Gravitational effects in bubble collisions

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We investigate the effects of gravitation in the collision of two bubbles in the very early universe, using the thin-wall approximation. In general, the collision of two bubbles gives rise to a modulus wall and a phase wave. The space-time metric and all physical quantities possess hyperbolic O(2,1) symmetry. We derive a generalized Birkhoff's theorem to show that the space-time in different regions must therefore be flat, de Sitter, pseudo-Schwarzschild, and pseudo-Schwarzschild—de Sitter, respectively. As in the spherically symmetric O(3) case, the space-time is Petrov type D, and so there is no gravitational radiation. Owing to the special symmetry of the space-time, the concentration of matter does not suffice to cause any gravitational collapse to a singularity no matter how severely the two bubbles collide. The modulus walls, viewed from the real vacuum region, eventually propagate outwards with kinks due to a series of collisions, in contrast to the situation in the absence of gravity.

I. INTRODUCTION

In the very early universe at very high temperature, the Higgs fields of any spontaneously broken gauge theory lose their expectation values, and this results in a gauge-symmetric phase.[1,2] When the temperature drops below that of the grand-unified-theory (GUT) and Weinberg-Salam energies, the symmetric phase becomes unstable and quantum tunneling effects cause bubbles of the stable broken-symmetry phase to form within the false vacuum. Coleman[3] shows that, in Minkowski space-time, the formation of a bubble may be thought of as an O(3,1)-invariant continuation from an O(4)-invariant "bounce" in a Euclidean manifold. Once the bubble materializes, it expands with a speed asymptotically approaching that of light, and the bubble wall, which separates the false-vacuum phase from the real vacuum, traces out the hyperboloid

$$|\vec{x}|^2-t^2=R^2 , \tag{1.1}$$

where R is the original radius of the bubble. If gravitation is present, one has to analytically continue the metric as well as the Higgs fields. Coleman[4] also discussed the effects of gravitation on the formation and expansion of a bubble.

The collision of two bubbles in flat space-time was discussed by Hawking, Moss, and Stewart.[5] They consider a Higgs scalar field Φ which is in the fundamental representation of a Yang-Mills group G. In general, the value of Φ in each bubble may be different, owing to the degeneracy of the global minimum of the effective potential $V(\Phi)$. Since it is always possible to find a U(1) subgroup W of G to connect the two Φ values, one can, without loss of generality, restrict attention to a model in which a single complex scalar field is coupled to an electromagnetic field, with an effective Lagrangian

$$\mathscr{L}=-\tfrac{1}{4}F^2-(D\Phi)^\dagger(D\Phi)-V(\Phi) . \tag{1.2}$$

Here D is the gauge-covariant derivative, and the effective potential $V(\Phi)$ which includes quantum and thermal corrections has two minima, the false vacuum at $\Phi=0$ and the broken-symmetric real-vacuum phase at $|\Phi| =\Phi_0$ (Fig. 1). In order that the so-called thin-wall approximation be valid, one needs the potential difference

$$V(0)-V(\Phi_0)=\epsilon^4 \tag{1.3}$$

to be much less than the height ξ^4 of the potential barrier between the two minima, since the initial radius of the bubble is of order $\xi^2\Phi_0\epsilon^{-4}$, while the thickness of the wall is initially of order $\frac{3}{2}\xi^{-2}\Phi_0$.

When two bubbles are present, the O(3,1) space-time symmetry is reduced to O(2,1), because of the preferred axis joining their centers. The respective values Φ_1, Φ_2 of a scalar field in each bubble are related through $\Phi_1=e^{i\alpha}\Phi_2$.

If $\alpha=0$, then there is no electromagnetic field. The walls of the two bubbles collide at the plane which perpendicularly bisects the line joining the centers of the two bubbles. In order that the energy be conserved, two modulus walls are formed in the collision region, which initially propagate outwards with almost the same speed as that of the incoming walls. The region between these outgoing modulus walls remains in the symmetric false-vacuum phase, while the outside region is unaffected. The pressure difference between the two phases and the wall tension bring the outgoing modulus walls back to collide

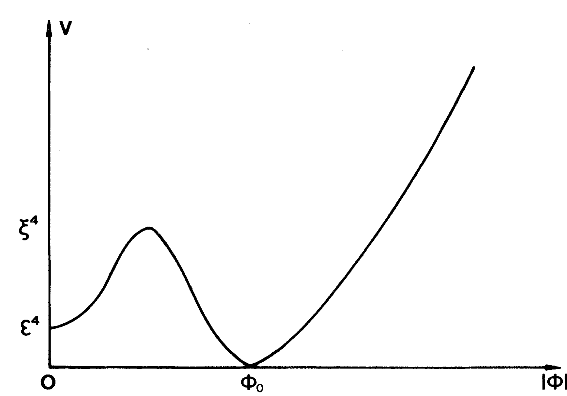

FIG. 1. The effective potential.

again and the process is repeated *ad infinitum* with the period and amplitude of oscillation decreasing asymptotically to zero.

If $\alpha \neq 0$, then in addition to the modulus walls two phase waves, which travel outwards at the speed of light, will be created by the collision. These phase waves carry off some of the energy of the incoming walls, thereby reducing the velocity of the outgoing modulus walls. If the phase difference is sufficiently large, then the modulus walls will completely disappear. Hawking *et al.*[5] argue that the electromagnetic field which appears after the collision with $\alpha \neq 0$, in general, does not play a significant role, and for simplicity we shall ignore it below. But,it seems straightforward to extend our discussion to include the electromagnetic field.

In this paper we investigate the effects of gravitation in the collision of two bubbles in the very early universe, using the thin-wall approximation. In Sec. II we show that before the creation of bubbles, the space-time is described by a de Sitter metric which is of O(4,1) symmetry. The existence of two bubbles reduces the symmetry to O(2,1). Any O(2,1)-symmetric Einstein space must be flat, de Sitter ($\Lambda > 0$), pseudo-Schwarzschild, or pseudo-Schwarzschild—de Sitter ($\Lambda > 0$). There is a naked curvature singularity in the last two cases.

Before the collision the metric in the false vacuum is de Sitter while in the real vacuum it is flat. In Sec. III we prove that the existence of the phase waves for the $\alpha > 0$ case causes the vacuum region after them to become pseudo-Schwarzschild with the metric parameter m determined by the energy radiated away at null infinity. It is found that the collision always happens after the Killing horizon of the pseudo-Schwarzschild space, and there is no singularity for the whole manifold. The physical significance of this phenomenon is that owing to the noncompactness of the hypersurface of the O(2,1) group transitivity, the concentration of matter does not suffice to cause any gravitational collapse.

In Sec. IV, adapting Israel's method[6,7] for spherical dust shells, i.e., matching the extrinsic geometry of the modulus walls through Einstein's equations, we derive an equation of motion of a modulus wall. It is possible, in principle, to determine the motion of the modulus walls and the pseudo-Schwarzschild-de Sitter metric in the false-vacuum regions after the collision by an iteration procedure. Although inside the false-vacuum regions the motion of the modulus walls looks similar to the motion in the absence of gravity, when viewed in the real-vacuum region, the walls eventually propagate outwards with kinks due to the series of collisions.

II. HYPERBOLICALLY SYMMETRIC SPACE-TIME AND THE GENERALIZED BIRKHOFF'S THEOREM

Before the bubbles appear, the space-time is in the false-vacuum phase, described by the de Sitter metric with O(4,1) symmetry. The cosmological constant is chosen to be $\Lambda = \delta\pi\epsilon^4$. When a single bubble materializes, the space-time inside must be flat because of the spherical symmetry, while the outside remains de Sitter.[4] The whole space-time is therefore of O(3,1) symmetry. For two bubbles the existence of the preferred axis (say x axis) through their centers reduces the O(3,1) symmetry to O(2,1), i.e., hyperbolic symmetry.

In general, this O(2,1)-symmetric space-time can be covered by four incomplete coordinate patches, using the two metric forms

$$dl^2 = -e^{2\alpha(s,x)}(ds^2 - dx^2) + e^{2\beta(s,x)}(d\theta^2 + \sinh^2\theta\, d\phi^2) \tag{2.1}$$

and

$$dl^2 = e^{2\alpha'(s,x)}(ds^2 + dx^2) + e^{2\beta'(s,x)}(-d\theta^2 + \cosh^2\theta\, d\phi^2)\ . \tag{2.2}$$

For flat space-time one has $\alpha = \alpha' = 0$, $\beta = \beta' = \ln s$. To derive the de Sitter space-time metric in this form we proceed as follows. It is easiest to visualize the space-time as a hyperboloid[8]

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2,\ \ \alpha^2 = \frac{3}{\Lambda}$$

in a flat five-dimensional space with metric

$$dl^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2\ .$$

If $|v| \geq |w^2 + x^2|^{1/2}$ one can introduce coordinates (s,θ,φ,ϕ) on the hyperboloid, defined by

$$v = \alpha \tan s \cosh\theta\ ,$$
$$w = \alpha \tan s \sinh\theta \cos\varphi\ ,$$
$$x = \alpha \tan s \sinh\theta \sin\varphi\ ,$$
$$y = \alpha \sec s \cos\phi\ ,$$
$$z = \alpha \sec s \sin\phi\ .$$

The metric then takes the form

$$dl^2 = \alpha^2 \sec^2 s(-ds^2 + d\phi^2) + \alpha^2 \tan^2 s(d\theta^2 + \sinh^2\theta\, d\varphi^2)\ , \tag{2.3}$$

where $0 \leq \theta < +\infty$, $0 \leq \varphi \leq 2\pi$, $-\pi/2 < s < \pi/2$, $0 \leq \phi < 2\pi$. If $|V| \leq |\omega^2 + x^2|^{1/2}$, then in a similar way the metric can be expressed as

$$dl^2 = \alpha^2 \mathrm{sech}^2 s(ds^2 + d\phi^2) + \alpha^2 \tanh^2 s(-d\theta^2 + \cosh^2\theta\, d\varphi^2)\ , \tag{2.4}$$

where $0 \leq \phi \leq 2\pi$, $0 \leq \varphi < 2\pi$, $0 \leq \theta < +\infty$, $-\infty < s < +\infty$.

In the following we concentrate our attention on the region described by the metric (2.1). If we take θ to be purely imaginary, and exchange the time and space coordinates s,x,

$$\theta \to i\theta'\ ,\quad x \to t\ ,\quad s \to r\ ,\quad \phi \to \phi\ , \tag{2.5}$$

the metric (2.1) becomes

$$dl'^2 = e^{2\alpha}(dr^2 - dt^2) - e^{2\beta}(d\theta'^2 + \sin^2\theta' d\phi^2)\ , \tag{2.6}$$

which possesses spherical symmetry. Clearly Einstein's field equations for the metric (2.1), which has O(2,1) symmetry, can be derived from those for the O(3) metric (2.6).

In fact, under the substitution (2.5) we have the following correspondence:

$$\begin{aligned} &O(2,1)\to O(3)\ ,\\ &G_{00}\to G_{11}\ ,\\ &G_{11}\to G_{00}\ ,\\ &G_{22}\to -G_{22}\ ,\\ &G_{33}\to -G_{33}\ ,\\ &G_{01}\to G_{01}\ . \end{aligned} \tag{2.7}$$

Following this procedure, we find that the metric coefficients α and β in (2.1) must satisfy the following.

(i) Time development equations:

$$\begin{aligned} -G^1_1 &= -e^{-2\beta}-2e^{-2\alpha}\dot\alpha\dot\beta+2e^{-2\alpha}\ddot\beta+3e^{-2\alpha}\dot\beta^2\\ &\quad -2e^{-2\alpha}\alpha'\beta'-e^{-2\alpha}\beta'^2\\ &= -8\pi p_1\ , \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{G}{2} &= e^{-2\alpha}(\ddot\alpha-\alpha''+\beta'^2-\dot\beta^2)+e^{-2\beta}\\ &= 4\pi(-\mu+p_1+p_2+p_3)\ . \end{aligned} \tag{2.9}$$

(ii) Constraint equations:

$$\begin{aligned} G^0_0 &= e^{-2\beta}+2e^{-2\alpha}\beta''-2e^{-2\alpha}\alpha'\beta'+3e^{-2\alpha}\beta'^2\\ &\quad -2e^{-2\alpha}\dot\beta\dot\alpha-e^{-2\alpha}\dot\beta^2\\ &= -8\pi\mu\ , \end{aligned} \tag{2.10}$$

$$G^1_0 = 2e^{-2\alpha}(\dot\beta'+\dot\beta\beta'-\dot\alpha\beta'-\alpha'\dot\beta) = 8\pi q\ . \tag{2.11}$$

Here μ is the energy density, $\vec{p}=(p_1,p_2,p_3)$ is the pressure, q is the energy flux, an overdot denotes $\partial/\partial s$, and a prime denotes $\partial/\partial x$.

Formulas (2.5) and (2.7) imply that all arguments applicable to the O(3) space-time will have a corresponding analog in the O(2,1) space-time, and the vice versa.

A form of Birkhoff's theorem states that the only spherically symmetric vacuum space-time with cosmological constant Λ is Schwarzschild—de Sitter space

$$\begin{aligned} dl^2 &= -\left[1-\frac{2m}{r}-\tfrac{1}{3}\Lambda r^2\right]dt^2\\ &\quad +\left[1-\frac{2m}{r}-\tfrac{1}{3}\Lambda r^2\right]^{-1}dr^2\\ &\quad +r^2(d\theta^2+\sin^2\theta\, d\phi^2) \end{aligned} \tag{2.12a}$$

or $S^2\times(2-d)$ de Sitter space[9]

$$\begin{aligned} dl^2 &= \Lambda^{-1}(d\theta^2-\cosh^2\theta\, d\phi^2\\ &\quad +d\theta'^2+\sin^2\theta' d\phi'^2)\quad (\Lambda>0)\ . \end{aligned} \tag{2.12b}$$

Thus the following generalized Birkhoff's theorem is self-evident: with a cosmological constant Λ, any C^2 solution of Einstein's equations in vacuum, which is of O(2,1) symmetry in an open set, is locally isometric to part of the pseudo-Schwarzschild—de Sitter spaces

$$\begin{aligned} dl^2 &= -\left[1-\frac{2m}{s}+\tfrac{1}{3}\Lambda s^2\right]^{-1}ds^2\\ &\quad +\left[1-\frac{2m}{s}+\tfrac{1}{3}\Lambda s^2\right]dx^2\\ &\quad +s^2(d\theta^2+\sinh^2\theta\, d\phi^2) \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} dl^2 &= \left[1-\frac{2m}{s}+\tfrac{1}{3}\Lambda s^2\right]^{-1}ds^2\\ &\quad +\left[1-\frac{2m}{s}+\tfrac{1}{3}\Lambda s^2\right]dx^2\\ &\quad +s^2(-d\theta^2+\cosh^2\theta\, d\phi^2)\ , \end{aligned} \tag{2.14}$$

except for two special cases

$$\begin{aligned} dl^2 &= |\Lambda^{-1}|(-d\theta^2+\cosh^2\theta\, d\phi^2\\ &\quad +d\theta'^2+\sinh^2\theta' d\phi'^2)\quad (\Lambda<0) \end{aligned} \tag{2.15}$$

and $S^2\times(2-d)$ de Sitter space (2.12b). Neither of these cases are relevant to the rest of this paper.

As in the case of spherical symmetry, the hyperbolically symmetric space-time must be Petrov type D. Solutions (2.13) and (2.14) are not new. They were given by Kinnersley[10] and Plebanski and Demianski.[11] As in the O(3) case there is no gravitational radiation in these space-times.

In the Schwarzschild solution (2.12), the Arnowitt, Deser, and Misner (ADM) mass[12] measured at spatial infinity, and the Bondi mass[13] measured at null infinity, are equal and take the value m. The collapsing of a thin spherical shell of matter with energy Δm per solid angle $\sin\theta\, d\theta\, d\phi$ measured at infinity will cause the parameter m to increase by $4\pi\Delta m$.[14] In the pseudo-Schwarzschild case, null infinity is a noncompact two-hyperboloid instead of a compact two-sphere as in the O(3) case. Under the correspondence (2.5) and (2.7), the pressure p_1 of a matter shell per unit pseudosolid angle $\sinh\theta\, d\theta\, d\phi$ measured at infinity $s=\infty$ will contribute to the pseudo-Schwarzschild parameter m in a similar manner.

The Penrose diagrams of Schwarzschild space and pseudo-Schwarzschild space are drawn in Fig. 2, for comparison. One can see that for O(2,1)-symmetry case, the Killing horizon $s=2m$ is no longer an event horizon, and the curvature singularity $s=0$ becomes naked.

III. SPACE-TIME METRIC

We shall now use the results of the previous section to investigate gravitational effects in two-bubble collisions. In the absence of gravity, Hawking *et al.*[5] use a hyperbolic coordinate system to describe the space-time

$$dl^2=-ds^2+dx^2+s^2(d\theta^2+\sinh^2\theta\, d\phi^2)\ , \tag{3.1}$$

where the x axis passes through the center of each bubble, and the origin of the coordinate is at the midpoint of the x axis. Because of the O(2,1) hyperbolic symmetry, the

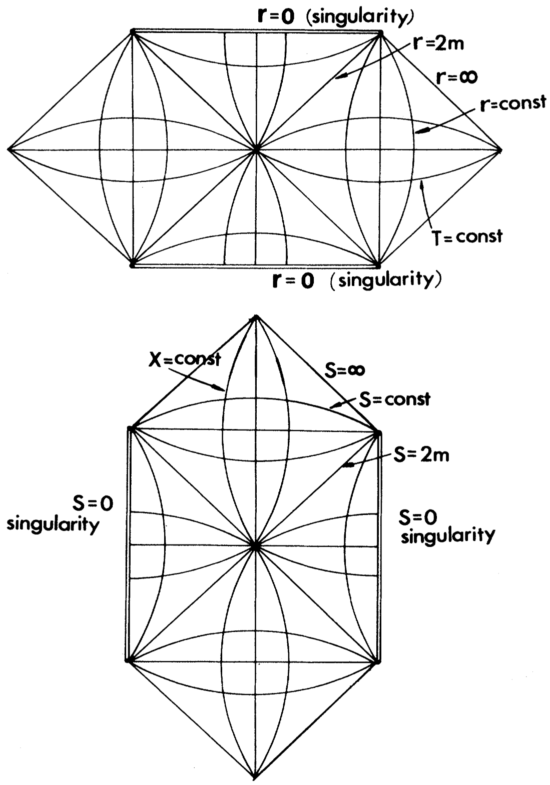

FIG. 2. (a) The Penrose diagram of Schwarzschild space. Each point represents a two-sphere except for $r=0$. (b) The Penrose diagram of pseudo-Schwarzschild space. Each point represents a two-hyperboloid except for $s=0$.

physics is independent of θ and ϕ, while by the reflection symmetry, only the region $x \geq 0$ needs to be considered. In the thin-wall approximation the energy-momentum tensor takes the form[5]

$$T^{ab} = -\epsilon^4 g^{ab}\theta(\bar{x}(s)-x) - \sigma(g^{ab}-V^a V^b)[1+(V^s)^2]^{-1/2}\delta(\bar{x}(s)-x) , \tag{3.2}$$

where $\bar{x}(s)$ denotes the position of the wall, $\sigma = \frac{1}{3}\xi^2\Phi_0$ is the surface energy density of the wall, which may be derived using Coleman's minimum-action principle,[3,4] and V^a is the unit vector normal to the wall. The first term represents the energy-momentum density of the false vacuum. Integrating the energy conservation equation,[5] one finds

$$\bar{x}' = \pm \frac{s}{(a^2+s^2)^{1/2}} , \tag{3.3}$$

where

$$a = \frac{(3\sigma/\epsilon^4)s^3}{s^3-s_0{}^3}, \quad (\ )' \equiv \frac{\partial}{\partial s}(\ ) \ (s_0=\text{const}) .$$

Equation (3.3) determines the motion of the bubble wall. If we set $s_0=0$, then $a=R=3\sigma/\epsilon^4$ is the size of the initial bubble, and Eq. (3.3) integrates to

$$(\bar{x}-b)^2 - s^2 = R^2 \quad (b=\text{const}) \tag{3.4}$$

as expected, which is exactly the same form as that found by Coleman.[3] The two bubbles will collide at $s=s_1 \equiv (b^2-R^2)^{1/2}$, $x=0$ with a Lorentz factor $\gamma=b/R$, while the wall thickness is reduced from $\Phi_0{}^2/2\sigma$ to $\Phi_0{}^2/2\sigma\gamma$ because of Lorentzian contraction.

In the presence of gravity, the O(3) Birkhoff's theorem implies that before the collision the space-time inside the bubble must be flat, while the outside remains de Sitter space with $\Lambda = 8\pi\epsilon^4$.[4] The O(3,1) symmetry implies that, viewed from the real-vacuum region inside the bubble, the wall must move along a hyperbola. Its motion will therefore be described by the same curve (3.4) as in the absence of gravity. The only effect of gravity is to reduce the radius of the inital bubble to[4]

$$R = \frac{\sigma}{2\pi\sigma^2 + \Lambda/24\pi} \tag{3.5}$$

but we defer the derivation of this formula from general relativity and all details of the motion of the modulus walls after the collision until the next section.

If the phase difference α is zero, then by local conservation of energy, as viewed from the false-vacuum region, two modulus walls are formed by the collision and initially move outwards with the same speed as the incoming walls. The modulus walls will be brought back to collide again at $s=s_2$, at $s=s_3$, and so on (Fig. 3). They will collide with shorter and shorter periods, because of the expansion in the s direction.

The region outside the modulus walls remains real vacuum, and because of the O(2,1) symmetry it can be described by a pseudo-Schwarzschild metric with an appropriate parameter m. Because the real-vacuum region is flat before the collision, it must, for consistency, also be flat after the collision, and therefore m must vanish.

For the $\alpha \neq 0$ case, two additional phase waves are created which propagate outward at the speed of light and carry away energy from the collision region (Fig. 4). If $\alpha < \alpha_c = 1$, the creation of the phase waves reduces the velocity of the outgoing modulus walls, while for $\alpha \geq \alpha_c$ the phase waves will take up all the energy of the incoming bubble walls and therefore no modulus wall will be created. For the $\alpha < \alpha_c$ case, the energy per unit area in the phase wave is $E_1 \equiv 2|\Delta\Phi|^2/d$, where $\Delta\Phi = \frac{1}{2}\alpha\Phi_0$, and the thickness of the phase wave is almost the same as

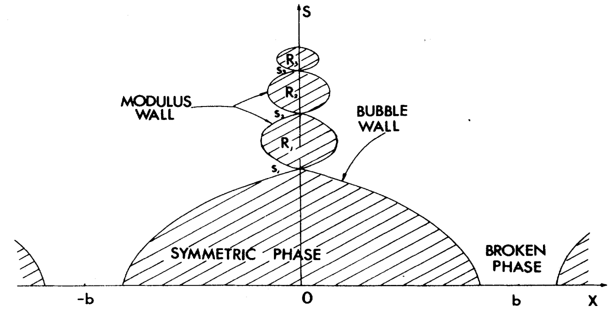

FIG. 3. The motion of the modulus walls in the absence of gravity ($\alpha=0$).

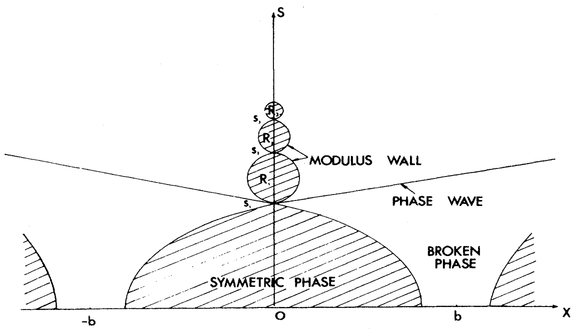

FIG. 4. The collision of two bubbles with $\alpha \neq 0$ in the absence of gravity.

that of the incoming walls $d=\Phi_0^2/2\sigma\gamma$. In the presence of gravity, one must take the Lorentz factor with respect to the de Sitter region, and γ is reduced to

$$\gamma_d=(R^2+s^2)^{1/2}\left[1+\frac{s^2\Lambda}{3}\right]^{-1/2}\Big/R \ .$$

If $\alpha>\alpha_c$, the energy of phase wave is equal to that of the incoming walls. One has therefore

$$E_1=\begin{cases}\alpha^2\sigma\gamma_d & (\alpha<\alpha_c)\ ,\\ \sigma\gamma_d & (\alpha\geq\alpha_c)\ .\end{cases} \tag{3.6}$$

The real-vacuum region to the future of the phase waves must be pseudo-Schwarzschild and its m parameter represents the pressure p_1 of the phase wave within 4π pseudosolid angle at the null infinity (in our case, the same as the energy). In order to determine the parameter m, one can approximate, at first stage, the phase wave as a dust shell which propagates outwards at very high speed. When the speed tends to that of light, the energy-momentum tensor of the dust shell will take the form of that for the phase wave, so that as far as their graviational effects are concerned, this replacement of the phase wave by a dust shell of light speed is reasonable. From Israel's result[6,7] on collapsing spherical dust shell and using the substitution (2.5), we have

$$4\pi s^2\sigma_d=\text{const} \tag{3.7}$$

along the propagation of the shell, where σ_d is the total mass energy per unit proper surface area, as measured by a "comoving observer." The energy of the dust shell within 4π pseudosolid angle can be expressed as $4\pi s^2\sigma_d V^s$, where V^s is the $\partial/\partial s$ component of the four-velocity of the dust. Letting the velocity of the dust shell tend to that of light while keeping the energy finite at null infinity, we have

$$m=\lim_{s\to\infty}(4\pi s^2\sigma_d V^s)\ . \tag{3.8}$$

What remains to be done is to connect the energy of the phase wave measured in the flat-space region with that measured in the static frame of the de Sitter region. The relative Lorentz factor of the flat region to the de Sitter region is

$$\gamma_r=\gamma\gamma_d-(\gamma^2-1)^{1/2}(\gamma_d^2-1)^{1/2}\ , \tag{3.9}$$

from a simple calculation in special relativity. The energy of the phase wave measured in the de Sitter region can be obtained by a Lorentz transformation using γ_r,

$$4\pi s_1^2E_1=m[\gamma_r+(\gamma_r^2-1)^{1/2}]\ . \tag{3.10}$$

By Eq. (3.6), m can be written as

$$m=\begin{cases}\dfrac{4\pi s_1^2\alpha^2\sigma\gamma_d}{\gamma_r+(\gamma_r^2-1)^{1/2}} & (\alpha<\alpha_c)\ ,\\[2ex] \dfrac{4\pi s_1^2\sigma\gamma_d}{\gamma_r+(\gamma_r^2-1)^{1/2}} & (\alpha\geq\alpha_c)\ .\end{cases} \tag{3.11}$$

After a tedious calculation one can prove that for both cases

$$s_1>2m\ . \tag{3.12}$$

If $\alpha\geq\alpha_c$, no modulus wall will appear, so at $s=s_1$, the direction $\partial/\partial s$ in the pseudo-Schwarzschild region is identified with that in the de Sitter region. One can use the following simpler method to determine the parameter m. The energy measured in the static frame of the pseudo-Schwarzschild region is

$$\frac{m}{(1-2m/s_1)^{1/2}}\ ,$$

where $(1-2m/s_1)^{1/2}$ can be explained as the "red-shift" factor, and

$$\frac{m}{(1-2m/s_1)^{1/2}}=4\pi s_1^2\sigma\gamma_d\ . \tag{3.13}$$

One can check that for $\alpha\geq\alpha_c$, formulas (3.11) and (3.13) are consistent. All the results can be obtained by using junction conditions along the null hypersurface of the phase wave.[15]

If $s_1 \gg 2m$, one can simply ignore the difference of the energy of the phase wave between the collision region and null infinity, and take

$$m=\begin{cases}4\pi s_1^2\sigma\gamma_d\alpha^2 & (\alpha<\alpha_c)\ ,\\ 4\pi s_1^2\sigma\gamma_d & (\alpha\geq\alpha_c)\ .\end{cases} \tag{3.14}$$

In the preceding paragraphs we have examined the pseudo-Schwarzschild region which occurs in the case $\alpha>0$, and have established the value of the parameter m [Eq. (3.11)]. For the region of false vacuum R_1, R_2, $R_3,\ldots$, which exists after the collision if $0\leq\alpha<\alpha_c$, it was shown in Sec. II that the appropriate metrics must be pseudo-Schwarzschild—de Sitter, parametrized by m_1, $m_2, m_3,\ldots$. The method of determining parameters m_1, m_2, $m_3,\ldots$ is discussed in Sec. IV.

In the absence of gravity, the initial velocity of the outgoing modulus walls is reduced to the phase waves

$$\bar{x}'=\left[1-\frac{R^2}{b^2(1-\alpha^2)^2}\right]^{1/2},\quad (\ )'\equiv\frac{\partial}{\partial s}(\ )\ . \tag{3.15}$$

In the presence of gravity, local conservation of energy causes the velocity of the modulus wall to suffer a jump in the collision region:

$$\left[1+\frac{s_1^2\Lambda}{3}\right]^{-1/2}\left[\frac{ds}{dl}\right]_- - \left[1-\frac{2m_1}{s_1}+\frac{s_1^2\Lambda}{3}\right]^{-1/2}\left[\frac{ds}{dl}\right]_+ = \alpha^2\gamma_d \;, \qquad (3.16)$$

where σ is assumed to be continuous that $+$ $(-)$ labels terms after (before) the collision.

The significance of the restriction $s_1 > 2m$ becomes apparent when one tries to draw the Penrose diagram (5b) for the case $s_1 \leq 2m$. The space-time would then include the part of the singularity shown in Fig. 2(b). However, owing to the special symmetry of space-time, the surface of the group transitivity is noncompact, so that the concentration of matter does not suffice to cause any gravitational collapse to a singularity no matter how severely the two bubbles collide [see the Penrose diagrams 5(a) and 5(b)].

IV. MOTION OF THE MODULUS WALLS

In this section we shall use the thin-wall approximation to investigate the motion of the modulus walls by adapting Israel's method dealing with a collapsing spherical shell of dust.[6,7]

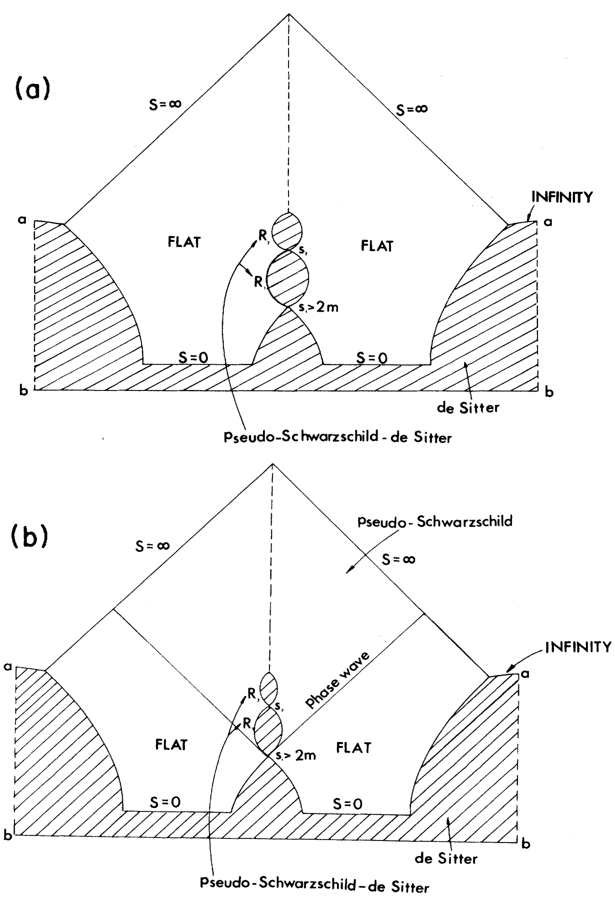

FIG. 5. (a) The Penrose diagram for the collision of two bubbles with $\alpha=0$. (b) The Penrose diagram for the collision of two bubbles with $\alpha\neq 0$.

The metric in the pseudo-Schwarzschild region V^- outside the modulus wall Σ and the pseudo-Schwarzschild—de Sitter region V^+ inside can be represented by

$$dl^2 = f_\pm^{-1}dx^2 + s^2(d\theta^2+\sinh^2\theta\, d\varphi^2) - f_\pm ds^2 \;, \qquad (4.1)$$

where

$$f_-^{-1} = 1-\frac{2m_-}{s}$$

and

$$f_+^{-1} = 1+\frac{s^2\Lambda}{3}-\frac{2m_+}{s} \;.$$

Since the intrinsic geometry of the modulus wall is the same viewed from either V^+ or V^-, we must have $s_+ = s_-$ at Σ, but in general $x_+ \neq x_-$. The unit normal n of the wall (directed from V^- to V^+) is everywhere spacelike. The surface energy-momentum density of Σ takes the form[5]

$$S^{ab} = -\sigma g^{ab} \quad (a,b=1,2,3) \;. \qquad (4.2)$$

Before the bubble walls collide σ is constant, owing to the O(3,1) symmetry; we can prove below that it remains constant even after collision. Because it has a nonzero surface energy density, the hypersurface Σ is singular. Therefore, the extrinsic curvature of Σ on either side of the wall will not be equal. They are defined by $K_{ab}^{\pm} = -\vec{n}^{\,\pm}\cdot(\partial\vec{e}_a/\partial\xi^b)$, where $\vec{e}_a$ are the orthonormal tangent vectors to Σ, and ξ^b are the intrinsic coordinates of Σ.

Einstein's equations take the form in V^+ and V^-, respectively,[16]

$$G^n{}_n = -\tfrac{1}{2}({}^3R + K_{ab}^{\pm}K_{\pm}^{ab} - K_{\pm}{}^2) = \begin{cases} -\Lambda, \\ 0 \,, \end{cases} \qquad (4.3)$$

$$G^n{}_i = +(K_{i\,|m}^{\pm m} - K_{\pm\,|i}) = 0 \;, \qquad (4.4)$$

$$G^i{}_j = {}^3G^i{}_j - (K^i_{\pm j} - \delta^i{}_j K_\pm)_{,n} - K_\pm K^i_{\pm j}$$
$$+\tfrac{1}{2}\delta^i{}_j K_\pm^2 + \tfrac{1}{2}\delta^i{}_z K_{ab}^{\pm}K_{\pm}^{ab} = \begin{cases} -\delta^i{}_j\Lambda, \\ 0 \,, \end{cases} \qquad (4.5)$$

where ${}^3R$, ${}^3G^i{}_j$, and the derivative symbol $|$ refer to the three-surface Σ. To discover the effect of the surface energy-momentum density of Σ on the space-time geometry, one should perform a "pill-box integration" of Einstein's field equations across Σ:

$$\lim_{\Sigma\to 0}\left[\int_{-\Sigma}^{+\Sigma} G^\alpha{}_\beta\, dn\right] = 8\pi S^\alpha{}_\beta \;. \qquad (4.6)$$

Define

$$\gamma_{ab} \equiv K_{ab}^+ - K_{ab}^- \;, \qquad (4.7)$$

$$\tilde{K}_{ab} \equiv \tfrac{1}{2}K_{ab}^+ + \tfrac{1}{2}K_{ab}^- \;. \qquad (4.8)$$

The "pill-box integration" of (4.5) and (4.4) implies

$$\gamma_{ab} - g_{ab}\gamma = -8\pi S_{ab} \qquad (4.9)$$

and

$$(\gamma^m{}_i - \delta^m{}_i\gamma)_{|m} = 0 \;. \qquad (4.10)$$

From Eqs. (4.9) and (4.10) one obtains

$$s^{im}{}_{|m}=0=(-\sigma g^{im})_{|m}=-\sigma^i\ ,$$

i.e., σ indeed remains constant as expected. By combining Eqs. (4.3) and (4.9), we find

$$\tilde{K}_{ab}g^{ab}=\frac{\Lambda}{8\pi\sigma}\ . \tag{4.11}$$

Let the motion of the modulus wall be parametrized by

$$x_\pm=x_\pm(\tau),\quad s=s(\tau)\ ,$$

where τ is the proper time measured along $\theta=$const, $\varphi=$const. Its tangent vector is

$$u^{a}_{\pm}=\frac{dx^{a}_{\pm}}{d\tau}\equiv(Y_\pm,0,0,\dot{s}),\quad (\ )^{\cdot}\equiv\frac{\partial}{\partial\tau}(\ )\ . \tag{4.12}$$

Using $n^{\pm}_{\alpha}u^{\alpha}_{\pm}=0$ and $u^{\pm}_{\alpha}u^{\alpha}_{\pm}=-1$, one finds

$$n^{\pm}_{\alpha}=(-\dot{s},0,0,Y_\pm) \tag{4.13}$$

and

$$Y_\pm=\pm(f_\pm{}^2\dot{s}^2-f_\pm)^{1/2}\ , \tag{4.14}$$

where the symbol $\pm$ in front of the square root denotes the velocity direction and it must change during the evolution of the modulus wall. The sign convention used below is to describe the wall motion before the collision. Differentiating $u^{\pm}_{\alpha}u^{\alpha}_{\pm}=-1$, we have

$$0=u^{\pm}_{\alpha}\frac{\delta u^{\alpha}_{\pm}}{d\tau}=f_\pm{}^{-1}Y_\pm\frac{\delta^2x_\pm}{d\tau^2}-f_\pm\dot{s}\frac{\delta^2 s}{d\tau^2}\bigg|_\pm\ ,$$

where the δ means covariant derivative. Therefore,

$$\begin{aligned} n^{\pm}_{\alpha}\frac{\delta u^{\alpha}_{\pm}}{d\tau}&=-\dot{s}\frac{\delta^2x_\pm}{d\tau^2}+Y_\pm\frac{\delta^2 s}{d\tau^2}\bigg|_\pm\\ &=\left[-\dot{s}\frac{f_\pm{}^2\dot{s}}{Y_\pm}+Y_\pm\right]\frac{\delta^2 s}{d\tau^2}\bigg|_\pm \end{aligned} \tag{4.15}$$

where

$$\frac{\delta^2 s}{d\tau^2}\bigg|_\pm=\ddot{s}-\frac{1}{2}\frac{df_\pm{}^{-1}}{ds}=\begin{cases}\ddot{s}-\dfrac{s\Lambda}{3}-\dfrac{m_+}{s^2},\\[2mm] \ddot{s}-\dfrac{m_-}{s^2}\ .\end{cases} \tag{4.16}$$

Let A^i be an arbitrary three-vector in Σ:

$$\vec{A}=A^i\vec{e}_i\ .$$

Then one can easily verify that

$$\frac{\partial\vec{A}}{\partial\xi^j}=A^i{}_{;j}\vec{e}_i-A^iK_{ij}\vec{n}\ . \tag{4.17}$$

Using Eq. (4.17) we find that the four-accelerations of the modulus wall, as measured in V^+ and V^-, are

$$\frac{\delta u^\alpha}{d\tau}\bigg|_\pm=-u^iK^{\pm}_{ij}u^jn^\alpha\ . \tag{4.18}$$

Equations (4.15), (4.16), and (4.18) imply

$$-\frac{f^+}{Y^+}\left[\ddot{s}-\frac{1}{2}\frac{df_+{}^{-1}}{ds}\right]-\frac{f^-}{Y^-}\left[\ddot{s}-\frac{1}{2}\frac{df_-{}^{-1}}{ds}\right]=-2u^au^b\tilde{K}_{ab}\ , \tag{4.19}$$

$$-\frac{f^+}{Y^+}\left[\ddot{s}-\frac{1}{2}\frac{df_+{}^{-1}}{ds}\right]+\frac{f^-}{Y^-}\left[\ddot{s}-\frac{1}{2}\frac{df_-{}^{-1}}{ds}\right]=-4\pi\sigma\ . \tag{4.20}$$

After a short calculation, one finds the components of the extrinsic curvature of Σ to be

$$\begin{aligned} K^{\pm}_{\theta\theta}&=s(\dot{s}^2-f_\pm{}^{-1})^{1/2}\ ,\\ K^{\pm}_{\phi\phi}&=s\sinh^2\theta\left[\dot{s}^2-f_\pm^{-1}\right]^{1/2}\ ,\\ K^{\pm}_{\theta\phi}&=0\ . \end{aligned} \tag{4.21}$$

Substituting Eqs. (4.11) and (4.21) into Eq. (4.19), we find

$$-\frac{f^+}{Y^+}\left[\ddot{s}-\frac{1}{2}\frac{df_+{}^{-1}}{ds}\right]-\frac{f^-}{Y^-}\left[\ddot{s}-\frac{1}{2}\frac{df_-{}^{-1}}{ds}\right]+\frac{2}{s}\left[\dot{s}^2-\frac{1}{f^+}\right]^{1/2}+\frac{2}{s}\left[\dot{s}^2-\frac{1}{f^-}\right]^{1/2}=\frac{\Lambda}{4\pi\sigma}\ . \tag{4.22}$$

One can easily check that in flat space-time the equation of motion for the modulus wall (4.22) reduces to that of Hawking *et al.*,[5] derived solely from requiring energy conservation. One can verify that Eqs. (4.20) and (4.22) possess the following first integral:

$$\begin{aligned} \frac{Y_-}{f_-}&=-(\dot{s}^2-f_-{}^{-1})^{1/2}\\ &=\frac{-1}{4sc}(f_+{}^{-1}-f_-{}^{-1})-sc\quad (c=2\pi\sigma)\ , \end{aligned} \tag{4.23a}$$

or equivalently

$$\begin{aligned} \frac{Y_+}{f_+}&=-(\dot{s}^2-f_+{}^{-1})^{1/2}\\ &=\frac{-1}{4sc}(f_+{}^{-1}-f_-{}^{-1})+sc\quad (c=2\pi\sigma)\ . \end{aligned} \tag{4.23b}$$

We can discuss the following cases.

(i) $\Lambda\neq0$, $m_\pm=0$ (which describes the expansion of one bubble before the collision):

$$(\dot{s}^2-1)^{1/2}=\frac{s}{R}\quad (R=\text{const})\ , \tag{4.24}$$

where R is the initial size of the bubble, one can show that

$$R=\frac{\sigma}{2\pi\sigma^2+\Lambda/24\pi}\ , \tag{4.25}$$

which is the same result as Coleman derives from the minimum-action principle.[4]

(ii) $\Lambda\neq0$, $m_-=0$, $m_+\neq0$ (which describes the motion of the modulus wall with $\alpha=0$ after the collision):

$$\pm(\dot{s}^2-1)^{1/2}=2\pi s\sigma+\frac{1}{8\pi\sigma s}\left[\frac{s^2\Lambda}{3}-\frac{2m_+}{s}\right], \qquad (4.26a)$$

$$\pm\left[\dot{s}^2-1-\frac{s^2\Lambda}{3}+\frac{2m_+}{s}\right]^{1/2}$$
$$=-2\pi s\sigma+\frac{1}{8\pi\sigma s}\left[\frac{s^2\Lambda}{3}-\frac{2m_+}{s}\right]. \qquad (4.26b)$$

After the first collision at $s=s_1$, the region $R_1,R_2,\ldots$ of false vacuum can be described by pseudo-Schwarzschild–de Sitter metrics with suitable parameters $m_1,m_2,\ldots$. At $s=s_1,s_2,\ldots$ the fact that the velocity of the outgoing modulus wall measured in the false-vacuum regions with respect to the symmetry plane remains the same as that of the incoming one, by requiring energy conservation in the local collision region,[5] yields

$$(f_{i-1}^+\dot{s}^2-1)^{1/2}_{s=s_{i-}}=(f_i^+\dot{s}^2-1)^{1/2}_{s=s_{i+}}. \qquad (4.27)$$

M_i^+ is related to M_{i-1}^+ through (4.26) and (4.27), and by such an iteration procedure the motion of the modulus walls and the metric of the whole manifold are determined.

It is not possible for any false-vacuum regions $R_1,R_2,\ldots$ to commence before their Killing horizon

$$1+\frac{s^2\Lambda}{3}-\frac{2m_i^+}{s}=0,$$

for the same reason we gave at the end of the last section.

If viewed in the real vacuum, the modulus wall comes to rest at one moment during a bounce, i.e., $\dot{x}_-=0$, Eqs. (4.14) and (4.26a) imply

$$\frac{s^2\Lambda}{3}-\frac{2m_i^+}{s}<0$$

at this moment. When this wall comes to rest, viewed from the false vacuum, i.e., $\dot{x}_+=0$, then it follows from (4.14) and (4.26b) that $s^2\Lambda/3-2m_i^+/s>0$. Since the quantity on the left-hand side of the above inequality increases monotonically with s during any interval between collisions, this must occur at a time which is later than the time at which the wall appears to be at rest viewed from the true vacuum. Because of the expansion of the space-time along the s direction and the conservation of energy, the oscillation magnitude of the modulus walls decreases and the wall velocity, viewed in the false-vacuum region, will eventually vanish.[5] The left-hand side of (4.26b) over

$$\left[1+\frac{s^2\Lambda}{3}-\frac{2m_+}{s}\right]^{1/2}$$

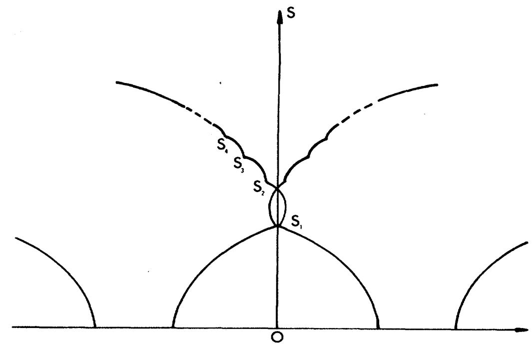

FIG. 6. The observer inside the real-vacuum region sees the modulus walls traveling outwards eventually with kinks due to a series of collisions.

represents the wall velocity with respect to a static observer in the false vacuum, so the right-hand side of (4.26b) over

$$\left[1+\frac{s^2\Lambda}{3}-\frac{2m_+}{s}\right]^{1/2}$$

must tend to 0 as $s\to\infty$. It follows that $m_+\propto O(s^3)$ as $s\to\infty$ (it should be mentioned that Coleman's argument in Ref. 4 is based on the condition implicitly: $48\pi^2\sigma^2<\Lambda$) and the difference of the two terms of its right-hand side is of lower order than $O(s)$ then. Thus, for sufficiently large s the right-hand side of (4.26a) will never vanish and according to our sign convention the modulus wall must travel outward from the real-vacuum region. Briefly, viewed in the real vacuum, the modulus wall eventually propagates outwards with kinks due to a series of collisions (see Fig. 6).

(iii) $\Lambda\neq0$, $m_+\neq0$, $m_-\neq0$ (which is the most general case of collision with $\alpha\neq0$): One can use similar methods except that at the first collision the velocity of the modulus walls suffers a jump due to the phase waves. The argument of the last paragraph of case (ii) can still be applied to this general case.

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