Conformally flat Einstein-Yang-Mills-Higgs solutions with spherical symmetry

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We solve the Einstein-Yang-Mills-Higgs equations in a conformally flat metric with spherical symmetry. Two solutions are obtained corresponding to magnetic monopoles in the Higgs vacuum and outside of it.

I. INTRODUCTION

The problem of Yang-Mills-Higgs solutions in a curved space-time remains of central interest for the understanding of the solutions of a future quantum theory of gravitation. The Higgs mechanism with its spontaneous symmetry breaking is going to be explored as the mass source in cosmological models.[1,2] The coupling constant of Yang-Mills theories was related to the cosmological constant[3] and remarks were made about the vanishing of the latter by the use of the fields in a Higgs vacuum.[4,5] We work here in a conformally flat space-time with spherical symmetry. The first of the solutions which we found represents a magnetic monopole in the Higgs vacuum, analogously to the solution found by Kasuya[4] in a Reissner-Nordström metric. The second one represents also a magnetic monopole but outside of the Higgs vacuum. We observe that this last solution reduces to the first one by a special choice of the vacuum expectation value of the Higgs field. Further, this solution suggests that the assertion that all finite-energy solutions in flat Minkowski space-time of this problem are in the Higgs vacuum[5,14] cannot be extended to the problem in a curved space-time.

II. THE FIELD EQUATIONS

We start from the action of a SO(3) Yang-Mills-Higgs theory in a curved space-time,

$$S=\int \sqrt{-g}\,d^4x\left[-\frac{1}{2k}(R-2\Lambda)-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a -\tfrac{1}{2}g^{\mu\nu}D_\mu\phi^a D_\nu\phi_a-\lambda V(\phi)\right], \quad (2.1)$$

where

$$F^a_{\mu\nu}=A^a_{\nu|\mu}-A^a_{\mu|\nu}+\epsilon\epsilon^a{}_{bc}A^b_\mu A^c_\nu\ , \quad a=1,2,3 \quad (2.2)$$

and

$$D_\mu\phi^a=\phi^a{}_{|\mu}+\epsilon\epsilon^a{}_{bc}A^b_\mu\phi^c \quad (2.3)$$

are the Yang-Mills (YM) fields and the internal covariant derivative, respectively. The last term in the integrand of Eq. (2.1) is the Higgs potential

$$V(\phi)=\tfrac{1}{4}(\phi^a\phi_a)^2-\tfrac{1}{2}\alpha^2\phi^a\phi_a\ , \quad (2.4)$$

where

$$\alpha^2=\frac{\mu^2}{\lambda}=\langle 0\,|\,\phi^a\phi_a\,|\,0\rangle \quad (2.5)$$

is the vacuum expectation value of the Higgs field, which is associated with mass creation, according to the Higgs symmetry-breaking mechanism.

We are looking for a spherically symmetric conformally flat static solution. We take for the line element[6]

$$ds^2=e^{2\sigma(r)}(-dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\varphi^2)\ . \quad (2.6)$$

The YM potentials and the fields can be written as

$$\begin{aligned}
&A^a_t=(J/\epsilon r)\hat r^a_t,\quad A^a_r=0,\\
&A^a_\theta=[(K-1)/\epsilon]\hat r^a_\varphi,\quad A^a_\varphi=[(1-K)/\epsilon]\sin\theta\,\hat r^a_\theta\ ,\\
&F^a_{rt}=(J/\epsilon r)'\,\hat r^a_r,\quad F^a_{\theta t}=(KJ/\epsilon r)\,\hat r^a_\theta\ ,\\
&F^a_{\varphi t}=(KJ/\epsilon r)\sin\theta\,\hat r^a_\varphi\ ,\\
&F^a_{\theta\varphi}=[(K^2-1)/\epsilon]\sin\theta\,\hat r^a_r,\quad F^a_{\varphi r}=(K'/\epsilon)\sin\theta\,\hat r^a_\theta\ ,\\
&F^a_{r\theta}=(K'/\epsilon)\hat r^a_\varphi\ ,
\end{aligned} \quad (2.7)$$

and the Higgs fields

$$\phi^a=(H/\epsilon r)\,\hat r^a_r\ , \quad (2.8)$$

where $\hat r_r$, $\hat r_\theta$, and $\hat r_\varphi$ are the components of the unit radial vector

$$\begin{aligned}
&\hat r_r=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\ ,\\
&\hat r_\theta=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta)\ ,\\
&\hat r_\varphi=(-\sin\varphi,\cos\varphi,0)\ .
\end{aligned} \quad (2.9)$$

The ansatz above may be easily deduced by the methods developed by Forgács and Manton.[7]

The Einstein, Yang-Mills, and Higgs equations are given by

$$\frac{\delta S}{\delta g_{\mu\nu}}=0\rightarrow R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=kT_{\mu\nu}\ , \quad (2.10)$$

$$\frac{\delta S}{\delta A^a_\mu}=0\rightarrow D_\nu(\sqrt{-g}F^{\mu\nu}_a)+\sqrt{-g}\,g^{\mu\nu}\epsilon\epsilon_{abc}\phi^bD_\nu\phi^c=0\ , \quad (2.11)$$

$$\frac{\delta S}{\delta\phi^a}=0\rightarrow D_\mu(\sqrt{-g}\,g^{\mu\nu}D_\nu\phi^a)-\sqrt{-g}\,\lambda(\phi^b\phi_b-\alpha^2)\phi^a=0\ , \quad (2.12)$$

where the energy-momentum tensor of (2.10) reads

$$T_{\mu\nu}=-F_{\mu a}^{\ \ \alpha}F_{\nu\alpha}^{a}+\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}^{a}F_{a}^{\alpha\beta}-D_{\mu}\phi^{a}D_{\nu}\phi_{a}+\tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_{\alpha}\phi^{a}D_{\beta}\phi_{a}+g_{\mu\nu}\lambda V(\phi)\ . \tag{2.13}$$

Using Eqs. (2.6) to (2.8), we can write the nonvanishing components of the field equations (2.10), (2.11), and (2.12):

$$(t\,t)\to 2\sigma''+\frac{4}{r}\sigma'+\sigma'^2-e^{2\sigma}\Lambda=-\frac{k}{\epsilon^2}\left[e^{-2\sigma}\left[\frac{J}{r}\right]'^2+2e^{-2\sigma}\frac{K^2J^2}{r^4}+\tfrac{1}{2}e^{2\sigma}L\right]-k\lambda e^{2\sigma}V(\phi)\ , \tag{2.14}$$

$$(rr)\to -\frac{4}{r}\sigma'-3\sigma'^2+e^{2\sigma}\Lambda=\frac{k}{\epsilon^2}\left[e^{-2\sigma}\left[\frac{J}{r}\right]'^2-2e^{-2\sigma}\frac{K'^2}{r^2}+\tfrac{1}{2}e^{2\sigma}L-\left[\frac{H}{r}\right]'^2\right]+k\lambda e^{2\sigma}V(\phi)\ , \tag{2.15}$$

$$(\theta\theta)\to -2r^2\sigma''-2r\sigma'-r^2\sigma'^2+r^2e^{2\sigma}\Lambda=\frac{k}{\epsilon^2}\left[e^{-2\sigma}\frac{K^2J^2}{r^2}-e^{-2\sigma}K'^2-e^{-2\sigma}\frac{(K^2-1)^2}{r^2}-\frac{H^2K^2}{r^2}+\tfrac{1}{2}r^2e^{2\sigma}L\right]$$
$$+k\lambda e^{2\sigma}r^2V(\phi)\ , \tag{2.16}$$

$$\left[r^2\left[\frac{J}{r}\right]'\right]'-2\frac{K^2J}{r}=0\ , \tag{2.17}$$

$$K''+\frac{KJ^2}{r^2}+\frac{K(1-K^2)}{r^2}-e^{2\sigma}\frac{KH^2}{r^2}=0\ , \tag{2.18}$$

$$\left[e^{2\sigma}r^2\left[\frac{H}{r}\right]'\right]'-2\frac{K^2H}{r}e^{2\sigma}-\frac{e^{4\sigma}\lambda}{\epsilon^2 r}(H^3-C^2r^2H)=0\ , \tag{2.19}$$

where L stands for

$$L=e^{-4\sigma}\left[-\left[\frac{J}{r}\right]'^2-2\frac{K^2J^2}{r^4}+2\frac{K'^2}{r^2}+\frac{(K^2-1)^2}{r^4}\right]+e^{-2\sigma}\left[\left[\frac{H}{r}\right]'^2+2\frac{K^2H^2}{r^4}\right]\ , \tag{2.20}$$

and

$$C^2=\epsilon^2\frac{\mu^2}{\lambda}=\epsilon^2\alpha^2\ . \tag{2.21}$$

The primes represent differentiation with respect to r. Let us observe that Eqs. (2.14)—(2.16) and (2.18) are not all independent. We have

$$(t\,t)+(rr)\to 2\sigma''-2\sigma'^2=\frac{k}{\epsilon^2}\left[-2e^{-2\sigma}\frac{K^2J^2}{r^4}-2e^{-2\sigma}\frac{K'^2}{r^2}-\left[\frac{H}{r}\right]'^2\right]\ , \tag{2.22}$$

$$(rr)-\frac{(\theta\theta)}{r^2}\to 2\sigma''-2\sigma'^2-\frac{2\sigma'}{r}=\frac{k}{\epsilon^2}\left[e^{-2\sigma}\left[\frac{J}{r}\right]'^2-e^{-2\sigma}\frac{K'^2}{r^2}-\left[\frac{H}{r}\right]'^2-e^{-2\sigma}\frac{K^2J^2}{r^4}+e^{-2\sigma}\frac{(K^2-1)^2}{r^4}+\frac{H^2K^2}{r^4}\right]\ , \tag{2.23}$$

$$(t\,t)+\frac{(\theta\theta)}{r^2}\to\frac{2\sigma'}{r}=\frac{k}{\epsilon^2}\left[-e^{-2\sigma}\left[\frac{J}{r}\right]'^2-e^{-2\sigma}\frac{K^2J^2}{r^4}-e^{-2\sigma}\frac{(K^2-1)^2}{r^4}-2e^{-2\sigma}\frac{K'^2}{r^2}-\frac{H^2K^2}{r^4}\right]\ . \tag{2.24}$$

Using Eq. (2.18) in (2.24) and subtracting Eq. (2.23) from (2.22) we get identical results. So we have to solve only Eqs. (2.17)—(2.19), (2.22), and (2.24).

III. THE TWO SOLUTIONS

A. A magnetic monopole in the Higgs vacuum

Owing to the complexity of the equations above, we try to get nontrivial solutions by simplifying them. Examining the ansatz (2.7), we see that a nontrivial choice will be $K=0$. Equation (2.17) gives then, after integration,

$$J(r)=a+br\ , \tag{3.1}$$

where a and b are integration constants. Equation (2.24) turns into

$$\frac{(e^{2\sigma})'}{r}=-\frac{k}{\epsilon^2}\frac{(1+a^2)}{r^4}\ , \tag{3.2}$$

which after integration gives

$$e^{2\sigma}=\beta+\gamma\frac{(1+a^2)}{r^2}\ , \tag{3.3}$$

where β is an integration constant and $\gamma=k/(2\epsilon^2)$. Using Eqs. (3.2) and (3.3), we can write for Eq. (2.22)

$$-\frac{3\gamma}{r^4}\frac{1}{\beta+\gamma/r^2}+\frac{3\gamma^2}{r^6}\frac{1}{(\beta+\gamma/r^2)^2}=\gamma\left[\frac{H}{r}\right]'^2 . \quad (3.4)$$

In order to have a Bertotti-Robinson—type solution[8–10] we take $\beta=0$ in Eq. (3.4). For $\beta=0$, the left-hand side of Eq. (3.4) vanishes and we have

$$H=Dr , \quad (3.5)$$

where D is a new integration constant.

The only equation which remains to be solved is (2.19). After substituting Eqs. (3.3) and (3.5), it turns into

$$-\frac{\lambda}{\epsilon^2}\frac{\gamma^2}{r^2}(D^3-C^2D)=0 . \quad (3.6)$$

A nontrivial solution will be given by

$$D=\pm C . \quad (3.7)$$

It can be seen that this solution corresponds to the minimum of the Higgs potential, Eq. (2.4), which can be written as

$$V(\phi)=\tfrac{1}{4}(\phi^a\phi_a-\alpha^2)^2-\tfrac{1}{4}\alpha^4 . \quad (3.8)$$

We have for solution I, by using Eq. (2.21),

$$V^{(\mathrm{I})}(\phi)=\frac{1}{4}\left[\frac{C^2}{\epsilon^2}-\alpha^2\right]-\tfrac{1}{4}\alpha^4=-\tfrac{1}{4}\alpha^4 . \quad (3.9)$$

By a direct calculation, we obtain

$$D_\mu\phi^a=0 \quad (3.10)$$

for this solution, which means, together with Eq. (3.9), that the field configurations are in the Higgs vacuum,[5] because we can define as the Higgs potential the first term of Eq. (3.8). In this case, the energy-momentum tensor will become traceless and the cosmological constant will be zero. In the general case, we consider Eq. (3.8) as the definition of the Higgs potential. By returning to the system of Eqs. (2.14) to (2.19) after the substitution of the solution found above, we have that Eqs. (2.14) and (2.15) become identical, or

$$1+\frac{k}{2\epsilon^2}\Lambda=1+a^2-\frac{\lambda k^2}{8\epsilon^2}\alpha^4 . \quad (3.11)$$

Equation (2.16) turns into

$$-1+\frac{k}{2\epsilon^2}\Lambda=-(1+a^2)-\frac{\lambda k^2}{8\epsilon^2}\alpha^4 . \quad (3.12)$$

These last equations are for determining a and Λ as functions of the known parameters k, ϵ, λ, and α. We can write then the solution

$$e^{2\sigma}=\frac{k}{2\epsilon^2}\frac{1}{r^2},\ H=\pm\epsilon\alpha r,\ K=0, \quad J=br,\ \Lambda=-\tfrac{1}{4}k\lambda\alpha^4 . \quad (3.13)$$

The electric and magnetic fields are characterized locally and gauge invariantly by the 't Hooft tensor,[11] or

$$f_{\mu\nu}=\hat{\phi}^aF_{\mu\nu a}-\frac{1}{\epsilon}\epsilon_{abc}\hat{\phi}^aD_\mu\hat{\phi}^bD_\nu\hat{\phi}^c , \quad (3.14)$$

where $\hat{\phi}^a$ are the unit Higgs vectors. Now calculating the fields given by Eq. (3.14) in the asymptotic orthonormal rest frame,[12] we have for the surviving component

$$B_{\hat{r}}^{(\mathrm{I})}=f_{\hat{\theta}\hat{\varphi}}^{(\mathrm{I})}=-\frac{2\epsilon}{k} . \quad (3.15)$$

We observe that we have only a constant magnetic field of a magnetic monopole.

B. A magnetic monopole outside of the Higgs vacuum

Another nontrivial choice to be made in order to simplify the field equations will be $J=0$ and $K=A$ where A is a constant. Then Eq. (2.18) gives

$$e^{2\sigma}H^2=1-A^2 . \quad (3.16)$$

Equation (2.24) turns into

$$\frac{(e^{2\sigma})'}{r}=\frac{k}{\epsilon^2}\left[-\frac{(1-A^2)^2}{r^4}-e^{2\sigma}\frac{A^2H^2}{r^4}\right] . \quad (3.17)$$

Substituting Eq. (3.16) into (3.17), we get after integration

$$e^{2\sigma}=\beta+\gamma\frac{(1-A^2)}{r^2} , \quad (3.18)$$

where β and γ have the same meaning as in Eq. (3.3). Equation (3.18), with $\beta=0$, and the function $H(r)$ calculated from Eq. (3.16) satisfies identically Eq. (2.22). We have

$$H=\gamma^{-1/2}r . \quad (3.19)$$

Equation (2.19) is then satisfied if γ is given by

$$\gamma=\frac{A^2(2\epsilon^2-\lambda)+\lambda}{(1-A^2)\lambda\alpha^2\epsilon^2},\ \lambda\neq0,\ A\neq1 . \quad (3.20)$$

Comparing Eq. (3.20) with $\gamma=k/(2\epsilon^2)$, we can find for the constant A,

$$A^2=\frac{(k\alpha^2-2)\lambda}{2(2\epsilon^2-\lambda)+k\alpha^2\lambda},\ \lambda\neq0,\ A\neq1 . \quad (3.21)$$

We observe that this solution does not correspond to the minimum of the Higgs potential. We have, by using Eq. (3.8),

$$V^{(\mathrm{I})}(\phi)=\frac{1}{4}\left[\frac{(1-A^2)\lambda\alpha^2}{A^2(2\epsilon^2-\lambda)+\lambda}-\alpha^2\right]^2-\tfrac{1}{4}\alpha^4 . \quad (3.22)$$

Calculating the internal covariant derivatives, we have that the field configurations related to this last solution are not in the Higgs vacuum for $A\neq0$, since Eq. (3.10) is not satisfied in this case.

We can go back to Eqs. (2.14) to (2.19) in order to determine the cosmological constant as a function of known parameters. Equations (2.14) and (2.15), after substitution of the last solution, become identical, and we have

$$1+\frac{k}{2\epsilon^2}(1-A^2)\Lambda=1+A^2+\frac{k\lambda}{2\epsilon^2}(1-A^2)\left[\frac{1}{k}-\alpha^2\right]. \tag{3.23}$$

Equation (2.16) is now

$$-1+\frac{k}{2\epsilon^2}(1-A^2)\Lambda=-1+A^2+\frac{k\lambda}{2\epsilon^2}(1-A^2)\left[\frac{1}{k}-\alpha^2\right]. \tag{3.24}$$

Adding up these last equations, we can determine the cosmological constant Λ. Our solution has the final form

$$e^{2\sigma}=\frac{k}{2\epsilon^2}\frac{(1-A^2)}{r^2},\quad H=\pm\epsilon\left[\frac{2}{k}\right]^{1/2}r,$$
$$K=A,\quad J=0,\quad \Lambda=-\tfrac{1}{2}\lambda\alpha^2, \tag{3.25}$$

where A is given by Eq. (3.22).

Calculating the electric and magnetic fields in the asymptotic orthonormal rest frame[12] by using the 't Hooft tensor (3.14), we get for the surviving component

$$B_{\hat{r}}^{(\mathrm{II})}=f_{\hat{\theta}\hat{\varphi}}^{(\mathrm{II})}=-\frac{2\epsilon}{k(1-A^2)}. \tag{3.26}$$

IV. DISCUSSION AND CONCLUSIONS

First of all, we observe that the two solutions found above are identical for $A=0$ $(\alpha^2=2/k)$, since the components of the vector potential may be gauge related. We have

$$0=A_t^{a(\mathrm{II})}=A_t^{a(\mathrm{I})}-D_t^{(\mathrm{I})}\varphi^a, \tag{4.1}$$

where φ^a means the parameters of the gauge transformation. Substituting here the solutions found in Sec. III, we get

$$\frac{b}{\epsilon}\hat{r}_r^a-\varphi^a_{|t}+\epsilon\epsilon^a{}_{bc}\frac{b}{\epsilon}\hat{r}_r^b\varphi^c=0, \tag{4.2}$$

which may be satisfied by

$$\varphi^a=t\frac{b}{\epsilon}\hat{h}_r^a. \tag{4.3}$$

The field energy is given by

$$E=\int\sqrt{-g}\,T^t{}_t dr\,d\theta\,d\varphi, \tag{4.4}$$

which becomes, for the two solutions above,

$$E^{(\mathrm{I})}=\frac{2\pi}{\epsilon^2}\left[1-\frac{\lambda}{2\epsilon^2}\right]\frac{1}{r_0}, \tag{4.5}$$

$$E^{(\mathrm{II})}=\frac{2\pi}{\epsilon^2}\left[1-A^4+\frac{\lambda}{2\epsilon^2}(1-A^2)^2(1-\alpha^2k)\right]\frac{1}{R_0}. \tag{4.6}$$

If we choose a particular value of the vacuum expectation value (VV) of the Higgs field, namely $\alpha^2=2/k$, we put the last solution in the Higgs vacuum by defining the Higgs potential as the first term in Eq. (3.8), and, as we saw above, we arrive at the first solution. Solution I seems to be a particular one compared to solution II. According to Eq. (3.21), to each value of the VV of the Higgs field, there is associated with it a value of A, which corresponds to a solution of class II. Let us observe that we have introduced spherical shells of radius r_0 and R_0 in which we consider the magnetic charges to be uniformly distributed. These cut-off radii are considered to be of the order of the Planck length.[13]

Owing to the criterion of positive definiteness of the energy density $(T^t{}_t\geq 0)$, we have a bound on the coupling constant of the Higgs field,

$$\lambda\leq 2\epsilon^2. \tag{4.7}$$

The only remaining components of the Riemann tensor, calculated in the orthonormal rest frame, are

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}}^{(\mathrm{II})}=R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}}^{(\mathrm{II})}=-\frac{2\epsilon^2}{k(1-A^2)}. \tag{4.8}$$

We get the components for the first solution by setting simply $A=0$ in Eq. (4.8), as was explained above. Equations (3.26) and (4.8) characterize the magnetic-monopole field configurations.

As a final remark, we want to observe that we got a finite-energy solution for $A\neq 0$, outside of the Higgs vacuum. Hence, the solution II does not allow the well-known assertion[5,14] in flat Minkowski space-time that all static finite-energy classical solutions to spontaneously broken gauge theories must be in the Higgs vacuum to be extended to a curved space-time.

The search for finite-energy solutions outside of the Higgs vacuum with other symmetries is in progress.

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