V - (\hbar^2/2m)\Delta\sqrt{\rho}/\sqrt{\rho} = 0 \;. \tag{8}$$

Therefore the constrained variational principle gives the gradient form (7) for the velocity field and the Hamilton-Jacobi-Madelung equation (8), which, together with (4), is the starting point of our treatment.

Let us remark that the physical interpretation of ρ forces us to assume the normalization

$$\int \rho_0(x)dx = 1 \;, \tag{9}$$

which will be preserved at all times as a consequence of (4).

Finally let us point out that the stochastic variational principle is equivalent to Nelson's[2] original assumptions given by (7) and the smoothed form of the second principle of dynamics.

III. THE SYMPLECTIC STRUCTURE IN PHASE SPACE

For the moment let us forget the particle structure and let us assume as basic variables the hydrodynamical fields given by the density $\rho(\cdot,t)$ and the phase function $S(\cdot,t)$, satisfying (4) and (8). It is immediately seen that (4) and (8) are the canonical Hamilton equations for a suitable symplectic phase-space structure and a particular Hamiltonian. In fact, let us assume a phase space Γ specified by the generic fields $\rho(\cdot)$ and $S(\cdot)$, acting as canonical variables. Here the field label $x \in R^n$ is the analog of the index specifying canonical variables in the discrete case (for example $q_i, p_i, i = 1, \ldots, s$). On Γ we introduce a symplectic structure given by the two-form

$$\omega_2(\delta\rho,\delta S;\delta'\rho,\delta'S) = \int [\delta\rho(x)\delta'S(x) - \delta'\rho(x)\delta S(x)]dx \;, \tag{10}$$

where δ and δ' are two generic systems of increments for the phase-space variables. For generic functions $\mathscr{A}(\rho,S), \mathscr{B}(\rho,S)$ on phase space the associated Poisson brackets are

$$\{\mathscr{A},\mathscr{B}\} = \int \{[\delta\mathscr{A}/\delta\rho(x)][\delta\mathscr{B}/\delta S(x)] - \cdots\}dx \;, \tag{11}$$

where the terms $\cdots$ contain $\mathscr{A}$ and $\mathscr{B}$ exchanged so that $\{\cdot,\cdot\}$ is antisymmetric in its argument. In particular, we have

$$\{\rho(x),S(x')\} = \delta(x - x') \;, \tag{12}$$

while all other terms vanish:

$$\{\rho,\rho\}=0,\{S,S\}=0\ . \quad (13)$$

Equations (12) and (13) show that ρ and S are canonically conjugated variables.

We would like to point out that the symplectic structure (10) is not arbitrary, but strictly connected with the basic variational principle for the action (6). In fact, let us recall how the symplectic structure is connected to the Lagrangian variational principle in classical mechanics.[7]

To this purpose let us introduce for a generic trajectory in configuration space $t \rightarrow q(t) \in R^n$, $t_0 \le t \le t_1$, the classical action

$$A=\int_{t_0}^{t_1}\{\tfrac{1}{2}m[\dot{q}(t)]^2-V(q(t))\}dt\ . \quad (14)$$

By a variation δq of the trajectory we have

$$\delta A=-\int_{t_0}^{t_1}[m\ddot{q}(t)+\nabla V(q(t))]\cdot\delta q(t)dt +[m\dot{q}(t_1)\cdot\delta q(t_1)-m\dot{q}(t_0)\cdot\delta q(t_0)]\ . \quad (15)$$

Therefore we can introduce the one-form

$$\omega_1=p\cdot\delta q,\ \ p\equiv m\dot{q}\ . \quad (16)$$

Then the stationary-action principle $\delta A=0$, under the constraint that the variations $\delta q(t)$ must be such that the ω_1-form takes the same value at t_0 and t_1, gives the Newton-Lagrange equations $m\ddot{q}+\nabla V=0$. In particular, one can assume $\delta q=0$ at t_0 and t_1 but this is not necessary. Then the two-form $\omega_2=d\omega_1$ is

$$\omega_2(\delta q,\delta p;\delta' q,\delta' p)=\sum_i(\delta q_i\delta' p_i-\delta' q_i\delta p_i)\ . \quad (17)$$

The ω_2 form gives rise to the Poisson brackets

$$\{q_i,p_j\}=\delta_{ij},\ \ \{q_i,q_j\}=0,\ \ \{p_i,p_j\}=0\ . \quad (18)$$

By starting from the stochastic variational principle for the action (6) and exploiting the same line of reasoning we can arrive at (11) and (12). In fact, the constraint $\delta\rho(\cdot,t_1)=0$ can be substituted by a suitable Lagrangian multiplier in the action in the form

$$\int S_1(x_1)\rho(x_1,t_1)dx_1\ . \quad (19)$$

Then under arbitrary variations of v_+ and $\rho_0(\cdot)$, we have

$$\delta A=\int S(x_1,t_1)\delta\rho(x_1,t_1)dx_1 -\int S(x_0,t_0)\delta\rho(x_0,t_0)dx_0 +\int_{t_0}^{t_1}dt\int(mv-\nabla S)\rho\cdot\delta v_+dx\ , \quad (20)$$

as can be shown by following the same methods as in Ref. 1, under the mild generalization that also the initial distribution ρ_0 can be varied. Formula (20) shows that in the stochastic case the role played by (16) is now assumed by

$$\omega_1=\int S(x,\cdot)\delta\rho(x,\cdot)dx\ . \quad (21)$$

For the validity of the stochastic variational principle and the derivation of (8) it is enough to assume the less stringent constraint that ω_1 takes the same value at t_0 and t_1. Notice that in (21) the integration $\int dx$ replaces the sum $\sum_i$ in (16). Clearly the two-form (10) is related to (21) (as a differential) in the same way as (17) is related to (16). This shows the connection between the stochastic variational principle and the symplectic structure (11) and (12), in analogy with the classical case.

Let us now introduce the Hamiltonian $\mathscr{H}$ as the phase-space function:

$$\mathscr{H}(\rho,S)=\int(\tfrac{1}{2}mv^2+\tfrac{1}{2}mu^2+V)(x)\rho(x)dx\ , \quad (22)$$

where v and u are defined in (5) and (7). Notice that the stochastic average of the Lagrangian at some fixed time reads in the same notations

$$\mathscr{L}(\rho,S)=\int(\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2-V)(x)\rho(x)dx\ . \quad (23)$$

As remarked in Ref. 1, this shows that the osmotic velocity term $mu^2/2$ acts like a potential energy; in fact, it changes sign going from the Lagrangian to the Hamiltonian. With the choice (22) the evolution equations (4) and (8) have the canonical Hamiltonian form

$$\partial_t\rho(x,t)=\{\rho(x,t),\mathscr{H}\}=\delta\mathscr{H}/\delta S(x,t)\ ,$$
$$\partial_t S(x,t)=\{S(x,t),\mathscr{H}\}=-\delta\mathscr{H}/\delta\rho(x,t)\ . \quad (24)$$

Therefore $\mathscr{H}$ acts as the generator of time translations according to the symplectic structure.

In order to complete the Hamiltonian-Lagrangian structure, it should be shown that the fields $S(\cdot,t)$ canonically conjugated to $\rho(\cdot,t)$ can be uniquely expressed in terms of $\partial_t\rho(\cdot,t)$. In fact, ρ,S are suitable variables for the Hamiltonian formalism, while $\rho,\partial_t\rho$ are involved in the underlying field Lagrangian structure. According to an old result,[8] this can be easily shown. In fact, if S and S' are two fields giving rise to the same time evolution for ρ according to (4) and (7), then for the difference $u=S'-S$ we have $\nabla(\rho\nabla u)=0$. If we multiply by u, integrate over all R^n and then integrate by parts, we then get $\int\rho(\nabla u)^2dx=0$. This shows that u must be constant all over configuration space where ρ is nonzero. Since in any case S is defined up to a constant, because only derivatives of S enter in (7) and (8), we have that ρ and $\partial_t\rho$ determine uniquely the essential physical content of S. While a description of the Lagrangian type, in terms of ρ and $\partial_t\rho$ is surely possible, as we have shown, nevertheless the Hamiltonian structure, expressed in terms of ρ and S, has some advantages, as for example the local nature of (4) and (8) in the configuration space.

Since the physical content of the theory must be invariant under uniform shifts of S, the corresponding canonical generator, which is $M=\int\rho(x)dx$, plays a special role. In fact, all phase functions having a physical meaning must be invariant under the canonical transformations generated by M (the rephasing group). Therefore M is a superselected quantity and its value, in this canonical frame, can be normalized to be equal to 1 in agreement with the interpretation of ρ given by the stochastic variational principle outlined in Sec. II.

Clearly $\mathscr{H}$ is invariant under the rephasing group. But we can easily construct other invariant functions, as will be shown in the following.

While the stochastic variational principle gives ρ and S as basic variables, some transformations in phase space are useful to consider. For example, we can introduce the canonical transformation

$$\psi_1=\sqrt{\rho}\cos(S/\hbar)\ ,\qquad \psi_2=\sqrt{\rho}\sin(S/\hbar) \tag{25}$$

with inverse

$$\rho=(\psi_1)^2+(\psi_2)^2,\quad \tan(S/\hbar)=\psi_2/\psi_1\ , \tag{26}$$

and easily check the following identity between one-forms:

$$\int \rho\,\delta S\,dx=\hbar\int(\psi_1\delta\psi_2-\psi_2\delta\psi_1)dx\ .$$

Therefore the symplectic structure associated to ψ_1,ψ_2 is the same as the one introduced before in terms of ρ and S. In fact, the Poisson brackets become

$$\{\mathscr{A},\mathscr{B}\}=\tfrac{1}{2}\hbar\int\{[\delta\mathscr{A}/\delta\psi_1(x)]\times[\delta\mathscr{B}/\delta\psi_2(x)]-\cdots\}dx\ , \tag{27}$$

in particular

$$\{\psi_1(x),\psi_2(x')\}=\delta(x-x')/2\hbar\ ,\qquad \{\psi_1,\psi_1\}=0,\quad \{\psi_2,\psi_2\}=0\ . \tag{28}$$

An equivalent representation can be introduced through the wave function

$$\psi=\sqrt{\rho}\exp(iS/\hbar) \tag{29}$$

and its complex conjugate ψ^*. Now we have

$$\int\rho\,\delta S\,dx=\tfrac{1}{2}i\hbar\int(\psi\delta\psi^*-\psi^*\delta\psi)dx\ , \tag{30}$$

$$\{\mathscr{A},\mathscr{B}\}=\frac{1}{i\hbar}\int\{[\delta A/\delta\psi(x)]\times[\delta\mathscr{B}/\delta\psi^*(x)]-\cdots\}dx\ , \tag{31}$$

$$\{\psi(x),\psi^*(x')\}=\delta(x-x')/i\hbar\ ,\qquad \{\psi,\psi\}=0,\quad\{\psi^*,\psi^*\}=0\ . \tag{32}$$

It is important to remark explicitly that (29)—(32) show that the appearance of complex numbers of quantum mechanics is strictly connected with the underlying symplectic structure of the state space.

The great advantage of the ψ_1,ψ_2 or ψ,ψ^* representations is that the Hamilton equations become linear. In fact, let us introduce the Hamiltonian operator of the usual Schrödinger theory,

$$H_{\rm op}=-(\hbar^2/2m)\Delta+V\ , \tag{33}$$

acting on the wave function ψ. Then the field Hamiltonian $\mathscr{H}$, defined in (22), can be expressed, according to (29), in the form

$$\mathscr{H}\equiv\mathscr{H}(\psi,\psi^*)=\langle\psi,H_{\rm op}\psi\rangle\equiv\int\psi^*(x)(H_{\rm op}\psi)(x)dx\ , \tag{34}$$

where $\langle\cdot,\cdot\rangle$ is the Lebesgue scalar product in the Hilbert space $L^2(R^n,dx)$. This formula is quite remarkable. It shows that the hydrodynamical Hamiltonian $\mathscr{H}$ can be written as the quantum average of the single-particle quantum Hamiltonian $H_{\rm op}$ in the state (ρ,S) expressed in the ψ representation. As a consequence, the Hamiltonian equations are linear and coincide with the Schrödinger equation (and its complex conjugate). In fact, we have

$$\partial_t\psi=\{\psi,\mathscr{H}\}=(1/i\hbar)[\delta\mathscr{H}/\delta\psi^*(x)]=(1/i\hbar)H_{\rm op}\psi\ . \tag{35}$$

Finally, we would like to point out that, while the canonical transformation (29) is quite general, it is only for the particular choice of the action (6) that we arrive at the form (22) for the Hamiltonian and can prove that (29) linearizes the Hamilton equation in the form (35). In general, we can start from the different action, perform the canonical transformation (29), and end up with a nonlinear Schrödinger equation, as is shown in Ref. 5, for example. From a physical point of view it would be necessary to explain for which reason quantum mechanics selects the particular stochastic action (6), or, alternatively, verify whether the nonlinear Schrödinger equations obtained from a different choice of the action have an effective physical meaning.

IV. THE INVARIANT POISSON SUBALGEBRA AND THE QUANTUM OPERATOR ALGEBRA

From the point of view of the field system defined by (ρ,S), an observable is a function $\mathscr{A}$ of (ρ,S) on phase space. A particular observable $\mathscr{A}$ generates canonical transformations on the Poisson algebra of all observables according to the following expression for infinitesimal changes:

$$\delta\mathscr{B}=\delta\epsilon\{\mathscr{B},\mathscr{A}\}\ , \tag{36}$$

where $\delta\epsilon$ is an infinitesimal parameter. Of special physical interest are the observables invariant under uniform shifts of S. In particular, the Hamiltonian $\mathscr{H}$ of (22) has this property.

Let us introduce those observables in phase space, which, when expressed in terms of the variables ψ,ψ^*, assume a bilinear form of the type

$$\mathscr{A}(\rho,S)=\int\int\psi^*(x)A(x,x')\psi(x')dx\,dx'\ , \tag{37}$$

or are suitable limits of observables of this form. Since we are interested only in real observables $\mathscr{A}$, we must have

$$A^*(x,x')=A(x',x)\ , \tag{38}$$

so that we can introduce the operator A defined by

$$(A\psi)(x)=\int A(x,x')\psi(x')dx' \tag{39}$$

and check, under suitable conditions on the kernel $A(\cdot,\cdot)$, that it is self-adjoint. Since we allow limits in (37), any self-adjoint operator can be obtained in this way. Obviously, in the case of unbounded operators, some care must be devoted to domain problems.

Clearly (37) provides invariant observables $\mathscr{A}$ and any phase-invariant observable can be obtained, or approximated, through polynomials of bilinear observables of the type (37). We will show that the algebra $\mathscr{a}$, made of the bilinear observables defined by (37), is closed under Poisson-bracket pairing. On the other hand, the Poisson brackets of polynomials can be easily expressed through those of multiplicands. Therefore we are motivated to introduce the subalgebra α of the bilinear invariant observables as a germinal nucleus of the physically relevant algebra of all invariant observables.

Let us point out some properties of the subalgebra α. On the basis of (37)—(39), any $\mathscr{A} \in \alpha$ can be written as

$$\mathscr{A}(\rho,S)=\langle \psi, A\psi\rangle \tag{40}$$

for some self-adjoint operator A.

Now we calculate the Poisson brackets of two elements of α. Exploiting the expressions (31) we have, through a simple calculation,

$$\{\mathscr{A},\mathscr{B}\}=(1/i\hbar)\langle \psi,[A,B]\psi\rangle \ , \tag{41}$$

where $[\cdot,\cdot]$ is the commutator of two operators. This shows that $\{\mathscr{A},\mathscr{B}\}$ has still the form (40); therefore α is closed under Poisson-bracket pairing, as stated before. Formulas (40) and (41) give the basic connection between a particular subalgebra α of the field symplectic structure (ρ,S) and the quantum operator algebra. Notice that there is equality in (41), while the standard connection between the commutators of the operator algebra and the Poisson brackets of the classical Hamiltonian particle theory has more the value of an analogy.

An important consequence of (41) is the following. If we consider canonical transformations generated by observables in α, then we can see that they are also unitary, in the sense that they preserve the scalar product in $L^2(R^n,dx)$ introduced in (34) and (40). Moreover, transformation groups whose generators are in α can be interpreted as a group of linear unitary transformations on L^2.

An interesting problem is to see whether the more general canonical transformation on the symplectic structure have some kind of physical meaning.

V. EXAMPLES OF OBSERVABLES

In this section, we give some examples of observables in the subalgebra α and show how they induce operator counterparts on L^2.

We have already discussed the case of the Hamiltonian, for which (34) holds.

Let us consider space translations,

$$x \to x' = x + a, \quad a \in R^n \ , \tag{42}$$

acting on the basic variables ρ,S as

$$\begin{aligned} &\rho(\cdot)\to\rho'(\cdot)=\rho(\cdot-a) \ , \\ &S(\cdot)\to S'(\cdot)=S(\cdot-a) \ . \end{aligned} \tag{43}$$

The canonical infinitesimal generator is

$$\mathscr{P}(\rho,S)=\int \rho\nabla S\, dx \ . \tag{44}$$

In fact, for infinitesimal δa we have

$$\begin{aligned} \delta\rho(x)&=\delta a\cdot\{\rho(x),\mathscr{P}\} \\ &=\delta a\cdot(\delta\mathscr{P}/\delta S(x))=-\delta a\cdot\nabla\rho \ , \\ \delta S(x)&=\delta a\cdot\{S(x),\mathscr{P}\} \\ &=-\delta a\cdot(\delta\mathscr{P}/\delta\rho(x))=-\delta a\cdot\nabla S \ , \end{aligned} \tag{45}$$

in complete agreement with (43). But let us now express (44) in terms of ψ,ψ^*, according to the canonical transformation (29). We immediately see that

$$\mathscr{P}(\rho,S)=\langle \psi,(\hbar/i)\nabla\psi\rangle \ . \tag{46}$$

Therefore $\mathscr{P}$ has the general form (40) with

$$\mathscr{P}=\langle \psi,p_{\rm op}\psi\rangle, \quad p_{\rm op}=(\hbar/i)\nabla \ , \tag{47}$$

which is the usual postulated quantum-mechanical expression for the momentum.

The same considerations hold for general space transformations. The resulting canonical generators are associated to quantum operators, which are found to agree with those constructed according to the usual operator quantization prescriptions. In the case of groups of transformations the Poisson Lie algebra of the symplectic structure goes to the operator algebra of the quantum generators.

In particular, we recover the average position observables connected to generators of Galilei boosts

$$\mathscr{X}=\int \rho x\, dx=\langle \psi,q_{\rm op}\psi\rangle \ , \tag{48}$$

where $q_{\rm op}$ is the multiplication operator $(q_{\rm op}\psi)(x)=x\psi(x)$.

VI. CONCLUSIONS AND OUTLOOK

We have seen that the stochastic variational principle of a controlled system in configuration space leads to a continuity equation (4) and a programming equation (8), which can be interpreted as canonical Hamilton equations, on a suitable phase space and for a particular Hamiltonian. On this phase space a very important role is played by a subalgebra of observables, minimally closed under Poisson-bracket pairing and invariant under the rephasing group. The Hamiltonian and the canonical generators associated to configuration-space transformations belong to this class, which therefore is, in particular, invariant under time development and geometric transformations in configuration space.

Observables in this class generate canonical transformations which are also unitary transformations in a naturally introduced Hilbert space of square integrable functions on configuration space. The corresponding self-adjoint generators are the operators of the usual quantum observable algebra. In general, the Poisson-bracket structure is strictly connected with the commutator operator structure. In some sense our results show that most of the peculiar features of quantum mechanics can be obtained in the frame of a stochastic variational principle with a suitable choice of the Lagrangian. In particular, we can derive the quantum operator algebra and the corresponding relevance of unitary transformations and linear unitary representations of groups.

Two areas of problems seem to suggest further expansion to this program.

First of all, it is clear that our results critically depend on the chosen form of the stochastic Lagrangian. Different forms may give rise not only to quantitative changes, but also to a drastic modification of the overall qualitative structure, in particular with a loss of the linear character of the Schrödinger equation. Therefore it is important to investigate whether there is a deeper physical motivation for the stochastic Lagrangian, which for the moment we have chosen "because it works" on a very pragmatic basis. On some alternative route one could see whether there is a physical interpretation for the nonlinear theories which came from a different choice of the Lagrangian.

There is another aspect which we would like to point out. It is clear that both the symplectic structure and the operator structure allow reductions to smaller observable subalgebras, consequently the state space reduces because some states become identified. For some examples of this reduction procedure and its consequences we refer to Ref. 9. It would be very interesting to see whether in general the reduced system also allows a formulation based on stochastic variational principles for control problem on a smaller configuration space. This would produce a very powerful tool for the exploitation of stochastic methods for systems which do not allow a direct application of the general methods of stochastic mechanics such as, for example, fixed-spin systems. We plan to consider these issues in a future report.

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