Anisotropic Bianchi types VIII and IX locally rotationally symmetric cosmologies

M. J. D. Assad
Universidade Federal da Paraíba, Departamento de Física, Cidade Universitária, 58.000, João Pessoa, PB Brasil

I. Damião Soares
Centro Brasileiro de Pesquisas Físicas, Departamento de Relatividade e Partículas, Rua Dr. Xavier Sigaud, 150 Urca 22290, Rio de Janeiro, Brasil
(Received 6 April 1982)

We present a class of exact cosmological solutions of Einstein-Maxwell equations, which are anisotropic and spatially homogeneous of Bianchi types VIII and IX, and class IIIb in the Stewart-Ellis classification of locally rotationally symmetric models. If we take the electromagnetic field equal to zero, a class of Bianchi types VIII/IX spatially homogeneous anisotropic cosmological solutions with perfect fluid is obtained.

I. INTRODUCTION

The present importance of spatially homogeneous anisotropic cosmological models is now related to the discovery of the anisotropy in the 3-K background radiation, which can be an indication that these models are a more realistic description of past eras in the history of the actual Universe. From the theoretical point of view, anisotropic models have a greater generality than isotropic solutions of the Einstein equations for the cosmological problem.

We present here a class of cosmological solutions of the Einstein equations with perfect fluid, which are spatially homogeneous and anisotropic, and belong to the types VIII and IX in the Bianchi classification. These are the first known anisotropic Bianchi types VIII/IX cosmological solutions with perfect fluid. Some preliminary results have been communicated in Ref. 1. Electromagnetic fields (solutions of Maxwell's equations in the cosmological background) can also contribute to the curvature of the models, and we examine their effect on the solutions. Other anisotropic spatially homogeneous models of Bianchi VIII/IX types have been obtained later, with perfect fluid and electromagnetic fields also. Collins *et al.*[2] presented some Bianchi type VIII models—solutions of the Einstein equations without the cosmological constant term—whose matter content is a perfect fluid but without electromagnetic field. From their solutions only one is guaranteed to satisfy reasonable energy conditions, namely, the mass-energy density ρ is positive, the pressure p is always negative but $|p| \leq \rho$. Lorenz[3] derived cosmological solutions of Bianchi types VIII and IX, with source-free electromagnetic fields and a perfect fluid whose equation of state is $p=\rho$. In Sec. II we describe the geometry of the models; we use a two-parameter line element which describes five different classes of geometries, for different values of the parameters. Curvatures are calculated and a detailed examination of the scalar curvature of homogeneity sections $t=\text{const}$ is done. In Sec. III, we describe the matter content of the model by the energy-momentum tensor of the perfect fluid; Maxwell's equations are solved for a special class of source-free electromagnetic fields and the corresponding energy-momentum tensor is constructed. Einstein's equations are calculated for the given geometry with the energy-momentum tensor of perfect fluid plus electromagnetic fields. Under certain assumptions the problem is reduced to solving one second-order differential equation for one metric function $B(t)$. The remaining sections are devoted to describing the explicit solutions for all cases, the properties of matter-energy density ρ and pressure p, and the behavior of the kinematic quantities associated with the four-velocity lines of the fluid.

II. THE GEOMETRY OF THE MODELS

Taking (χ,θ,φ) as local coordinates on the homogeneity sections $t=\text{const}$, we choose a tetrad field $e_\alpha^{(A)}(x)$ such that the line element can be expressed as

$$ds^2=\eta_{AB}\theta^A\theta^B$$
$$=(\theta^0)^2-(\theta^1)^2-(\theta^2)^2-(\theta^3)^2\ , \tag{2.1}$$

where the $\theta^A=e_\alpha^{(A)}dx^\alpha$ are given by[4]

$$\begin{aligned} &\theta^0 = dt \ , \\ &\theta^1 = A(t)[d\chi + 4m^2(\theta)d\varphi] \ , \\ &\theta^2 = B(t)K(\theta)d\theta \ , \\ &\theta^3 = B(t)K(\theta)\sin\theta\, d\varphi \ . \end{aligned} \tag{2.2}$$

The functions $m(\theta)$ and $K(\theta)$ satisfy

$$\frac{4m}{K^2\sin\theta}\frac{dm}{d\theta} = \lambda_1 \ , \tag{2.3}$$

$$\frac{d^2K}{d\theta^2} - \frac{1}{K}\left[\frac{dK}{d\theta}\right]^2 + \cot\theta\frac{dK}{d\theta} - K = \lambda K^3 \ , \tag{2.4}$$

where λ_1 and λ are constants, λ being proportional to the curvature of the two-dimensional surface[5] with line element

$$d\Sigma^2 = K^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2) \ .$$

We remark that for $A \neq B$ there is a preferred local spatial direction determined by the one-form θ^1. We have models of Bianchi II, VIII, or IX types[6] according to $\lambda = 0, 1, -1$. For $\lambda \neq 0$, $\lambda_1 = 0$, spatially homogeneous models of the Kantowski-Sachs type[6,7] are obtained. The case $\lambda = 0, \lambda_1 \neq 0$ without electromagnetic fields was examined by Collins[8] and $\lambda = 0, \lambda_1 \neq 0$ with electromagnetic fields was discussed by one of us.[9] All cases covered by the line element are summarized in Table I.

In this paper we shall treat cases $\lambda = \pm 1$ only. For $\lambda = 1$ we take for (2.4) the solution $K(\theta) = \tan\theta$ and (2.3) can be integrated to

$$m^2 = \frac{\lambda_1}{2}\left[\frac{1}{\cos\theta} + \cos\theta\right] \ ,$$

up to an additive constant. Defining the coordinate $\bar{\theta} = -\ln(\cos\theta)$, $0 \leq \bar{\theta} < \infty$, we can express the line element of the manifold as

$$\begin{aligned} ds^2 = dt^2 &- A^2(t)(d\chi + 2\lambda_1\cosh\bar{\theta}\, d\varphi')^2 \\ &- B^2(t)(d\bar{\theta}^2 + \sinh^2\bar{\theta}\, d\varphi'^2) \ , \end{aligned} \tag{2.5}$$

which has the standard form for spatially homogeneous Bianchi type VIII models. For $\lambda = -1$, we take $K(\theta) = 1$ in (2.4) and we have from (2.3)

$$m^2 = \frac{-\lambda_1\cos\theta}{2} \ .$$

We can express the line element as

$$\begin{aligned} ds^2 = dt^2 &- A^2(t)(d\chi - 2\lambda_1\cos\theta\, d\varphi)^2 \\ &- B^2(t)(d\theta^2 + \sin^2\theta\, d\varphi^2) \end{aligned} \tag{2.6}$$

which has a standard form for Bianchi type IX spatially homogeneous models.

Besides the respective Bianchi type VIII/IX groups, the line elements (2.5) and (2.6) admit one additional isometry generated by the Killing vector $\partial/\partial\chi$. The solutions thus admit a G_4 acting transitively on the $t = \text{const}$ sections, and are locally rotationally symmetric models belonging to class III*b* in the Stewart and Ellis classification scheme.[10]

The components of the Ricci tensor for the geometry (2.1) are given in the Appendix. We now discuss some interesting features of the models, connected to the sign of the scalar curvature ${}^{(3)}R$ of the three-dimensional hypersurfaces of homogeneity $t = \text{const}$. We obtain

$${}^{(3)}R = \frac{2}{B^2}\left[\lambda + \frac{\lambda_1{}^2A^2}{B^2}\right] \ , \tag{2.7}$$

where the scale functions $A(t)$ and $B(t)$ are calculated at $t = \text{const}$. For Bianchi types II/VIII ($\lambda = 0, 1$, $\lambda_1 \neq 0$) and for Bianchi type I ($\lambda = 0$, $\lambda_1 = 0$) cases, we have ${}^{(3)}R > 0$ and ${}^{(3)}R = 0$, respectively, and the homogeneity sections are open. For the Bianchi type IX case ($\lambda = -1$, $\lambda_1 \neq 0$) we have ${}^{(3)}R < 0$ if $B^2 > \lambda_1{}^2A^2$, ${}^{(3)}R = 0$ if $B^2 = \lambda_1{}^2A^2$, and ${}^{(3)}R > 0$ if $B^2 < \lambda_1{}^2A^2$—the sign of the intrinsic curvature ${}^{(3)}R$ changes along the evolution of the model. In case of isotropic expansion ($A = B$) the sign of ${}^{(3)}R$ can be positive, zero, or negative according to $\lambda_1{}^2 > 1$, $\lambda_1{}^2 = 1$, or $\lambda_1{}^2 < 1$. However, the Einstein equations for *isotropic models* ($A = B$) with perfect fluid require that $\lambda_1 = \frac{1}{2}$, and ${}^{(3)}R < 0$ always[11]—in which case we have the closed Robertson-Walker-

TABLE I. Cases covered by the line elements (2.1)—(2.4).

Parameters \ Type	Bianchi type I	Bianchi type II	Bianchi type VIII	Bianchi type IX	Kantowski-Sachs
λ	0	0	1	-1	$\neq 0$
λ_1	0	$\neq 0$	$\neq 0$	$\neq 0$	0

Friedmann model. For the *anisotropic* Bianchi type IX class for solutions presented here, ${}^{(3)}R$ changes sign due to the anisotropy in the expansion, but the homogeneity sections $t=$const are always compact. For the anisotropic Bianchi type VIII class of solutions ${}^{(3)}R>0$ always and the homogeneity sections are open.

The relevant fact here is that the sign of ${}^{(3)}R$ is not in general connected to the topological properties (closed, open[12]) of the homogeneity sections; also the knowledge of ${}^{(3)}R>0$ at a given time t does not determine the future sign of ${}^{(3)}R$ in the model, contrary to the case of Robertson-Walker-Friedmann models.[13] Detailed observational consequences of this fact will be examined in a future publication.

III. THE MATTER CONTENT OF THE MODELS AND THE FIELD EQUATIONS

The matter content of the models is a perfect fluid plus eventually source-free electromagnetic fields. In the local Lorentz frame determined by (2.2) we assume that an observer comoving with the fluid has four-velocity

$$u^A=\delta_0^A \tag{3.1}$$

and we denote, respectively, by ρ and p the density of matter energy and pressure of the fluid, as measured locally by the observer (3.1). The energy-momentum tensor for the fluid is then expressed as

$$T_{AB}=(\rho+p)u_A u_B-p\eta_{AB} \ . \tag{3.2}$$

The source-free Maxwell equations are expressed as

$$e^{\rho}_{[(P)}F_{QR]\,|\rho}+2F_{A[R}\gamma^{A}{}_{PQ]}=0 \ , \tag{3.3a}$$

$$e^{\rho}_{(P)}F^{P}{}_{D\,|\rho}-F_{AD}\gamma^{AP}{}_{P}-F_{PB}\gamma_D{}^{PB}=0 \ . \tag{3.3b}$$

From spatial homogeneity and the existence of a preferred direction determined by θ^1 (for $A\neq B$) we restrict the electromagnetic tensor F_{AB} to the form

$$F_{01}=-F_{10}=E(t) \ , \qquad F_{23}=-F_{32}=H(t) \ , \tag{3.4}$$

all other $F_{AB}=0$. For (2.2) and (3.4), Maxwell equations (3.3) reduce to

$$(HB^2)^{\cdot}-2EA\lambda_1=0 \ , \qquad (EB^2)^{\cdot}+2HA\lambda_1=0 \tag{3.5}$$

and introducing a new variable $\tilde{t}$ defined by $d\tilde{t}=AB^{-2}dt$ we obtain the independent solutions

$$E=\frac{1}{B^2}(\Sigma_1\cos2\lambda_1\tilde{t}+\Sigma_2\sin2\lambda_1\tilde{t}) \ , \qquad H=\frac{1}{B^2}(\Sigma_1\sin2\lambda_1\tilde{t}-\Sigma_2\cos2\lambda_1\tilde{t}) \ , \tag{3.6}$$

where Σ_1 and Σ_2 are integration constants. For (3.6) the electromagnetic energy-momentum tensor

$$T_{AB}(\text{e.m.})=-F_{AC}F^{C}{}_{B}+\tfrac{1}{4}\eta_{AB}F_{CD}F^{CD}$$

has non-null components

$$T_{00}=-T_{11}=T_{22}=T_{33}=\tfrac{1}{2}(E^2+B^2)=\frac{\Sigma^2}{2B^4} \ , \tag{3.7}$$

where $\Sigma^2=\Sigma_1{}^2+\Sigma_2{}^2$; for a given value of B, Σ is proportional to the intensity of the electromagnetic field.

Einstein field equations for the model

$$R_{AB}-\tfrac{1}{2}\eta_{AB}R+\Lambda\eta_{AB}=k[T_{AB}(\text{perfect fluid})+T_{AB}(\text{e.m.})]$$

reduce to the set of independent equations

$$R_{00}+3R_{11}+\frac{k\Sigma^2}{B^4}+2\Lambda=2k\rho \ , \tag{3.8a}$$

$$R_{00}-R_{11}-\frac{k\Sigma^2}{B^4}-2\Lambda=2kp \ , \tag{3.8b}$$

$$R_{11}+\frac{k\Sigma^2}{B^4}-R_{22}=0 \ . \tag{3.8c}$$

We consider Eqs. (3.8a) and (3.8b) as defining ρ and p. Equation (3.8c) is then one differential equation for the two metric functions A and B. Instead of imposing an equation of state $p=p(\rho)$, we assume here a relation between A and B, and from (3.8a) and (3.8b) we have $p=p(\rho)$.[14] We take

$$A=A_0B^{1/2} \tag{3.9}$$

and Eq. (3.8c) reduces then to

$$\frac{\ddot{B}}{B}+\frac{3}{2}\left[\frac{\dot{B}}{B}\right]^2-\frac{8A_0{}^2\lambda_1{}^2}{B^3}-\frac{2k\Sigma^2}{B^4}-\frac{2\lambda}{B^2}=0 \ . \tag{3.10}$$

Defining a new variable $\tilde{t}$ by

$$d\tilde{t}=B^{-3/2}dt \ , \tag{3.11}$$

Eq. (3.10) can be expressed as

$$B''-8A_0{}^2\lambda_1{}^2B-2\lambda B^2-2k\Sigma^2=0 \ , \tag{3.12}$$

where a prime denotes the $\tilde{t}$ derivative. Equation (3.12) has as first integral

$$(B')^2=\tfrac{4}{3}\lambda B^3+8A_0{}^2\lambda_1{}^2B^2+4k\Sigma^2B+C\ , \tag{3.13}$$

where C is an integration constant which must be always greater than or equal to zero, in order to have real physical solutions [cf. also (5.1) and (5.2)].

IV. SOLUTIONS AND PROPERTIES OF THE MODELS

The solutions are defined only for $B(\tilde{t})\geq 0$, so that the coordinate t assumes only real values [cf. (3.11)] and the signature of the metric remains unaltered [cf. (3.9)]. The singularities of the models occur for $B=0$ as we shall see. We remark that the class of solutions with $\Sigma=0$ corresponds to cosmological models with perfect fluid only.

Equation (3.13) can be expressed in the standard form which defines Jacobi elliptic functions, and which have trigonometric and hyperbolic functions as limiting cases. For a complete treatment of elliptic functions and integrals, as well as of Eq. (3.12), we refer to Davis,[15] Erdélyi *et al.*,[16] and Abramowitz and Stegun.[17]

We have obtained explicit solutions $B(\tilde{t})$ for the case $C=0$ in (3.13), which we describe now.

A. Bianchi type IX models ($\lambda_1\neq 0$, $\lambda=-1$, $C=0$)

Equation (3.13) takes the form

$$(B')^2=-\tfrac{4}{3}B[B-\epsilon(1+\Delta)][B-\epsilon(1-\Delta)]\ , \tag{4.1}$$

where

$$\epsilon=3\lambda_1{}^2A_0{}^2,\quad \Delta=(1+3k\Sigma^2/\epsilon^2)^{1/2}\ . \tag{4.2}$$

Considering $0\leq B\leq\epsilon(1+\Delta)$ as the physical domain of $B(\tilde{t})$, we can see that the electromagnetic intensity parameter Σ affects the amplitude $B_{\max}$ of the model, since $B=\epsilon(1+\Delta)$ is the maximum of $B(\tilde{t})$. In case of perfect fluid only ($\Sigma=0$), the amplitude $B_{\max}$ has its lowest value $B_{\max}=2\epsilon$.

Defining

$$b=\epsilon-B\ , \tag{4.3}$$

where $-\epsilon\Delta\leq b\leq\epsilon$, (4.1) reduces to

$$(b')^2=\tfrac{4}{3}(b-\epsilon)(b^2-\epsilon^2\Delta^2)\ ,$$

and introducing the new variable

$$z^2=\frac{b+\epsilon\Delta}{\epsilon(1+\Delta)},\quad 0\leq z^2\leq 1 \tag{4.4}$$

we obtain

$$(z')^2=\frac{2\epsilon\Delta}{3}(1-z^2)(1-q^2z^2) \tag{4.5}$$

or

$$\pm\left[\frac{2\epsilon\Delta}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)=\int_0^{[(b+\epsilon\Delta)/\epsilon(1+\Delta)]^{1/2}}\frac{dz}{[(1-z^2)(1-q^2z^2)]^{1/2}}\ . \tag{4.6}$$

This is the Jacobi form for elliptic integrals of the first kind, where the parameter

$$q=\left[\frac{1+\Delta}{2\Delta}\right]^{1/2}\leq 1 \tag{4.7}$$

is the modulus of the elliptic integral. We have from (4.6)

$$\left[\frac{b+\epsilon\Delta}{\epsilon(1+\Delta)}\right]^{1/2}=\mathrm{sn}\left[\pm\left[\frac{2\epsilon\Delta}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)\right]$$

and solving for $B(\tilde{t})$ [cf. (4.3)],

$$B(\tilde{t}-\tilde{t}_0)=\epsilon(1+\Delta)\mathrm{cn}^2\left[\left[\frac{2\epsilon\Delta}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)\right]\ , \tag{4.8}$$

where we have used the properties $\mathrm{cn}(-x)=\mathrm{cn}(x)$ and $\mathrm{cn}^2(x)+\mathrm{sn}^2(x)=1$. Jacobian elliptic functions $\mathrm{sn}(x)$ and $\mathrm{cn}(x)$ are periodic functions—namely, $\mathrm{sn}(x+4K)=\mathrm{sn}(x)$, $\mathrm{cn}(x+4K)=\mathrm{cn}(x)$—and the period is determined by the complete elliptic integral

$$K(q)=\int_0^1\frac{dz}{[(1-z^2)(1-q^2z^2)]^{1/2}}\ , \tag{4.9}$$

which is obviously a function of the modulus q of the elliptic integral of Jacobi (cf. Refs. 17 and 18 for numerical values). Figure 1 represents the solution (4.8), restricted to the interval (between two successive singularities)

$$-K\leq\left[\frac{2\epsilon\Delta}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)\leq K$$

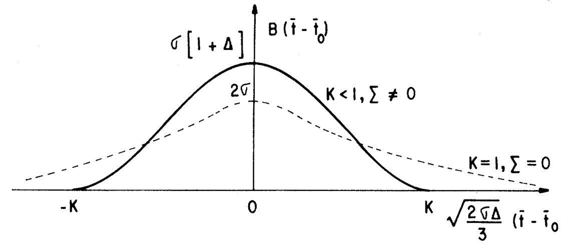

FIG. 1. Graphic representation of Eqs. (4.8) and (4.10): The presence of electromagnetic fields ($\Sigma\neq 0$) reduces the period of Bianchi type IX models.

since $B(\tilde{t})=0$ corresponds to the singularities of the models.[19]

The solution (4.8) in fact contains a solution of the Kantowski-Sachs type for $\lambda_1=0$. In this limit Eq. (4.8) reduces to[20]

$$B(\tilde{t}-\tilde{t}_0)=\mathscr{T}\,\mathrm{cn}^2\left[\left[\frac{2\mathscr{T}}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)\right],$$

where $\mathscr{T}^2=3k\Sigma^2$. In this case we have $q^2=\frac{1}{2}$ and now the period is exactly determined as $K(\frac{1}{2})$ and the electromagnetic intensity parameter Σ affects only the amplitude $B_{\max}$ of the model ($B_{\max}=\mathscr{T}$). We remark that solutions with field and matter, for the present case, exist only if $\Sigma\neq0$. The corresponding Kantowski-Sachs ($\lambda_1=0$) solutions for the Bianchi type VIII case do not exist.

For perfect fluid only (no electromagnetic field present), $\Sigma=0$, $\Delta=1$, $q=1$, and the function $\mathrm{cn}(x)$ (elliptic cosine) becomes the hyperbolic secant; our solution (4.8) takes the form

$$B(\tilde{t}-\tilde{t}_0)=2\epsilon\,\mathrm{sech}^2\left[\left[\frac{2\epsilon}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)\right] \tag{4.10}$$

(see Fig. 1).

The Bianchi type IX models evolve between two pointlike singularities, starting from one singularity and expanding continuously until the half-period of the models, and then contracting back to the other singularity. The period of the model, $2K$, is determined by (4.9) and depends on the modulus of the elliptic integral (4.7)—in other words, the presence of electromagnetic fields ($\Sigma\neq0$) affects basically the period of Bianchi type IX models.

B. Bianchi type VIII models ($\lambda_1\neq0$, $\lambda=1$, $C=0$)

In this case Eq. (3.13) assumes the form

$$(B')^2=\tfrac{4}{3}B[B+\epsilon(1+\delta)][B-\epsilon(1-\delta)]\,, \tag{4.11}$$

where $\epsilon=3A_0{}^2\lambda_1{}^2$ and

$$\delta=\left[1-\frac{k\Sigma^2}{3\lambda_1{}^4A_0{}^4}\right]^{1/2}. \tag{4.12}$$

In order to have real roots for $(B')^2=0$, the electromagnetic intensity parameter Σ must be restricted by

$$0\le k\Sigma^2\le3\lambda_1{}^4A_0{}^4\,. \tag{4.13}$$

For $\Sigma\neq0$, we introduce the variable

$$z^2=\frac{B}{B+\epsilon(1-\delta)} \tag{4.14}$$

with $0\le z^2\le1$ for $0\le B<\infty$. Equation (4.11) is then expressed as

$$(z')^2=\frac{\epsilon(1+\delta)}{3}(1-z^2)(1-q^2z^2)\,, \tag{4.15}$$

where

$$q^2=\frac{2\delta}{1+\delta}\le1 \tag{4.16}$$

and we have

$$\left[\frac{\epsilon(1+\delta)}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)=\int_0^{\{B/[B+\epsilon(1-\delta)]\}^{1/2}}\frac{dz}{[(1-z^2)(1-q^2z^2)]^{1/2}}$$

or

$$B(\tilde{t})=\epsilon(1-\delta)\frac{\mathrm{sn}^2[\epsilon(1+\delta)/3]^{1/2}(\tilde{t}-\tilde{t}_0)}{\mathrm{cn}^2[\epsilon(1+\delta)/3]^{1/2}(\tilde{t}-\tilde{t}_0)}\,. \tag{4.17}$$

Figure 2 shows the behavior of this solution, where the period of the elliptic functions involved in (4.17) is determined by (4.9) for q^2 given in (4.16). The solution (4.17) is restricted to the domain

$$0\le\left[\frac{\epsilon(1+\delta)}{3}\right]^{1/2}(\tilde{t}-\tilde{t}_0)\le K\,, \tag{4.18}$$

which corresponds to $0\le B<\infty$.[21]

For $\Sigma=0$ (perfect fluid only) the solutions are integrated directly in the variable t. Using (3.11), Eq. (3.13) is expressed as (recall $C=\Sigma=0$)

$$\left[\frac{dB}{dt}\right]^2=\frac{4}{3}\left[\frac{B+2\epsilon}{B}\right]$$

whose integral is given in parametric form by

$$B(\eta)=2\epsilon\tan^2\eta\,, \tag{4.19}$$

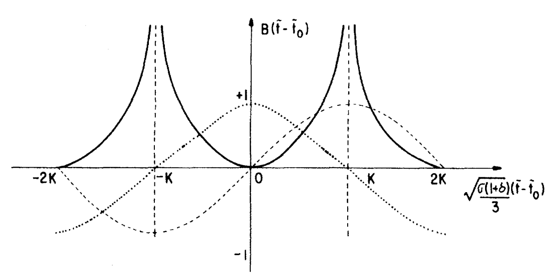

FIG. 2. The function (4.17) is represented by a solid line. The dashed line represents $\mathrm{sn}(x)$ and the dotted line $\mathrm{cn}(x)$.

$$t(\eta)=\epsilon\sqrt{3}[\sec\eta\tan\eta-\ln(\sec\eta+\tan\eta)] .$$

For $\eta=0$ we have $B=0$, $t=0$, and for $\eta\to\pi/2$ we have $B\to\infty$ and $t\to\infty$.

The evolution of Bianchi type VIII models starts from the singularity $B=0$ and expands continuously to $B=\infty$, for all cases. The presence of electromagnetic fields ($\Sigma\neq0$) affects basically the rate of expansion of the model.

V. THE MASS-ENERGY DENSITY AND PRESSURE. KINEMATIC QUANTITIES

The behavior of the mass-energy density ρ and pressure p can be completely described in terms of $B(\tilde{t})$. In fact, from our definitions (3.8a) and (3.8b) of ρ and p, and using (3.9) and (3.11)–(3.13), we obtain

$$k\rho=\left[\frac{5\lambda}{3}B+15A_0{}^2\lambda_1{}^2+\tfrac{15}{2}k\Sigma^2B^{-1}+8CB^{-2}+\Lambda B^3\right]B^{-3} , \qquad (5.1)$$

$$kp=\left[-\frac{\lambda}{3}B+3A_0{}^2\lambda_1{}^2+\tfrac{9}{2}k\Sigma^2B^{-1}+8CB^{-2}-\Lambda B^3\right]B^{-3} . \qquad (5.2)$$

For $B\to0$ we can see directly that the density and pressure diverge, which actually corresponds to a singularity in our models. To have physically significant solutions we must guarantee that $\rho>0$ (positivity of energy) and that $|p/\rho|<1$ and $|dp/d\rho|<1$,[22] for values of $\tilde{t}$ for which $B(\tilde{t})\geq0$. The condition $\rho>0$ holds for all cases and the others can also always be satisfied, and we discuss each case separately.

A. Bianchi type IX ($\lambda=-1$, $\lambda_1\neq0$)

The possible solutions of (3.13) for $\lambda=-1$ are periodic and bounded functions [cf. for instance (4.8)], and we select only a period between two successive zeros of B such that $B>0$ in this interval. If $C=\Sigma=0$ the singularities are located at $t=\pm\infty$ [cf. (4.10)]. Consider the following.

(a) $C=\Sigma=0$: in this case

$$k\rho=\tfrac{1}{3}(-5B+15\epsilon+3\Lambda B^3)B^{-3} ,$$

$$kp=\tfrac{1}{3}(B+3\epsilon-3\Lambda B^3)B^{-3} ,$$

where $\epsilon=3A_0{}^2\lambda_1{}^2$ and $0\leq B\leq2\epsilon$. The density ρ and pressure p have their behavior $k\rho\sim5\epsilon B^{-3}$, $kp\sim\epsilon B^{-3}$, in the neighborhood of the singularities $B\sim0$. They assume their lowest values

$$k\rho=\frac{5}{24\epsilon^2}+\Lambda,\quad kp=\frac{5}{24\epsilon^2}-\Lambda$$

for $B_{max}=2\epsilon$. To have the pressure positive definite we choose $0\leq\Lambda\leq5/24\epsilon^2$. It then follows that $\rho>0$, $p>0$, and $p/\rho<1$. Near the singularity we have the equation of state $p\sim\rho/5$; $dp/d\rho<1$ for the allowable domain of $\tilde{t}$.

(b) $C=0$, $\Sigma\neq0$: in this case

$$k\rho=\tfrac{5}{3}[-B+3\epsilon+\tfrac{3}{2}\epsilon^2(\Delta^2-1)B^{-1}+\tfrac{3}{5}\Lambda B^3]B^{-3} ,$$

$$kp=\tfrac{1}{3}[B+3\epsilon+\tfrac{9}{2}\epsilon^2(\Delta^2-1)B^{-1}-3\Lambda B^3]B^{-3} ,$$

where $0\leq B\leq\epsilon(1+\Delta)$ [cf. (4.8)]. The density ρ and pressure p behave like $k\rho\sim\frac{5}{2}\epsilon^2(\Delta^2-1)B^{-4}$ and $kp\sim\frac{3}{2}\epsilon^2(\Delta^2-1)B^{-4}$ near the singularities $B\sim0$, with the corresponding equation of state $p/\rho\sim\frac{3}{5}$. They reach their lowest values

$$k\rho=\frac{5}{6}\frac{1}{\epsilon^2(1+\Delta)^2}+\Lambda ,$$

$$kp=\frac{11\Delta-1}{6\epsilon^2(1+\Delta)^3}-\Lambda$$

for $B_{max}=\epsilon(1+\Delta)$. We choose $\Lambda=(11\Delta-1)/6\epsilon^2(1+\Delta)^3$ to have $p=0$ at B_{max}, in order that $p/\rho<1$ always. For the allowable values of $\tilde{t}$, $\rho>0$, $p>0$, and $dp/d\rho<1$.

(c) $\Sigma\neq0$, $C\neq0$: In this case $\rho\sim8CB^{-5}$, $p\sim8CB^{-5}$ near the singularities with equation of state $p/\rho\sim1$.

Large values of Σ and C could imply $p>\rho$ at $B=B_{max}$. We then choose Λ such that $p=0$ at B_{max} and we have $\rho>0$, $p>0$, $p/\rho<1$, and $dp/d\rho>1$.

B. Bianchi type VIII ($\lambda=1$, $\lambda_1\neq0$)

In these models, the solutions B increase monotonically from $B=0$ and we have $\rho>0$ always. From (5.2) we see that for large B the occurrence of negative values of pressure is inevitable, $p\sim-\frac{1}{3}B^{-2}$. For these Bianchi type VIII solutions we take $\Lambda=0$; the results $|p|<\rho$ and $|dp/d\rho|<1$ can be verified immediately.

C. Kinematical parameters[23]

The expansion of the model is anisotropic, and can be measured by the total averaged (over angles of the observational sphere) expansion θ of the congruence (3.1) of comoving observers,

$$\theta=-\gamma_0{}^A{}_A=\frac{\dot{A}}{A}+2\frac{\dot{B}}{B} .$$

Using (2.7), (3.9), and (3.11)—(3.13) we obtain

$$\theta^2=\tfrac{25}{4}(\tfrac{4}{3}\lambda+8A_0{}^2\lambda_1{}^2B^{-1}+4k\Sigma^2B^{-2}+CB^{-3})B^{-2}\,. \tag{5.3}$$

The anisotropy can be measured by the shear of the congruence (3.1),

$$\sigma^2=\tfrac{1}{2}\gamma_{0(AB)}\gamma_0{}^{(AB)}=\tfrac{1}{3}\left[\frac{\dot B}{B}-\frac{\dot A}{A}\right]^2.$$

We obtain for our solutions

$$\sigma=\frac{\sqrt{3}}{15}\theta\,. \tag{5.4}$$

VI. CONCLUSIONS

The models described in this paper are the first known cosmological solutions of the Einstein equations with perfect fluid, spatially homogeneous of Bianchi types VIII and IX, and anisotropic. The Bianchi type IX solutions evolve between two pointlike singularities, starting from one singularity and expanding continuously until reaching a maximum volume and then recontracting towards the other singularity—a behavior analogous to that of the Friedmann closed universe. The curvature of the three-dimensional homogeneity sections $t=$const changes sign along the evolution, although these sections remain compact.

The Bianchi type VIII solutions start from the singularity $B=0$ and expand monotonically to infinite volume. Electromagnetic fields (solutions of Maxwell equations on the cosmological background) can also contribute to the curvature but do not change the basic evolution features of the models, their presence modifying only the period between two singularities (Bianchi type IX case) or changing the expansion rate of the models (Bianchi type VIII case). The metric function $B(t)$ depends critically on the parameters C (integration constant) and Σ (electromagnetic intensity parameter), as does the thermodynamic behavior of the fluid near the singularity: for $C\neq 0$ the equation of state is approximately $p=\rho$, while for $C=0$ and $\Sigma\neq 0$ we have $p\sim 3\rho/5$; for $C=0=\Sigma$, $p\sim\rho/5$. For all cases, the relative anisotropy in the Hubble expansion is a constant, $\Delta H/H=\sqrt{3}/15$ [cf. (5.4)].

APPENDIX

For the metric (2.1) the Ricci tensor R_{AB} has non-null components

$$R_{00}=-\frac{\ddot A}{A}-2\frac{\ddot B}{B}\,,$$

$$R_{11}=\frac{\ddot A}{A}+2\frac{\dot A\dot B}{AB}+\frac{2A^2\lambda_1{}^2}{B^4}\,,$$

$$R_{22}=R_{33}=\frac{\ddot B}{B}+\frac{\dot A\dot B}{AB}+\left[\frac{\dot B}{B}\right]^2-\frac{\lambda}{B^2}-\frac{2A^2\lambda_1{}^2}{B^4}$$

in the local Lorentz frame determined by (2.2).

---

[1]I. Damião Soares and M. J. D. Assad, Phys. Lett. 66A, 359 (1978).

[2]C. B. Collins, E. N. Glass, and D. A. Wilkinson, Gen. Relativ. Gravit. 12, 805 (1980).

[3]D. Lorenz, Phys. Rev. D 22, 1848 (1980).

[4]Notations and conventions: Capital Latin indices are tetrad indices and run from 0 to 3; they are raised and lowered with the Minkowski metric η^{AB}, $\eta_{AB}=\text{diag}(+1,-1,-1,-1,)$. Greek indices run from 0 to 3 and are raised and lowered with $g^{\alpha\beta},g_{\alpha\beta}$. Square and round brackets denote antisymmetrization and symmetrization, respectively. A bar denotes partial derivative. Ricci rotation coefficients are defined by $\gamma_{ABC}=-e^{\alpha}_{(A)||\beta}\,e_{\alpha(B)}e^{\beta}_{(C)}$. The Riemann tensor is defined by $V_{\alpha||[\beta||\gamma]}=R^{\lambda}{}_{\alpha\beta\gamma}V_\lambda$, and the Ricci tensor by $R_{\alpha\beta}=R^{\lambda}{}_{\alpha\lambda\beta}$, which implies that the Einstein constant k is positive.

[5]We always normalize $\lambda=0,\pm 1$.

[6]M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, New Jersey, 1975).

[7]R. Kantowski and R. K. Sachs, J. Math. Phys. 7, 443 (1966).

[8]C. B. Collins, Commun. Math. Phys. 23, 137 (1971).

[9]I. Damião Soares, Rev. Brasil. Fis. 8, 336 (1978).

[10]J. M. Stewart and G. F. R. Ellis, J. Math. Phys. 9, 1072 (1968).

[11]A general study of the curvatures of the transitivity hypersurfaces for all Bianchi types—without regard to the dynamics of the model as given by Einstein's equations—has been made by A. Harvey, J. Math. Phys. 21, 870 (1980).

[12]A more exact terminology would be compact/noncompact.

[13]We refer here to standard Robertson-Walker-Friedmann models which have the three-dimensional spatial sections $t=$const maximally symmetric, namely, the sections $t=$const admit a G_6 isometry group [cf. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1970)]. The maximal symmetry condition rules out the possibility of modifying the connectivity-in-the-large properties of the spatial sections by identification of certain point sets. By this procedure we could for instance obtain a Bianchi type V isotropic Friedmann model with a compact section and $^{(3)}R>0$ always [cf. G. F. R. Ellis, Gen. Relativ.

Gravit. 2, 7 (1971)].

[14]In fact, using (3.9), (3.12), and (3.13), Eqs. (3.8a) and, (3.8b) are third-degree algebraic equations in $1/B$ with real coefficients which depend on $(A_0,\lambda,\lambda_1,\Sigma,\rho,p)$. Each admits then one real root at least, which, equated, gives $p=p(\rho)$. Obviously this function $p(\rho)$ depends on the parameters $(A_0,\lambda,\lambda_1,\Sigma)$.

[15]H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962).

[16]*Higher Transcendental Functions,* edited by A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi (McGraw-Hill, New York, 1953).

[17]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), p. 596.

[18]E. Jahnke, F. Emde, and F. Lösh, *Tables of Higher Functions* (McGraw-Hill, New York 1960).

[19]Indeed the matter-energy density ρ, pressure p, and curvature scalar R diverge for $B\rightarrow 0$.

[20]We remark that in (4.2) the limit $\lambda_1{}^2\rightarrow 0$ implies $\epsilon\Delta\rightarrow\mathscr{T}$.

[21]For this class of solutions, $\Sigma=0$ implies $\delta=1$ and (4.17) reduces to $B=0$. The class of perfect-fluid solutions is obtained below, by direct integration in the coordinate t.

[22]These conditions imply that the velocity of sound is always smaller than the velocity of light.

[23]G. F. R. Ellis, in *General Relativity and Cosmology,* proceedings of the International School of Physics, "Enrico Fermi," Course XLVII edited by R. K. Sachs (Academic, New York, 1971).