Smarr's zero-frequency-limit calculation

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We show that Smarr's calculation of the zero-frequency limit of the energy spectrum for the gravitational radiation produced during the scattering or collision of two particles is a linearized approximation valid only when the radiation is weak. In particular, it cannot be applied to the head-on collision of two black holes, so Smarr's conclusions concerning the isotropy of the radiation emitted during the high-speed, equal-mass, black-hole collision have no firm foundation.

I. INTRODUCTION

A paper by Smarr[1] appeared in the Physical Review some years ago, discussing the zero-frequency-limit technique, which purports to be able to predict exactly the zero-frequency limit (ZFL) of the gravitational radiation spectrum produced by a scattering process for which the asymptotic trajectories of the colliding bodies are known. In other words, by following the algorithm in Ref. 1, one is supposed to derive the zero-frequency limit of $dE/d\omega\, d\Omega$, the energy radiated per unit frequency per unit solid angle as gravitational waves. One can show that the energy spectrum is flat as $\omega \rightarrow 0$ (Ref. 1) [i.e., $(d/d\omega)dE/d\omega\, d\Omega\,|_{\omega=0}=0$], so by crude extrapolation from $\omega=0$ one obtains some idea of the angular distribution, and polarization, of the radiation at nonzero frequencies. In addition, by introducing a cutoff frequency ω_c, determined by the physical parameters of the problem, one obtains an estimate for the total energy radiated during the collision, namely,

$$E_{\text{total}} \approx \omega_c \int \left. \frac{dE}{d\omega\, d\Omega} \right|_{\omega=0} d\Omega \tag{1}$$

which should certainly be of the correct order of magnitude, if the ZFL formula is correct.

In particular, Smarr applied the ZFL method to the axisymmetric high-speed, equal-mass, black-hole collision and found that apart from some detailed structure very near the axis $\theta=0,\pi$, $dE/d\omega\, d\Omega\,|_{\omega=0}$ is isotropic. He conjectured that $dE/d\omega\, d\Omega$ is also isotropic at nonzero frequencies, and that therefore the isotropic part of the news function calculated by D'Eath[2] is in fact the full news function. This would mean that in the high-speed collision, gravitational waves are generated with an efficiency of 25%, well below the upper bound of 50% placed on the efficiency by the arguments of Penrose[3] (which rest on the assumption that cosmic censorship is valid).

We show here that the ZFL formula in Ref. 1 is in fact a linearized approximation, valid only when the gravitational radiation is weak, and which cannot predict the strong-field radiation generated by fully nonlinear gravitational interactions. Thus it cannot be applied to the axisymmetric two-black-hole encounter, so Smarr's conjecture concerning the isotropy of the radiation generated by this collision has no firm foundation.

II. THE EQUAL-MASS, AXISYMMETRIC COLLISION

We shall now demonstrate explicitly that for the axisymmetric collision of two bodies of equal mass coalescing to form one body at rest, the ZFL formula in Ref. 1 is incorrect. We do this as follows. First, we show that a knowledge of $dE/d\omega\, d\Omega\,|_{\omega=0}$ is sufficient to determine *exactly* the mass of the body formed as the product of the collision. Then we show that the explicit form (2.16) for the ZFL predicts that the mass of the final Schwarzschild field will be equal to the incident energy of the colliding particles (we use dimensionless units, $G=c=1$). Since gravitational waves carry away energy, it is therefore predicting that no gravitational waves will be produced by the collision: that is to say, there will be no "news." But if there is no news, then $dE/d\omega\, d\Omega\,|_{\omega=0}$ must vanish. This is a contradiction.

Since this system is axisymmetric and reflection-symmetric, we may use the results of Bondi *et al.*[4] In particular, we shall make use of the supplementary condition [Eq. (35) in Ref. 4]

$$\frac{\partial M}{\partial u} = -\left[\frac{\partial c}{\partial u}\right]^2 + \frac{1}{2}\frac{\partial}{\partial u}\left[\frac{\partial^2 c}{\partial \theta^2} + 3\cot\theta \frac{\partial c}{\partial \theta} - 2c\right] . \tag{2}$$

Here $M(u,\theta)$ is the mass aspect of the system, and $\partial c(u,\theta)/\partial u$ is the news function. In the Bondi metric, M appears in

$$g_{uu} = 1 - \frac{2M(u,\theta)}{r} + \cdots , \tag{3}$$

while

$$\begin{aligned} g_{\theta\theta} &= r^2\left[1 + \frac{2c}{r} + \cdots\right] , \\ g_{\phi\phi} &= r^2 \sin^2\theta \left[1 - \frac{2c}{r} + \cdots\right] . \end{aligned} \tag{4}$$

M is thus a generalized mass suitable for nonstatic systems. The mass aspect of a particle of rest mass m, moving with speed v [and Lorentz factor $\gamma=(1-v^2)^{-1/2}=\cosh\lambda$] in the $\theta=\pi$ direction is (Ref. 4, Appendix 3)

$$M(\theta) = \frac{m}{(\cosh\lambda + \cos\theta \sinh\lambda)^3} . \tag{5}$$

Reexpressing this in terms of v, we find

$$M=\frac{m(1-v^2)^{3/2}}{(1+v\cos\theta)^3} . \tag{6}$$

Let the rest mass of each body be m. Before the collision, we have one particle moving with speed v in the $\theta=0$ direction, while the other moves with the same speed in the $\theta=\pi$ direction. The respective mass aspects of these two particles are

$$M_1=\frac{m(1-v^2)^{3/2}}{(1-v\cos\theta)^3},\quad M_2=\frac{m(1-v^2)^{3/2}}{(1+v\cos\theta)^3} . \tag{7}$$

In the distant past, the total mass aspect of the system is simply the linear superposition of M_1 and M_2:

$$M(u=-\infty,\theta)=M_1+M_2 . \tag{8}$$

On the axis $\theta=0,\pi$, the news function $\partial c/\partial u$ and its first angular derivative $(\partial/\partial\theta)\partial c/\partial u$ must both vanish, to ensure the regularity of the metric. Thus on the axis, Eq. (2) becomes

$$\left.\frac{\partial M}{\partial u}\right|_{\theta=0,\pi}=\frac{1}{2}\frac{\partial}{\partial u}\left.\left[\frac{\partial^2 c}{\partial\theta^2}+3\cot\theta\,\frac{\partial c}{\partial\theta}\right]\right|_{\theta=0,\pi} . \tag{9}$$

Integrating over retarded time, we find

$$[M(\infty,\theta)-M(-\infty,\theta)]\,|_{\theta=0,\pi}=\frac{1}{2}\left[\frac{\partial^2}{\partial\theta^2}+3\cot\theta\frac{\partial}{\partial\theta}\right][c(\infty,\theta)-c(-\infty,\theta)]\Bigg|_{\theta=0,\pi} . \tag{10}$$

The initial mass aspect is known: it is given by (8). After the collision, the nonspherical perturbations will die away and the field will approach the Schwarzschild field asymptotically. This means that the final mass aspect of the system will be isotropic, and equal in magnitude to the mass of the body formed by the collision: $M(\infty,\theta)=m_{\text{final}}$, independent of angle. Hence

$$m_{\text{final}}=M(-\infty,\theta)\,|_{\theta=0,\pi}+\frac{1}{2}\left[\frac{\partial^2}{\partial\theta^2}+3\cot\theta\frac{\partial}{\partial\theta}\right][c(\infty,\theta)-c(-\infty,\theta)]\Bigg|_{\theta=0,\pi} . \tag{11}$$

Thus to calculate m_{final}, we need only know $c(\infty,\theta)-c(-\infty,\theta)$. However [see Eq. (24) below]

$$c(\infty,\theta)-c(-\infty,\theta)=2\pi\left[\left.\frac{dE}{d\omega\,d\Omega}\right|_{\omega=0}\right]^{1/2} , \tag{12}$$

where the sign of the square root must be chosen appropriately (in fact, in Ref. 1 $h_{jk}^{TT}|_{-\infty}^{\infty}$ is calculated directly, so there is no ambiguity). Therefore the ZFL *alone* is sufficient to determine the final mass. From Eq. (2.19) in Ref. 1 [or directly from (2.15)], we find

$$c(\infty,\theta)-c(-\infty,\theta)=-2\gamma m v^2\frac{\sin^2\theta}{(1-v^2\cos^2\theta)} . \tag{13}$$

A brief calculation yields

$$\frac{1}{2}\left[\frac{\partial^2}{\partial\theta^2}+3\cot\theta\frac{\partial}{\partial\theta}\right][c(\infty,\theta)-c(-\infty,\theta)]\Bigg|_{\theta=0}=\frac{-8\gamma m v^2}{(1-v^2)} . \tag{14}$$

Also

$$M(-\infty,0)=m\gamma(1-v^2)^2\left[\frac{1}{(1-v)^3}+\frac{1}{(1+v)^3}\right] . \tag{15}$$

Substituting (14) and (15) into (11), and simplifying, we find that

$$m_{\text{final}}=2m\gamma . \tag{16}$$

Thus Smarr's formula for the ZFL predicts that the final mass of the system will be equal to the initial energy of the colliding bodies, which cannot be correct.

III. THE SUPERTRANSLATION FREEDOM

In Ref. 1, the ZFL formula is derived using standard linearized theory (see Misner, Thorne, and Wheeler[5]). Here we use instead the entirely analogous news-function method of Bondi[4] and Sachs.[6] Using this approach, the error in the derivation of the ZFL formula (A10) in Ref. 1 is more readily apparent.

The gravitational radiation in an asymptotically flat space-time is described by a complex function (the so-called "news function"[4,6])

$$\frac{\partial c(u,\theta,\phi)}{\partial u}=\frac{\partial A_+(u,\theta,\phi)}{\partial u}+i\frac{\partial A_\times(u,\theta,\phi)}{\partial u} , \tag{17}$$

where u is retarded time, and θ and ϕ are the polar and azimuthal angles, respectively. $\partial A_+/\partial u$ and $\partial A_\times/\partial u$ are the amplitudes of the two polarization modes of the radiation field. The energy flux per steradian is[6]

$$\frac{dE}{du\,d\Omega}=\frac{1}{4\pi}\left|\frac{\partial c}{\partial u}\right|^2=\frac{1}{4\pi}\left[\left[\frac{\partial A_+}{\partial u}\right]^2+\left[\frac{\partial A_\times}{\partial u}\right]^2\right] . \tag{18}$$

Thus the total energy radiated per steradian is

$$\frac{dE}{d\Omega}=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\left[\frac{\partial A_+}{\partial u}\right]^2+\left[\frac{\partial A_\times}{\partial u}\right]^2\right]du . \tag{19}$$

Define $\dot a_+(\omega)$ and $\dot a_\times(\omega)$ by

$$\dot a_+(\omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{\partial A_+}{\partial u}e^{i\omega u}du ,$$

$$\dot a_\times(\omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{\partial A_\times}{\partial u}e^{i\omega u}du . \tag{20}$$

Using Parseval's theorem, we find that

$$\frac{dE}{d\Omega}=\frac{1}{2\pi}\int_0^{\infty}[\,|\dot{a}_+(\omega)|^2+|\dot{a}_\times(\omega)|^2]d\omega\ . \qquad (21)$$

Clearly ω is to be identified with frequency, and so the energy radiated per unit frequency per steradian is

$$\frac{dE}{d\omega\, d\Omega}=\frac{1}{2\pi}[\,|\dot{a}_+(\omega)|^2+|\dot{a}_\times(\omega)|^2]\ . \qquad (22)$$

At zero frequency

$$\dot{a}_+(0)=(2\pi)^{-1/2}(A_+\,|_{u=\infty}-A_+\,|_{u=-\infty})\ ,$$
$$\dot{a}_\times(0)=(2\pi)^{-1/2}(A_\times\,|_{u=\infty}-A_\times\,|_{u=-\infty})\ . \qquad (23)$$

Thus the ZFL of $dE/d\omega\, d\Omega$ is

$$\left.\frac{dE}{d\omega\, d\Omega}\right|_{\omega=0}=\frac{1}{4\pi^2}[(A_+\,|_{-\infty}^{\infty})^2+(A_\times\,|_{-\infty}^{\infty})^2]$$
$$=\frac{1}{4\pi^2}\,|\,c(+\infty,\theta,\phi)-c(-\infty,\theta,\phi)\,|^2\ . \qquad (24)$$

Equation (24) is entirely equivalent to Eqs. (A8) and (A9) in Ref. 1, since

$$A_+=\frac{1}{2\sqrt{2}r}h_{jk}e_+^{jk},\quad A_\times=\frac{1}{2\sqrt{2}r}h_{jk}e_\times^{jk} \qquad (25)$$

and the argument is certainly correct up to this point.

The crucial question now is whether or not, in some scattering process, one can calculate the total change in $c(u,\theta,\phi)$ [or equivalently $h_{jk}^{TT}(t,r,\theta,\phi)$] in going from the infinite past to the infinite future, knowing only the asymptotic four-momenta of the gravitating bodies. In Ref. 1, it is assumed that the metric perturbation $\bar{h}_{\mu\nu}$ induced at a space-time point x, by a particle of rest mass m at x', moving with four-velocity u_ν, necessarily has the explicit form (A2), i.e.,

$$\bar{h}_{\mu\nu}=\frac{4mu_\mu u_\nu}{-k\cdot u}\ , \qquad (26)$$

where $k=(|\vec{x}-\vec{x}\,'|,\vec{x}-\vec{x}\,')$ is the null vector joining x and x'. From (26) one deduces that in the gravitational scattering of N particles (incoming and outgoing particles are counted separately), the change in the transverse-traceless metric perturbations induced at x, between the initial and final states, is

$$\left.h_{jk}e^{jk}_{\{\times \atop +\}}\right|_{-\infty}^{\infty}=\lim_{|\vec{x}_N|\to\infty,\forall N}\left[4\sum_N \eta_N\left[\frac{P_\mu{}^N e^{\mu\nu}_{\{\times \atop +\}}P_\nu{}^N}{k^N\cdot p^N}\right]\right]\ , \qquad (27)$$

where $k^N=(|\vec{x}-\vec{x}_N|,\vec{x}-\vec{x}_N)$, P^N is the asymptotic four-momentum of the Nth particle, and $\eta_N=+1\ (-1)$ for incoming (outgoing) particles. The ZFL formula (A10) in Ref. 1 follows directly from (27). However, we know from the work of Bondi[4] and Sachs[6] that $c(u,\theta,\phi)$ (and hence h_{jk}^{TT}) has no invariant significance, and in fact can be changed by an arbitrary amount by making an appropriate supertranslation. More explicitly, if we change our origin of retarded time, letting $u=u'+\alpha(\theta,\phi)$, then $c(u,\theta,\phi)$ transforms as[6]

$$c'=c-\tfrac{1}{2}\sin\theta\Delta[(\sin\theta)^{-1}\Delta\alpha]\ , \qquad (28)$$

where $\Delta=\partial/\partial\theta+i(\sin\theta)^{-1}\partial/\partial\phi$. It is true that for each particle there will exist a set of coordinates for which the $\bar{h}_{\mu\nu}$ induced by that particle has the explicit form (26). However, *a priori* there is no reason why these individual coordinate systems should be in the same supertranslation state. In general they will not, and there will exist no coordinate system in which the $\bar{h}_{\mu\nu}$ induced by *every* particle has the explicit form (26). When this happens, the formula (A10) for the ZFL will be incorrect.

IV. THE LINEARIZED APPROXIMATION

It is significant that the ZFL formula (A10) predicts that there will be no mass loss in this collision, rather than some other result. It is exactly what one would obtain if one assumed that the gravitational radiation was so weak that all nonlinearities could be neglected. For computational simplicity, let us consider the example of the two equal-mass black-hole collision. Let us assume, although we know it to be false, that the radiation produced by the collision is very weak ($|\partial c/\partial u|\ll 1$), so that the nonlinear $(\partial c/\partial u)^2$ term in (2) can be neglected. Then

$$\frac{\partial M}{\partial u}=\frac{1}{2}\frac{\partial}{\partial u}\left[\frac{\partial^2 c}{\partial\theta^2}+3\cot\theta\frac{\partial c}{\partial\theta}-2c\right]\ . \qquad (29)$$

Integrating (29) over retarded time, we have

$$M(u,\theta)\,|_{-\infty}^{\infty}=\frac{1}{2}\left[\frac{\partial^2}{\partial\theta^2}+3\cot\theta\frac{\partial}{\partial\theta}-2\right][c(u,\theta)\,|_{-\infty}^{\infty}]\ . \qquad (30)$$

Under this assumption there is no loss of mass, since it is the nonlinear term which carries away the energy. Therefore the mass of the residual black hole formed by the collision will be equal to the initial energy. Equation (30) may be thought of as a second-order inhomogeneous differential equation in $c(u,\theta)\,|_{-\infty}^{\infty}$. The source term $M(u,\theta)\,|_{-\infty}^{\infty}$ is known, since

$$M(-\infty,\theta)=m\gamma(1-v^2)^2\left[\frac{1}{(1+v\cos\theta)^3}+\frac{1}{(1-v\cos\theta)^3}\right] \qquad (31)$$

and

$$M(\infty,\theta)=2m\gamma\ . \qquad (32)$$

The boundary conditions

$$c(u,\theta)\,|_{-\infty}^{\infty}\,|_{\theta=0,\pi}=\left.\left.\frac{\partial c(u,\theta)}{\partial\theta}\right|_{-\infty}^{\infty}\right|_{\theta=0,\pi}=0 \qquad (33)$$

are sufficient to uniquely determine the solution. Smarr's formula for $c\,|_{-\infty}^{\infty}$ is given by (13). By substitution, one may easily verify that the explicit form (13) for $c\,|_{-\infty}^{\infty}$ is the solution to the differential equation (30). We are led to conjecture that Smarr's formula for $c\,|_{-\infty}^{\infty}$ is in each case the solution to the linearized version of the supplementary condition (2), or its nonaxisymmetric generalization (we shall confirm this below). If there is strong-field gravitational radiation present in the space-time

($|\partial c/\partial u| \approx 1$), inclusion of the nonlinear $|\partial c/\partial u|^2$ term will clearly lead to a different $c|_{-\infty}^{\infty}$. However, when the radiation really is weak ($|\partial c/\partial u| \ll 1$), the quadratic term will be very small, so (A10) will be very close to the true ZFL. It seems that the presence of strong-field radiation somehow induces a supertranslation between the initial and final states, as viewed from $\mathscr{I}^+$, spoiling the simple formula (A2). However, it is not possible to quantify this statement.

It is well known that general relativity is equivalent to the quantum theory of a massless spin-2 field in Minkowski space, if one sums over all tree graphs (Deser,[7] Deser and Boulware[8]). The ZFL formula in Ref. 1 was originally derived from the quantum approach (see references in Ref. 1). The argument is briefly as follows. To find the total amplitude for the generation of low-frequency gravitational radiation, one sums the amplitudes for all Feynman diagrams representing emission of soft gravitons by the collision. One finds that the only contribution to the sum is from those diagrams in which the soft gravitons are emitted from external particle lines. This is why the total amplitude (2.1) in Ref. 1 depends only on the asymptotic momenta of the colliding bodies. However, (2.1) completely neglects the contribution to the total amplitude from soft graviton emission by external graviton lines. The effective coupling constant for the emission of a very soft graviton from an external graviton line with energy E is proportional to E (Weinberg[9]). If there is a large quantity of gravitational radiation present, such as in the two-black-hole collision, then a large proportion of the outgoing momentum will be carried by gravitons of nonzero energy E. Soft graviton emission from these gravitons of nonzero energy will then make a significant contribution to the ZFL amplitude. Thus one cannot calculate the zero-frequency part of the graviton energy spectrum without knowing the entire spectrum itself. This provides a nice parallel to the classical picture, in which, because of the supertranslation freedom, one needs to know the entire news function $\partial c/\partial u$ before one can calculate its zero-frequency part $c|_{-\infty}^{\infty}$.

We see that in the quantum picture, we obtain the formula (2.1) for the emission amplitude only if we neglect all contributions from graviton-graviton interactions. However, it is well known that if one ignores gravitational self-interactions in the spin-2 quantum field theory, so that in calculating amplitudes one includes only those Feynman diagrams which have no graviton-graviton vertices, then one obtains a theory in which particles interact exactly as in linearized general relativity. This shows that Smarr's formula for the ZFL, which follows directly from the quantum amplitude (2.1), will in each case provide, through Eq. (24), the solution to the linearized supplementary condition, confirming our conjecture. It also explains why in the gravitational scattering of two bodies with a large impact parameter, in which gravitational self-interactions *are* unimportant, Smarr found such good agreement between his ZFL calculation and the more elaborate investigations of previous authors.

V. CONCLUSIONS

Even when nonlinear effects are important, the ZFL formula (A10) should still be able to give an order-of-magnitude estimate of the total energy carried off by gravitational waves. Inclusion of the $\int_{-\infty}^{\infty} c_0{}^2$ term in (30) will of course change $c|_{-\infty}^{\infty}$, but one expects the exact $c|_{-\infty}^{\infty}$ to be of the same order of magnitude as the linearized $c|_{-\infty}^{\infty}$. Thus the estimate (1), using (A10), should still be reasonable. The angular distribution and polarization of the true ZFL may of course be quite different from that calculated in Ref. 1.

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