PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOLUME 28, NUMBER 8 15 OCTOBER 1983

Cosmological perturbations in the early universe

Robert Brandenberger, Ronald Kahn, and William H. Press
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
and Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138
(Received 10 May 1983)

We elucidate and somewhat extend Bardeen's gauge-invariant formalism for calculating the growth of linear gravitational perturbations in a Friedmann-Robertson-Walker cosmological background. We show that the formalism can be derived from the usual gravitational Lagrangian, by variation with respect to a restricted set of metric perturbation functions. This approach produces a natural decomposition of an arbitrary matter field (whose constitutive equations need not resemble the usual cosmological perfect fluid) into a spatially homogeneous piece, which couples to the background metric, plus a spatially inhomogeneous piece, which is not necessarily small and which is the source term in a second-order differential equation which evolves the gauge-invariant metric perturbation potential. We show how the complete perturbed metric can be reconstructed in arbitrary gauge from the single gauge-independent metric potential, so that the evolution of the matter fields can be concurrently calculated in the usual manner (i.e., in a perturbed coordinate frame). The approach of this paper is designed to be particularly suited to the study of fluctuations generated by classical scalar or gauge fields in "inflationary" cosmological models.

I. INTRODUCTION

Many papers have, by now, been written on the evolution of density perturbations in a Friedmann-Robertson-Walker (FRW) cosmology. The pioneering work is that of Lifschitz,[1] as summarized by Lifschitz and Khalatnikov.[2] The texts of Weinberg[3] and Peebles[4] treat the subject in some detail. Although thus standardized, the subject has continued to be plagued by difficulties in interpretation, particularly in the subtle area of choice of gauge, that is, infinitesimal coordinate transformations. The analyses mentioned thus far are all gauge dependent (or, more precisely, "gauge specific"). A particular choice of gauge is made, the so-called "synchronous gauge," where, without loss of generality, the perturbation in the metric tensor is taken to have zero time-time and time-space components, nonzero space-space components.

In the synchronous gauge, the interpretation of perturbations whose wavelength is larger than the horizon size is not always straightforward. Press and Vishniac[5] proposed a scheme for keeping track of gauge modes (modes representing only coordinate changes) while continuing to calculate, following the textbooks, in the standard synchronous gauge. The proposed scheme clarified a certain number of "tenacious myths" that had crept into the literature. For example, the Press-Vishniac scheme shows clearly that pressure perturbations on scales larger than the horizon size do not give rise, at lowest order, to growing density perturbations.

At about the same time as Ref. 5 was published, Bardeen[6] published an entirely different solution to the same sorts of problems. Bardeen's approach, based on previous work of Hawking[7] and Olson,[8] is to eliminate the gauge rather than just to specify and understand it. One does this by finding, at the very start of the analysis, gauge-invariant quantities which completely specify the nature of the metric perturbation. The advantages of this approach are that it is conceptually straightforward and mathematically elegant. The apparent disadvantage is that it is not *technically* straightforward; indeed, Bardeen's paper is a computational *tour de force*. Also, it is not straightforward to make the reverse connection back from the gauge-independent formalism to a particular (e.g., synchronous) coordinate system where other, coupled, physical processes may be computed.

The purpose of this paper (its overlapping authorship with Ref. 5 notwithstanding) is to remedy the apparent disadvantages of the Bardeen formalism. We give a new, and in our opinion simpler, derivation of the gauge-independent Bardeen equations, starting in fact from a synchronous-gauge formulation of the problem. The relation between the gauge-specific and gauge-independent formulations is thus elucidated. Our formulation of the problem is Lagrangian based. This, we think, allows for a closer investigation of the coupling between the gauge-independent quantities in the metric perturbation and the various matter fields which act as source terms. In particular, we move away from the fluid formulation of the matter sources (perfect, imperfect, or otherwise), and toward a more general formulation in which the cosmological perturbations are coupled "automatically," in correct manner, to any other matter fields that one chooses to include in one's master Lagrangian.

The particular relevance of this reformulation, and impetus to write this paper, is the recent interest in perturbations of the "inflationary universe" and "new inflationary universe" models of Guth,[9] Linde,[10] Albrecht and Steinhardt,[11] and others. Several authors[12–16] have obtained tantalizing results suggesting that perturbations on the required scales of galaxies and clusters of galaxies, and perhaps someday (though not yet) with the correct amplitudes, can be obtained by coupling the quantum fluctuations of a grand unified theory (GUT) of matter in an early, inflationary epoch, to the classical growing modes of the FRW model.

The methodology of this paper is designed for maximum usefulness in investigating the coupling of cosmologically interesting modes to GUT or other unified theories of matter. It is designed to supplement the existing analyses which are, in our opinion, incomplete: Some neglect the first-order perturbation equations for the metric in the initial phases of the evolution of the universe (Refs. 12—14), while others replace the quantum field by a perfect fluid outside the horizon (Refs. 15 and 16). The circumstances under which these idealizations may or may not be justified is the subject of a second paper,[17] where we will give a consistent analysis of the growth of perturbations in the new inflationary cosmology, based on the formalism of this paper.

We will now briefly outline the derivation to be presented: We consider the so-called "scalar" perturbations of a FRW background metric, which are known to be the only class of perturbations which couple to matter and contain growing modes. We derive zeroth-order (background metric) and first-order (metric perturbation) equations of motion by considering appropriate variations of the action

$$I=\int \left[\frac{R}{16\pi G}+\mathcal{L}_M\right]\sqrt{-g}\,d^4x\ . \tag{1.1}$$

Here R is the Ricci scalar curvature, while $\mathcal{L}_M$ is an arbitrary matter Lagrangian density. Next, we combine our variational equations to produce one second-order differential equation for a gauge-invariant metric potential ϕ_H. For a fluid formulation of the matter source, our equation reduces to Bardeen's equation [Eq. (4.9) in Ref. 6].

The plan of our paper is as follows: In the next section we introduce our notation and derive the basic variational equations for gauge-variant (i.e., *noninvariant*) quantities. In Sec. III we analyze the effect of gauge transformations, identify the gauge-invariant metric potential ϕ_H, and combine the initial variational equations to yield an explicitly gauge-invariant equation of motion for ϕ_H. Section IV discusses the matter source terms for three different examples of matter; a scalar field with arbitrary effective potential, a perfect fluid, and a fluid with shear stresses. In Sec. V we compare our method to previous work, in particular to Lifschitz and Khalatnikov,[2] Press and Vishniac,[5] and Bardeen.[6]

A word about the notation we use: Greek indices run from 0 to 3; Roman, from 1 to 3. Indices are raised and lowered with the full metric except where otherwise stated. The Einstein summation convention is assumed throughout; repeated Roman indices are summed even when both are lowered, or both raised. A comma indicates an ordinary partial derivative with respect to a coordinate. We employ units in which $\hbar=c=k_B=1$.

II. THE VARIATIONAL EQUATIONS

A general perturbation of the Friedmann-Robertson-Walker (FRW) metric can be decomposed into the sum of three decoupled pieces, called "scalar," "vector," and "tensor" according to how they transform under spatial coordinate transformations at constant time.[3–6] As shown in Refs. 3 and 6, there are no instabilities in either the vector or the tensor perturbations. The former decay kinematically in an expanding universe. The latter are gravitational waves, whose coupling to matter is so weak as to render them, for all interesting purposes, decoupled; hence they red-shift away. If we are interested in *growing* inhomogeneities in the early universe, it is therefore sufficient to restrict attention to scalar perturbations. We will now show how to obtain these perturbations, uncluttered by the vector and tensor modes, from the variation of a constrained Lagrangian, i.e., the true gravitational Lagrangian restricted to a certain class of metrics.

The Einstein equations

$$G_{\mu\nu}=8\pi G T_{\mu\nu} \tag{2.1}$$

are obtained as the variational equations of the action (1.1) when the metric $g_{\mu\nu}$ ranges over all possible metrics. Let us first, however, restrict attention to the class of metrics that describes a homogeneous and isotropic universe, namely, the FRW metrics

$$g_{\mu\nu}=\mathrm{diag}[-1,a^2(t),a^2(t),a^2(t)]\ . \tag{2.2}$$

It is well known that the equation of motion for the scale factor $a(t)$, and the equation for the change in energy density, can be directly obtained by varying (1.1) in the class of "generalized FRW metrics"

$$g_{\mu\nu}=\mathrm{diag}[-\alpha^2(t),a^2(t),a^2(t),a^2(t)]\ . \tag{2.3}$$

If we "unfreeze" the constrained metric (2.3) somewhat more by adding aptly chosen degrees of freedom, then we can obtain the linear gravitational perturbation equations for the scalar perturbation modes. The most general perturbed metric is

$$g_{\mu\nu}=\mathrm{diag}[-\alpha^2(t),a^2(t),a^2(t),a^2(t)]+\epsilon a^2 g^{(1)}_{\mu\nu}\ . \tag{2.4}$$

Here the superscript (1) denotes that the term is of the first order of smallness, and ϵ is a formal expansion parameter introduced to keep track of the order of magnitude of terms in power-series expansions in the order of smallness. One always sets ϵ equal to 1 at the end of any computation.

The only ways that scalar quantities can enter linearly into a three-tensor are (i) by multiplying the unit tensor or (ii) by contributing their covariant derivative. Therefore, the most general form of (2.4) containing only scalar perturbations is

$$g^{(1)}_{\mu\nu}=\begin{bmatrix} E & F_{,i} \\ F_{,i} & A\delta_{ij}+B_{,ij}\end{bmatrix}, \tag{2.5}$$

where A, B, E, and F are functions of all four coordinates:

$$A=A(t,\vec{x}),\quad B=B(t,\vec{x}),$$
$$E=E(t,\vec{x}),\quad F=F(t,\vec{x}). \tag{2.6}$$

We will in due course fix the gauge to the synchronous gauge by setting

$$\alpha(t)=1,\quad E=F=0, \tag{2.7}$$

and *then* derive an explicitly gauge-invariant framework (which will therefore be independent of the synchronous derivation). However, all the degrees of freedom in (2.5) are necessary when we vary the Lagrangian: In the language of the Arnowitt-Deser-Misner (ADM) formalism,[18] the function E is a lapse function and the function $F_{,i}$ is a shift function. These are not dynamical degrees of freedom, but parametrize how the spacetime is sliced in time, and how spatial coordinates are transported from one slice to the next. The two dynamical degrees of freedom for scalar metric perturbations are the two free functions $A(t,\vec{x})$ and $B(t,\vec{x})$.

There are two conceptually equivalent but computationally distinct ways to vary Eqs. (2.4) and (2.5) with respect to any one of their free functions. The first is to express R and $\mathcal{L}_M$ explicitly in terms of the free functions and their derivatives, thus reducing the problem to a standard variational problem with 1 degree of freedom. The second method is to reduce each variation to a combination of the known variations with respect to $g_{\mu\nu}$. We will follow the latter approach. The crucial formulas are

$$\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\,R)=\sqrt{-g}\,G^{\mu\nu}, \tag{2.8}$$

$$\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\,\mathcal{L}_M)=-\tfrac{1}{2}\sqrt{-g}\,T^{\mu\nu}. \tag{2.9}$$

Here $\delta/\delta g_{\mu\nu}$ stands for variation with respect to one component of a general metric, with all other components held fixed.

The Einstein tensor for the metric (2.5), to first order in ϵ and evaluated at $\alpha(t)=1, E(t,\vec{x})=F(t,\vec{x})=0$ is

$$G^0_0=-3\left[\frac{\dot a}{a}\right]^2+\epsilon\left[\frac{\nabla^2 A}{a^2}-3\frac{\dot a}{a}\dot A-\frac{\dot a}{a}\nabla^2\dot B\right],$$
$$G^i_0=-\epsilon\frac{1}{a^2}\dot A_{,i}, \tag{2.10}$$
$$G^j_i=-\left[2\frac{\ddot a}{a}+\left[\frac{\dot a}{a}\right]^2\right]\delta^j_i-\epsilon\delta^j_i\left[\ddot A+3\frac{\dot a}{a}\dot A+\nabla^2 D\right]$$
$$+\epsilon D_{,ij},$$

where

$$D\equiv\frac{1}{2}\left[\ddot B+3\frac{\dot a}{a}\dot B-\frac{A}{a^2}\right] \tag{2.11}$$

and the "overdot" denotes $\partial/\partial t$.

We now perform each desired variation, keeping in each case terms of lowest nonvanishing order in ϵ. Since $\alpha(t)$ and $a(t)$ are functions of time only, variation with respect to these functions does not probe the spatially varying degrees of freedom of the system. Therefore, such variations automatically produce *space-averaged* equations. To lowest order in ϵ these will be exactly the FRW equations for the smooth, zeroth-order, background. A key feature of this variational approach is that we do *not* need to assume that the matter fields consist of a smooth background plus a *small* inhomogeneous perturbation. The ordering of terms in the metric by powers of ϵ automatically induces a decomposition of an arbitrary matter field into a smooth part that couples to the background metric, plus a spatially varying part that couples to the metric perturbations. The linear perturbation expansion fails only when the metric perturbations become large, not when the matter perturbations become large. (For spatial scales much smaller than the horizon size, large matter perturbations can induce small metric perturbations.)

Consider first varying with respect to $a(t)$. The action (1.1) is

$$I=\int d^4x\,L\{g_{\mu\nu}[a(t)]\}, \tag{2.12}$$

$$L=\frac{1}{16\pi G}\sqrt{-g}\,R+\sqrt{-g}\,\mathcal{L}_M. \tag{2.13}$$

We demand that I be constant under arbitrary variations $\delta a(t)$ which vanish at infinity:

$$0=\delta I=\int d^4x\left[\frac{\partial L}{\partial g^{\mu\nu}}\delta g^{\mu\nu}+\frac{\partial L}{\partial g^{\mu\nu}_{,\alpha}}\delta g^{\mu\nu}_{,\alpha}+\frac{\partial L}{\partial g^{\mu\nu}_{,\alpha\beta}}\delta g^{\mu\nu}_{,\alpha\beta}\right]$$
$$=\int d^4x\left[\frac{\partial L}{\partial g^{\mu\nu}}-\left[\frac{\partial L}{\partial g^{\mu\nu}_{,\alpha}}\right]_{,\alpha}+\left[\frac{\partial L}{\partial g^{\mu\nu}_{,\alpha\beta}}\right]_{,\alpha\beta}\right]\delta g^{\mu\nu}$$
$$=-\int d^4x\frac{\delta L}{\delta g^{\mu\nu}}g^{\mu\alpha}\frac{\partial g_{\alpha\beta}}{\partial a}g^{\beta\nu}\delta a(t). \tag{2.14}$$

This gives the space-averaged equation [using (2.8), (2.9), and (2.13)]

$$\int d^3x\sqrt{-g}\,(G_{\mu\nu}-8\pi G T_{\mu\nu})g^{\mu\alpha}\frac{\partial g_{\alpha\beta}}{\partial a}g^{\beta\nu}=0. \tag{2.15}$$

To lowest order in ϵ

$$\int d^3x\,(G^i_i-8\pi G T^i_i)\,|_{\epsilon=0}=0. \tag{2.16}$$

It is natural to define the background pressure as a space- and angle-averaged combination

$$P\equiv\tfrac{1}{3}\langle T^i_i\rangle, \tag{2.17}$$

where angle brackets denote spatial averaging. Formally, this averaging is to be done with $\epsilon=0$, i.e., in the background, unperturbed, metric, and with unperturbed matter fields. In practice it is usually more convenient to average the perturbed matter fields, in either the unperturbed or perturbed metrics. Doing this will introduce errors only of order ϵ^2, so it is allowed. By (2.10), the variational equation (2.16) now reads

$$2\frac{\ddot{a}}{a}+\left[\frac{\dot{a}}{a}\right]^2=-8\pi GP\ . \tag{2.18}$$

This is one of the two usual FRW equations.

The other FRW equation follows from variation with respect to $\alpha(t)$. Analogously to (2.14),

$$0=\delta I=-\int d^4x\frac{\delta L}{\delta g^{\mu\nu}}g^{\mu 0}\frac{\partial g_{00}}{\partial \alpha}g^{0\nu}\delta\alpha(t)$$
$$=\frac{1}{8\pi G}\int d^4x(G^{00}-8\pi GT^{00})\sqrt{-g}\ \alpha(t)\delta\alpha(t)\ . \tag{2.19}$$

To lowest order in ϵ we get

$$\int d^3x(G^{00}-8\pi GT^{00})\,|_{\epsilon=0}=0\ . \tag{2.20}$$

Defining the background energy density as

$$\rho\equiv-\langle T^0_0\rangle\ , \tag{2.21}$$

(2.20) becomes

$$3\left[\frac{\dot{a}}{a}\right]^2=8\pi G\rho \tag{2.22}$$

which is the other FRW equation.

The induced decomposition of an arbitrary matter field, mentioned above, is

$$T^{\nu}_{\mu}=T^{\nu(0)}_{\mu}+T^{\nu(1)}_{\mu}\ , \tag{2.23}$$

where

$$T^{\nu(0)}_{\mu}\equiv\text{diag}(-\rho,P,P,P)\ , \tag{2.24}$$

where ρ and P are *defined* by Eqs. (2.21) and (2.17).

The equations of motion for $A(t,\vec{x})$ and $B(t,\vec{x})$, that is, the linear gravitational perturbation equations, follow from varying the action (1.1) with respect to $A(t,\vec{x})$ and $B(t,\vec{x})$. Consider first the A variation:

$$\delta I=-\int d^4x\frac{\delta L}{\delta g^{\mu\nu}}g^{\mu\alpha}\frac{\partial g_{\alpha\beta}}{\partial A}g^{\beta\nu}\delta A(t,\vec{x})$$
$$=-\frac{1}{16\pi G}\int d^4x(G_{\mu\nu}-8\pi GT_{\mu\nu})$$
$$\times g^{\mu k}\delta_{kl}g^{l\nu}\sqrt{-g}\ \epsilon a^2\delta A(t,\vec{x})\ . \tag{2.25}$$

Expanding the parentheses to order ϵ we obtain

$$G^{l(0)}_i g^{ik(0)}+G^{l(1)}_i g^{ik(0)}+G^{l(0)}_i g^{ik(1)}-8\pi G(T^{l(0)}_i g^{ik(0)}+T^{l(1)}_i g^{ik(0)}+T^{l(0)}_i g^{ik(1)}+T^{l(1)}_i g^{ik(1)})\delta_{kl} \tag{2.26}$$

(superscripts in parentheses on g and G indicating the order in ϵ). The zeroth-order terms as well as the third and sixth terms cancel by (2.18). Thus to lowest nonvanishing order in ϵ the A variation gives

$$G^{i(1)}_i=8\pi G(T^{i(1)}_i+T^{l(1)}_i g^{ik(1)}g^{(0)}_{kl}) \tag{2.27}$$

or using (2.10)

$$(3A+\nabla^2B)_{,tt}+3\frac{\dot{a}}{a}(3A+\nabla^2B)_{,t}-\frac{\nabla^2A}{a^2}=-8\pi G(T^{i(1)}_i+T^{l(1)}_i g^{ik(1)}g^{(0)}_{kl})\ . \tag{2.28}$$

Analogously for the B variation,

$$\delta I=-\int d^4x\frac{\delta L}{\delta g^{\mu\nu}}g^{\mu\alpha}\frac{\partial g_{\alpha\beta}}{\partial B_{,\alpha\beta}}g^{\beta\nu}\delta(B_{,\alpha\beta})$$
$$=-\frac{1}{16\pi G}\int d^4x[(G^{ik}-8\pi GT^{ik})\sqrt{-g}\ ]_{,ik}\delta B(t,\vec{x})\epsilon a^2\ . \tag{2.29}$$

The corresponding equation of motion is

$$[(G^{ik}-8\pi GT^{ik})\sqrt{-g}\ ]_{,ik}=0\ . \tag{2.30}$$

The lowest order nonvanishing terms are of order ϵ:

$$0=(G^{k(1)}_j-8\pi GT^{k(1)}_j)_{,ik}g^{ij(0)}+(G^{k(0)}_j-8\pi GT^{k(0)}_j)g^{ij(1)}_{,ik}$$
$$+(G^{k(0)}_j-8\pi GT^{k(0)}_j)g^{ij(0)}(\sqrt{-g}\ )^{(1)}_{,ik}a^{-3}-8\pi GT^{k(1)}_{j,i}(g^{ij}\sqrt{-g}\ )^{(1)}_{,k}a^{-3}\ . \tag{2.31}$$

The second and third parentheses vanish by (2.18). Thus by (2.10)

$$\nabla^2(\ddot{A}+3\frac{\dot{a}}{a}\dot{A})$$
$$=-8\pi G[T^{k(1)}_{j,ik}\delta^{ij}+T^{k(1)}_{j,i}(g^{ij}\sqrt{-g}\ )^{(1)}_{,k}a^{-1}]\ . \tag{2.32}$$

We now make use of the ϵ ordering and drop all source terms of order ϵ^1 in the presence of those of order ϵ^0. This is *not* the same as ordering in the superscript (0)'s and (1)'s: Note that $T^{(0)}_{\mu\nu}$ and $T^{(1)}_{\mu\nu}$ by construction are of the same order of magnitude. The ordering by ϵ therefore correctly drops both $T^{(0)}g^{(1)}$ and $T^{(1)}g^{(1)}$ terms compared to $T^{(1)}g^{(0)}$ terms. This is an important point, not always treated correctly elsewhere in the literature.

Both (2.28) and (2.32) now simplify; the second source term in each can be neglected. The equations are then

$$\ddot{A}+3\frac{\dot{a}}{a}\dot{A}=-8\pi G\nabla^{-2}(a^2T^{ij(1)}_{,ij}) , \quad (2.33)$$

$$(3A+\nabla^2 B)_{,tt}+3\frac{\dot{a}}{a}(3A+\nabla^2 B)_{,t}-\frac{\nabla^2 A}{a^2} = -8\pi G(a^2\delta_{ij}T^{ij(1)}) . \quad (2.34)$$

The use of ∇^{-2} is unambiguous and represents solving the flat-space three-dimensional Laplace equation. ∇^2 has no bounded analytic zero modes on $\mathscr{R}^3$, except for constant functions. Since the space average of a perturbation vanishes [definition of "(1)" quantities] ∇^{-2} in (2.33) is uniquely determined. The same argument applies to all subsequent uses of the operator ∇^{-2} in this paper: it always represents the solution of an auxiliary Laplace equation for the unique bounded solution with vanishing spatial average; or else, in Fourier space, it is simply algebraic division by $-k^2$.

The components of the stress tensor $T^{ij(1)}$ only enter the final equations in the following two combinations:

$$\mathscr{P}_1\equiv\nabla^{-2}a^2T^{ij(1)}_{,ij} , \quad (2.35)$$

$$\mathscr{P}_2\equiv\nabla^{-2}a^2(\delta_{ij}T^{ij(1)}-3\nabla^{-2}T^{ij(1)}_{,ij}) . \quad (2.36)$$

With these two definitions we can combine (2.33) and (2.34) to yield

$$\ddot{A}+3\frac{\dot{a}}{a}\dot{A}=-8\pi G\mathscr{P}_1 , \quad (2.37)$$

$$\ddot{B}+3\frac{\dot{a}}{a}\dot{B}-\frac{A}{a^2}=-8\pi G\mathscr{P}_2 . \quad (2.38)$$

These two equations are the main result of this section.

To derive, in the following section, the equation of motion for the gauge-invariant metric potential we will also need to use the variational equations with respect to E and F. We obtain these by repeating the steps outlined above. The E equation gives to lowest order in ϵ

$$G^{0(1)}_0=\frac{\nabla^2 A}{a^2}-3\frac{\dot{a}}{a}\dot{A}-\frac{\dot{a}}{a}\nabla^2\dot{B}=8\pi GT^{0(1)}_0 . \quad (2.39)$$

The F equation is

$$[(G^{0i}-8\pi GT^{0i})\sqrt{-g}\,],i=0 \quad (2.40)$$

or by (2.10) to lowest order in ϵ

$$\nabla^2\dot{A}=-8\pi Ga^2T^{0i(1)}_{,i}\equiv-8\pi Ga^2\mathscr{P}_3 , \quad (2.41)$$

where the last equality defines $\mathscr{P}_3$ as one final combination of source terms that enters the analysis.

The full set of equations of motion consists of (2.18), (2.22), (2.37), (2.38), and in addition the matter equations obtained by varying the action (1.1) with respect to the matter degrees of freedom (see Sec. IV).

III. A GAUGE-INVARIANT EQUATION FOR FLUCTUATIONS

Not all solutions of the system of Eqs. (2.37) and (2.38) correspond to physical perturbations. Some can be obtained by a coordinate transformation of the background system and are thus gauge artifacts rather than physical perturbations. Before studying perturbations we must therefore eliminate the gauge degrees of freedom. We do this by deriving an explicitly gauge-invariant equation of motion for a gauge-invariant metric potential ϕ_H. First, a brief analysis of the change of the metric and matter variables under a coordinate transformation:

A coordinate transformation

$$x^{\mu'}=x^\mu+\xi^\mu(t,\vec{x}) \quad (3.1)$$

induces the following change in the metric:

$$\delta g_{\mu\nu}=\xi_{\mu;\nu}+\xi_{\nu;\mu} . \quad (3.2)$$

The covariant derivative is with respect to the background metric.

Since we are only interested in scalar metric perturbations, we can restrict our attention to coordinate transformations which, via (3.2), generate scalar perturbations of the FRW background metric. Later in this paper we will use the terms "gauge transformation" and "gauge invariance" to refer only to these restricted transformations.

The general gauge transformation generated by scalars takes the form

$$\begin{aligned} t'&=t+f^0(t,\vec{x}) ,\\ x_i'&=x_i+f_{,i}(t,\vec{x}) , \end{aligned} \quad (3.3)$$

where f^0 and f are two arbitrary functions. The induced changes in the metric coefficients are

$$\begin{aligned} \delta g_{00}&=-2\dot{f}^0 ,\\ \delta g_{0i}&=-f^0_{,i}+\dot{f}_{,i}-2\frac{\dot{a}}{a}f_{,i} ,\\ \delta g_{ij}&=2(f_{,ij}+a\dot{a}\delta_{ij}f^0) . \end{aligned} \quad (3.4)$$

The gauge transformation (3.3) thus yields a scalar perturbation of the FRW background metric with

$$\begin{aligned} A&=2\frac{\dot{a}}{a}f^0 ,\\ B&=2a^{-2}f ,\\ E&=-2a^{-2}\dot{f}^0 ,\\ F&=-a^{-2}f^0+(a^{-2}f)_{,t} . \end{aligned} \quad (3.5)$$

It therefore follows immediately that the following is a gauge-invariant metric potential:

$$\phi_H\equiv\tfrac{1}{2}(A-a\dot{a}\dot{B}+2a\dot{a}F) , \quad (3.6)$$

or, in the synchronous gauge,

$$\phi_H=\tfrac{1}{2}(A-a\dot{a}\dot{B}) . \quad (3.7)$$

We have chosen notation and constant factors to obtain agreement with Bardeen (Ref. 6).

Turn now to the source terms. The following two quantities are conventionally used to describe the equation of state of the matter and its time change:

$$w \equiv \frac{P}{\rho} , \tag{3.8}$$

$$c_s{}^2 \equiv \frac{\dot P}{\dot \rho} . \tag{3.9}$$

In the present context, these are to be viewed as merely derived quantities from the definitions (2.17) and (2.21); they need not have the physical interpretation as enthalpy and speed of sound squared. Note that w and c_s^2 have *by definition* no spatial variation, but are functions of t only.

In the Appendix we derive the transformation laws for matter perturbations. We obtain

$$\mathscr{P}'_1 = \mathscr{P}_1 - 3(\rho+P)c_s^2 \frac{\dot a}{a} f^0 , \tag{3.10}$$

$$\mathscr{P}'_2 = \mathscr{P}_2 , \tag{3.11}$$

$$\mathscr{P}'_3 = \mathscr{P}_3 + (\rho+P)a^{-2}\nabla^2 f^0 . \tag{3.12}$$

Equation (3.12) holds for gauge transformations preserving the synchronous gauge condition. The above formulas will be useful to show that the inhomogeneous terms in the equation of motion for ϕ_H are jointly gauge invariant.

After these preliminaries we can now start the derivation of the gauge-invariant equation of motion for ϕ_H. We will proceed in two steps. First we make an ansatz of the form

$$\ddot\phi_H = C_1 \dot\phi_H + C_2 \phi_H + C_3 . \tag{3.13}$$

Using the equations of motion (2.37) and (2.38) for $A(t,\vec x)$ and $B(t,\vec x)$ we will express ϕ_H, $\dot\phi_H$, and $\ddot\phi_H$, as functions of A, $\dot A$, $\dot B$, and matter perturbations. C_1 and C_2 are determined by comparing the coefficients of $\dot A$ and $\dot B$ on both sides of (3.13); they will turn out to be gauge invariant. C_3 will still contain individually gauge-dependent, but jointly gauge-independent, pieces, most notably A.

In the second step we will use an additional equation, namely, (2.41), to replace the gauge-dependent terms by explicitly gauge-invariant functions of ϕ_H, $\dot\phi_H$, and gauge-invariant matter terms. Rearranging terms will give the final equation for ϕ_H.

For notational simplicity all calculations are performed in the synchronous gauge. The final equation however is manifestly gauge invariant.

We will start with step 1: ϕ_H is given by (3.7). Differentiating and using (2.38) to eliminate $\ddot B$ we obtain

$$2\dot\phi_H = \dot A - \frac{\dot a}{a} A + [3\dot a^2 - (a\dot a)_{,t}]\dot B + 8\pi G a \dot a \mathscr{P}_2 . \tag{3.14}$$

Differentiating once more and using (2.37) and (2.38) to replace $\ddot A$ and $\ddot B$ we get

$$2\ddot\phi_H = -4\frac{\dot a}{a}\dot A + \left\{ \frac{3\dot a^2 - (a\dot a)_{,t}}{a^2} - \left[\frac{\dot a}{a}\right]_{,t} \right\} A + \left\{ [3\dot a^2 - (a\dot a)_{,t}]_{,t} - 3\frac{\dot a}{a}[3\dot a^2 - (a\dot a)_{,t}] \right\} \dot B + 8\pi G\{ -\mathscr{P}_1 - [3\dot a^2 - 2(a\dot a)_{,t}]\mathscr{P}_2 + a\dot a \dot{\mathscr{P}}_2 \} . \tag{3.15}$$

Comparing the $\dot A$ coefficient yields

$$C_1 = -4\frac{\dot a}{a} \tag{3.16}$$

and similarly for the $\dot B$ coefficient

$$C_2 = -\frac{1}{a\dot a}\left\{ [3\dot a^2 - (a\dot a)_{,t}]_{,t} + \frac{\dot a}{a}[3\dot a^2 - (a\dot a)_{,t}] \right\} . \tag{3.17}$$

Finally

$$C_3 = \ddot\phi_H - C_1\dot\phi_H - C_2\phi_H = -\frac{1}{2}A\left\{ -\left[\frac{\dot a}{a}\right]^2 + 3\frac{\ddot a}{a} - (a\dot a)^{-1}[3\dot a^2 - (a\dot a)_{,t}]_{,t} \right\} + 4\pi G\{ -\mathscr{P}_1 + (3\dot a^2 + 2a\ddot a)\mathscr{P}_2 + a\dot a \dot{\mathscr{P}}_2 \} . \tag{3.18}$$

The coefficients of Eq. (3.18) have been written entirely in terms of higher-order time derivatives of the scale factor a. One can alternatively write them in terms of only a, its first derivative written as the Hubble "constant"

$$H(t) \equiv \frac{\dot a}{a} \tag{3.19}$$

and equation-of-state functions P, ρ, and $c_s{}^2$, by using the FRW equations (2.18) and (2.22), the definitions (3.8) and (3.9), and the continuity equation

$$\dot\rho = -3H(\rho+P) . \tag{3.20}$$

We obtain, e.g.,

$$3\dot a^2 - (a\dot a)_{,t} = 4\pi G(P + \tfrac{5}{3}\rho)a^2 . \tag{3.21}$$

Thus (3.13) becomes

$$\ddot\phi_H + 4H\dot\phi_H - 4\pi G[2P + 3c_s{}^2(P+\rho)]\phi_H = -4\pi G[\mathscr{P}_1 + \tfrac{3}{2}c_s{}^2(P+\rho)A] + 4\pi G[8\pi G(\tfrac{2}{3}\rho - P)a^2\mathscr{P}_2 + Ha^2\dot{\mathscr{P}}_2] . \tag{3.22}$$

All terms except for those in the first brackets on the right-hand side of (3.22) are individually gauge invariant. It is a consistency check to verify that the two terms in the first brackets are jointly gauge invariant; this follows immediately from (3.5) and (3.10).

Equation (3.22) still contains the gauge-variant metric perturbation $A(t,\vec x)$. In the second step we replace A by a function of gauge-invariant metric perturbations and gauge-variant matter terms. We do this by considering the system of Eqs. (3.7), (3.14), and (2.41).

Equation (2.41) is used to eliminate $\dot A$ from the system:

$$\dot A = -8\pi G a^2 \nabla^{-2}\mathscr{P}_3 \tag{3.23}$$

(3.14) gives $\dot B$ as a function of $\dot\phi_H$, $\dot A$, and A:

$$\dot B = [3\dot a^2 - (a\dot a)_{,t}]^{-1}\left[2\dot\phi_H + \frac{\dot a}{a}A - 8\pi G a^2\left[\frac{\dot a}{a}\mathscr{P}_2 - \nabla^{-2}\mathscr{P}_3\right]\right] . \tag{3.24}$$

Finally, we insert both (3.23) and (3.24) into (3.7) to obtain

$$A=2\phi_H+\nu\left[2\dot{\phi}_H+\frac{\dot{a}}{a}A-8\pi Ga^2\left[\frac{\dot{a}}{a}\mathcal{P}_2-\nabla^{-2}\mathcal{P}_3\right]\right] \quad (3.25)$$

with

$$\nu\equiv a\dot{a}[3\dot{a}^2-(a\dot{a})_{,t}]^{-1}=H[4\pi G(P+\tfrac{5}{3}\rho)]^{-1}\ . \quad (3.26)$$

Solving (3.25) for A, we find

$$A=\left[1-\frac{\dot{a}}{a}\nu\right]^{-1}\left[2\phi_H+2\nu\dot{\phi}_H-8\pi Ga^2\nu\left[\frac{\dot{a}}{a}\mathcal{P}_2-\nabla^{-2}\mathcal{P}_3\right]\right]\ . \quad (3.27)$$

Introducing the abbreviation

$$\omega=3\frac{\ddot{a}}{a}-\left[\frac{\dot{a}}{a}\right]^2-(a\dot{a})^{-1}[3\dot{a}^2-(a\dot{a})_{,t}]_{,t}$$
$$=12\pi Gc_s^2(P+\rho) \quad (3.28)$$

and inserting our result for A in (3.22) we obtain

$$\ddot{\phi}_H+\left[4H+\frac{\omega\nu}{1-\nu\dot{a}/a}\right]\dot{\phi}_H+\left\{-4\pi G[2P+3c_s^2(P+\rho)]+\frac{\omega}{1-\nu\dot{a}/a}\right\}\phi_H$$
$$=4\pi G\left[-\mathcal{P}_1+\frac{a^2\nu\omega}{1-\nu\dot{a}/a}\left[\frac{\dot{a}}{a}\mathcal{P}_2-\nabla^{-2}\mathcal{P}_3\right]\right]+4\pi G[8\pi G(\tfrac{2}{3}\rho-P)a^2\mathcal{P}_2+Ha^2\dot{\mathcal{P}}_2]\ . \quad (3.29)$$

This result can be easily simplified to yield

$$\ddot{\phi}_H+(4+3c_s^2)H\dot{\phi}_H+8\pi G\rho(c_s^2-\omega)\phi_H=4\pi G[-\mathcal{P}_1-3c_s^2Ha^2\nabla^{-2}\mathcal{P}_3+a^2H\dot{\mathcal{P}}_2+8\pi G(\tfrac{2}{3}\rho-P+\rho c_s^2)a^2\mathcal{P}_2]\ . \quad (3.30)$$

As a consistency check we observe that by (3.10) and (3.12), the $\mathcal{P}_1$ and $\mathcal{P}_3$ terms are together gauge invariant, and they are the only ones which are not individually so. Thus (3.30) is a manifestly gauge-invariant equation of motion for ϕ_H. It constitutes the main result of this paper.

The initial conditions for ϕ_H are determined by a gauge-invariant initial-value constraint equation which we now derive. By (2.10)

$$G_0^{0(1)}-3a\dot{a}\nabla^{-2}G_{0,k}^{k(1)}=2a^{-2}\nabla^2\phi_H\ . \quad (3.31)$$

Inserting the E and F equations (2.39) and (2.40) we obtain

$$\phi_H=4\pi Ga^2\nabla^{-2}[T_0^{0(1)}-3a\dot{a}\nabla^{-2}T_{0,k}^{k(1)}]\ . \quad (3.32)$$

Equation (3.32) and its first time derivative can be thought of as providing, for specified densities and energy fluxes on an initial time surface, a consistent set of starting conditions which are then evolved by the second-order dynamical equation (3.30).

One might well ask the question, why not use *only* Eq. (3.32) to obtain ϕ_H, by evolving the matter fields in time by their own equations of motion, computing the implied stress tensor, and substituting into (3.32) to infer the geometric inhomogeneities. In that case, the use of the dynamical equation (3.30) seems superfluous. The answer to this question depends on the specific application: For perturbations well inside their horizon, the metric perturbation ϕ_H is a small "tail" being wagged by a large "dog," the matter field $T_0^{0(1)}$. In this case (3.32), which is the relativistic generalization of the Poisson equation of Newtonian gravity, is most appropriate. For perturbations outside of the horizon, on the other hand, whose wave number k satisfies

$$k/a \ll H\ , \quad (3.33)$$

the source terms on the right-hand side of (3.30) are negligible in comparison to the homogeneous terms (see next section). The homogeneous, dynamical evolution of the metric perturbation is now the "dog", while the matter terms are the "tail." Any attempt to use (3.32) in this regime would lead to a very delicate problem in computing the evolution of the matter source terms: all the metric evolution will be buried in the small differences between covariant derivatives and partial derivatives in the matter equations, as due to the nonzero value of ϕ_H. While it is formally possible to compute in this manner, it would not be easy to read off any useful order-of-magnitude estimates of quantities.

Phrased in different terms: Inside the horizon, the matter evolution is dominated by matter self-interactions. Thus it is appropriate to solve the matter equations of motion and infer the geometric fluctuations from (3.32). Outside the horizon on the other hand, the matter evolution is dominated by gravitational effects. Therefore it is more convenient to take a simple model for matter and solve (3.30) in place of the dynamical equations for matter.

Equation (3.30) is equivalent, but not algebraically identical to Bardeen's principal equation (4.9) in Ref. 6. The two equations differ by adding and subtracting the term

$$\frac{1}{a^2}\nabla^2 c_s{}^2 \Phi_H \ . \tag{3.34}$$

Adding (3.34) to both sides of (3.30) and using (3.32) to rewrite the right-hand side gives

$$\ddot{\phi}_H+(4+3c_s{}^2)H\dot{\phi}_H+\left[\left[-\frac{\nabla^2}{a^2}+8\pi G\rho\right]c_s{}^2-8\pi GP\right]\phi_H = 4\pi G[-\mathscr{P}_1-c_s{}^2T_0^{0(1)}+a^2H\dot{\mathscr{P}}_2+8\pi G(\tfrac{2}{3}\rho-P+\rho c_s{}^2)a^2\mathscr{P}_2] \ . \tag{3.35}$$

This equation coincides with Bardeen's equation, but with the source terms written in general form rather than in the form of a fluid with shear stresses. We work out the latter example in the next section.

A few words are in order about which of the equivalent equations (3.30) or (3.35) is "best." Our view is that both are flawed (though mathematically correct). One would like an equation in which physical processes at different cosmological epochs cause source terms (on the equation's right-hand side) to generate perturbations in ϕ_H which subsequently grow homogeneously (by the equation's left-hand side). Thus cosmological perturbations today, for example, would be the sum of homogeneous solutions, each of which could be traced back to a physical "cause."

For the above picture to hold, one would need an equation equivalent to (3.30) or (3.35) whose source terms depend only on the spatial stresses T_{ij}, and not on the energy fluxes T_{0j} or the density T_{00}. The reason is that only the spatial stresses are freely specifiable (by the underlying microscopic theory of matter) at each instant of time. Fluxes are not freely specifiable: they evolve from the influence of spatial stresses over time according to the momentum conservation equations; likewise the density is even less freely specifiable, since it evolves over time according to the conservation equation from the accumulation of fluxes. The trouble with both (3.30) and (3.35) is that a physical "cause" of a perturbation acts not only instantaneously through its stress terms to create a nonzero ϕ_H, but then it also continues to act over time as matter, set into motion, builds up fluxes and density perturbations. There is no clean separation between homogeneous and inhomogeneous pieces of an evolving ϕ_H. We have tried unsuccessfully to find a more "perfect" form of the Bardeen equation, one coupled only to T_{ij}, and commend this problem to the reader.

Until a better form of the equation is found, Eqs. (3.30) and (3.35) each have minor advantages and disadvantages. Equation (3.30) is an ordinary differential equation. Since an easy Green's function method is available,[17] the order of magnitude estimates of the effects of source terms become very simple. Equation (3.35) on the other hand shows more clearly the two ways in which classical matter can generate metric fluctuations, via entropy perturbations [the first two terms on the right-hand side of (3.35) represent a nonadiabatic stress perturbation] or via anisotropic stress perturbations.

IV. EXAMPLES OF MATTER SOURCE TERMS

In Secs. II and III we derived the equation of motion for the single variable ϕ_H which completely characterizes the metric perturbations. Matter terms appear as inhomogeneous sources in that equation. In this section, we want to write those matter terms explicitly for several useful cases. This will complete the description of the evolution of ϕ_H. In Sec. VI, below, we will show how ϕ_H acts back on the matter *evolution* equations to produce a completely closed set of equations.

A. Scalar field with arbitrary effective potential

The case where the matter content of the universe is idealized as a single scalar field, with arbitrary effective potential, is of interest in inflationary-universe scenarios.

The matter Lagrangian for a scalar field is

$$\mathscr{L}_M=-[\tfrac{1}{2}\phi_{,\alpha}\phi_{,\beta}g^{\alpha\beta}+V(\phi)] \ . \tag{4.1}$$

Its stress-energy-momentum tensor is

$$T^\nu_\mu=\phi_{,\mu}\phi^{,\nu}-\delta^\nu_\mu[\tfrac{1}{2}\phi_{,\alpha}\phi_{,\beta}g^{\alpha\beta}+V(\phi)] \ . \tag{4.2}$$

Using (2.17) and (2.21) the background energy and pressure are readily computed as

$$\rho=\tfrac{1}{2}\langle\dot{\phi}^2\rangle+\tfrac{1}{2}a^{-2}\langle(\nabla\phi)^2\rangle+\langle V(\phi)\rangle \ , \tag{4.3}$$

$$P=\tfrac{1}{2}\langle\dot{\phi}^2\rangle-\tfrac{1}{6}a^{-2}\langle(\nabla\phi)^2\rangle-\langle V(\phi)\rangle \ , \tag{4.4}$$

where angle brackets again denote spatial averaging. The matter source terms $\mathscr{P}_1,\mathscr{P}_2,\mathscr{P}_3$ are given by

$$\mathscr{P}_1=a^{-2}\nabla^{-2}(\phi_{,i}\phi_{,j})_{,ij}+\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}a^{-2}(\nabla\phi)^2-V(\phi) -\tfrac{1}{2}\langle\dot{\phi}^2\rangle+\tfrac{1}{6}a^{-2}\langle(\nabla\phi)^2\rangle+\langle V(\phi)\rangle \ , \tag{4.5}$$

$$\mathscr{P}_2=a^{-2}\nabla^{-2}(\phi_{,i}\phi_{,i})-3a^{-2}\nabla^{-4}(\phi_{,i}\phi_{,j})_{,ij} \ , \tag{4.6}$$

$$\mathscr{P}_3=-a^{-2}(\dot{\phi}\phi_{,i})_{,i} \ . \tag{4.7}$$

Variation of the action with respect to $\phi(t,\vec{x})$ leads to the Klein-Gordon equation in curved spacetime:

$$\Box_g\phi=\frac{1}{\sqrt{-g}}(\sqrt{-g}\ g^{\alpha\beta}\phi_{,\beta})_{,\alpha}=V'(\phi) \ . \tag{4.8}$$

Expanding the metric as in (2.4), using the synchronous gauge and keeping terms to order ϵ, (4.8) becomes

$$\left[-\frac{\partial^2}{\partial t^2}-3\frac{\dot{a}}{a}\frac{\partial}{\partial t}+a^{-2}\nabla^2+\epsilon D^{(1)}\right]\phi=V'(\phi)\ . \tag{4.9}$$

Here $D^{(1)}$ is the perturbation-order differential operator

$$D^{(1)}\phi\equiv -a^{-2}g^{ij(1)}\phi_{,ij}-a^{-2}g^{ij(1)}_{,i}\phi_{,j} -\tfrac{1}{2}\dot{h}\phi_{,t}+\tfrac{1}{2}a^{-2}h_{,i}\phi_{,i}\ , \tag{4.10}$$

where

$$h\equiv g^{(1)}_{ii}=3A+\nabla^2 B\ . \tag{4.11}$$

If ϕ is spatially uniform, so that $\phi(\vec{x},t)=\phi(t)$, then (4.10) becomes

$$D^{(1)}\phi=-\tfrac{1}{2}(3\dot{A}+\nabla^2\dot{B})\dot{\phi}\ . \tag{4.12}$$

Once we learn, in Sec. VI, how to obtain A and B from ϕ_H, then Eqs. (4.9) and either (4.10) or (4.12) will close the set of evolution equations for all quantities. It is important to note at this stage, however, that the evolution equation for ϕ, the matter field, is *not* gauge invariant. It is true in general that the matter equations, computed as they are in some coordinate background, are not gauge invariant. This is no cause for alarm: Having eliminated any gauge dependence from the gravitational equations, we are free to evolve the matter equations in *any* convenient gauge, or to make arbitrary changes of gauge at any time-slice.

In a companion paper,[17] we will consider the evolution of fluctuations during the de Sitter phase of an inflationary cosmological model (see Refs. 12—16). For that and similar purposes, it is useful at this point to make order-of-magnitude estimates of some of the above expressions.

The inflationary scenario is characterized by an energy scale σ of grand unified symmetry breaking. This, in turn, determines the Hubble constant during the inflationary era, by the relation

$$H\sim\frac{\sigma^2}{m_{\text{Planck}}}\ . \tag{4.13}$$

Here m_{Planck} is the Planck mass, and one has $\sigma/m_{\text{Planck}}\sim 10^{-5}$ for typical grand unified theories.

In the final stages of the inflationary de Sitter stage, it turns out that $\langle\phi\rangle$ increases, at first slowly but then rapidly, from $O(H)$ to $O(\sigma)$, with approximately the functional form[17]

$$\langle\phi\rangle\sim Hf(t)\ , \tag{4.14}$$

$$f(t)\equiv(H^{-1}-t)^{-1}H^{-1}\ . \tag{4.15}$$

During this period

$$V(\phi)\sim\sigma^4\ ,\ \ \delta\phi(t)\sim Hf(t)\ . \tag{4.16}$$

Also, using (4.3)—(4.7),

$$\rho\sim\sigma^4\ ,\ \ P\sim\sigma^4\ , \tag{4.17}$$

$$\mathscr{P}_1\sim H^4f^4(t)\ ,\ \ \mathscr{P}_2\sim a^{-2}H^2f^3(t)\ ,$$

$$\mathscr{P}_3\sim\frac{k^2}{a^2}H^3f^3(t)\ , \tag{4.18}$$

where k/a is the characteristic wave number of the perturbation in physical units.

It follows that the total source term on the right-hand side of (3.30) is of order

$$\sim\left[\frac{H}{m_{\text{Planck}}}\right]^2 H^2f^4(t)\ . \tag{4.19}$$

By (3.32), the initial condition for ϕ_H when it previously crossed the de Sitter event horizon is

$$\phi_H\sim\left[\frac{H}{m_{\text{Planck}}}\right]^2\ . \tag{4.20}$$

We will make use of these scalings in Ref. 17 when we come to decide whether the homogeneous evolution of ϕ_H dominates or is dominated by the source terms at various epochs.

B. A perfect fluid

In the inflationary cosmology, the matter content is well modeled by a perfect fluid after grand unified symmetry breaking, and concomitant reheating, have occurred.

Consider small perturbations of a fluid at rest in the FRW background, so that

$$T^{\mu\nu}=(P^0+P^1)g^{\mu\nu(0)}+(P^0+\rho^0+P^1+\rho^1)u^\mu u^\nu\ . \tag{4.21}$$

To linear order in the perturbations P^1, ρ^1, and v:

$$T^{\mu\nu}=\text{diag}(\rho^0,a^{-2}P^0,a^{-2}P^0,a^{-2}p^0)+T^{\mu\nu(1)} \tag{4.22}$$

with

$$T^{00(1)}=\rho^1\ ,$$
$$T^{0i(1)}=(\rho^0+P^0)v^i\ , \tag{4.23}$$
$$T^{ij(1)}=a^{-2}P^1\delta^{ij}\ .$$

Therefore

$$\mathscr{P}_1=\nabla^{-2}a^2T^{ii(1)}_{,ii}=P^1\ ,$$
$$\mathscr{P}_2=0\ , \tag{4.24}$$
$$\mathscr{P}_3=(\rho^0+P^0)v^i_{,i}\ .$$

For the evolution equation in the form of (3.35), the right-hand side becomes simply

$$4\pi G\left[{c_s}^2-\frac{P^1}{\rho^1}\right]\rho^1\ . \tag{4.25}$$

This is just an entropy perturbation. If the equations of state for the background and the perturbation are identical, then (4.25) vanishes. In this case the gauge-invariant equation of motion for ϕ_H is homogeneous and the perturbations evolve in a simple way. Even if (4.25) does not vanish, outside the horizon it is negligible compared to the homogeneous terms in (3.35) by a factor a^2H^2/k^2.

C. A fluid with shear stresses

The energy-momentum tensor of a fluid with shear stresses is equivalent to the most general scalar perturbation:

$$T^{\mu\nu}=\mathrm{diag}(\rho,Pa^{-2},Pa^{-2},Pa^{-2})+T^{\mu\nu(1)}\ , \tag{4.26}$$

$$T^{\mu\nu(1)}=\begin{bmatrix} \rho\delta & (\rho+P)v_j a^{-2} \\ (\rho+P)v_j a^{-2} & a^{-2}P[\{\Pi_L-\frac{1}{3}\nabla^2\Pi_T\}\delta^{ij}+\Pi_{T,ij}] \end{bmatrix}\ , \tag{4.27}$$

with v_j equal to the gradient of some scalar velocity potential $v_j \equiv \Psi_{,j}$. Here, δ is the fractional energy density perturbation, Π_L the fractional pressure perturbation, and Π_T the fractional anisotropic stress. Equation (4.27) is the form of matter perturbations used in Bardeen (Ref. 6). In the context of first-order matter perturbations it is completely general; nevertheless we think that the notation of this paper for general source terms [Eqs. (2.35), (2.36), and (2.41)] makes clearer how to calculate with matter sources that do not resemble a perfect fluid.

The matter inhomogeneities do not now vanish as they did in (4.25). By Eqs. (2.35) and (2.36) they are

$$\mathscr{P}_1=P\Pi_L+\tfrac{2}{3}P\nabla^2\Pi_T\ , \tag{4.28}$$

$$\mathscr{P}_2=-2P\Pi_T\ . \tag{4.29}$$

Thus the equation of motion (3.35) becomes

$$\ddot{\phi}_H+(4+3c_s^{\,2})H\dot{\phi}_H+\left[\left[-\frac{\nabla^2}{a^2}+8\pi G\rho\right]c_s^{\,2}-8\pi GP\right]\phi_H=4\pi G(\rho\delta c_s^{\,2}-P\Pi_L)-8\pi Ga^2HP\dot{\Pi}_T$$
$$+8\pi G\left[-\frac{\nabla^2}{3}+8\pi G\rho a^2\left[\frac{c_s^{\,2}}{w}-\frac{2}{3}+w\right]\right]P\Pi_T\ . \tag{4.30}$$

It is easy to check that this corresponds exactly to Eq. (4.9) in Ref. 6.

To complete the dynamical system of equations of motion we need the time evolution of the fluid variables. The energy-momentum conservation equations

$$T^{\mu\nu}_{\ \ ;\nu}=0 \tag{4.31}$$

will not, in general, be sufficient. One will also need "constitutive equations" which give the various off-diagonal stresses in terms of the material state. These can only come from a more detailed model of the matter.

V. RELATION TO PREVIOUS WORK

In view of the long history of the study of linear cosmological perturbations, and because a variety of different formalisms have been put forth, we particularly want to make explicit the relation between the methods of this paper and some previous ones. We have selected three papers for comparison: Lifschitz and Khalatnikov,[2] as the most often quoted review of the entire subject; Press and Vishniac,[5] as typical of the standard treatment in synchronous gauge; and Bardeen,[6] as the original derivation of the gauge-invariant formalism used here (extending previous gauge-invariant work[7,8]).

A. Lifschitz and Khalatnikov

Lifschitz and Khalatnikov[2] begin in the synchronous gauge, but obtain final equations that are gauge invariant. Their framework is more general than this paper in that it is developed for the general case $k=-1$, 0, or $+1$. (Our restriction to $k=0$ is one of simplicity only; the extension to ± 1 is straightforward, and is carried out in Bardeen.[6])

Perturbations are decomposed into scalar harmonics:

$$g_{ij}^{(1)}(t,\vec{x})=\lambda(t)P_{ij}(\vec{x})+\mu(t)Q_{ij}(\vec{x})\ , \tag{5.1}$$

where

$$P_{ij}\equiv k^{-2}Q_{,ij}+\tfrac{1}{3}g_{ij}^{(0)}Q\ , \tag{5.2}$$

$$Q_{ij}\equiv\tfrac{1}{3}g_{ij}^{(0)}Q\ , \tag{5.3}$$

and $Q(\vec{x})$ is the scalar harmonic of wave number k. In a flat spacetime, e.g.,

$$Q(\vec{x})=\exp(ik_jx^j)\ . \tag{5.4}$$

In other words, P_{ij} is the traceless tensor that can be formed from Q,Q_{ij} its trace. Note that k^2 denotes the scalar product in the background three-metric.

The relation to our free functions $A(t,\vec{x})$ and $B(t,\vec{x})$ is

$$A(t,\vec{x})=\tfrac{1}{3}[\lambda(t)+\mu(t)]Q(\vec{x})\ , \tag{5.5}$$

$$B(t,\vec{x})=\frac{1}{k^2}\lambda(t)Q(\vec{x})\ . \tag{5.6}$$

This follows immediately by comparing (5.1) and (2.5).

The equations of motion for λ and μ are obtained by considering various combinations of

$$T_i^{j(1)}=-\delta_i^j\frac{\Pi_L}{\delta}wT_0^{0(1)} \tag{5.7}$$

(a relation which is true for a perfect fluid only) and using the Einstein equations to express $T_i^{j(1)}$ and $T_0^{0(1)}$ in terms of metric perturbations. The result is

$$\ddot{\lambda}+3\frac{\dot{a}}{a}\dot{\lambda}-\frac{k^2}{3a^2}(\lambda+\mu)=0\ , \tag{5.8}$$

$$\ddot{\mu}+3\frac{\dot{a}}{a}\dot{\mu}\left[1+\frac{\Pi_L}{\delta}w\right]+\frac{k^2}{3a^2}(\lambda+\mu)\left[1+3\frac{\Pi_L}{\delta}w\right]=0\ . \tag{5.9}$$

A disadvantage of these equations is that the matter perturbations do not only appear as source terms.

Lifschitz and Khalatnikov eliminate the residual gauge freedom by defining two new invariant variables $\xi(t)$ and $\zeta(t)$ by

$$\lambda+\mu=(\lambda_0+\mu_0)\int \xi(t)dt \ , \quad (5.10)$$

$$\dot{\lambda}-\dot{\mu}=(\dot{\lambda}_0-\dot{\mu}_0)\int \xi(t)dt+\zeta(t) \ . \quad (5.11)$$

ξ and ζ are manifestly invariant under coordinate transformations preserving the synchronous gauge, since the integrals of (5.8) and (5.9) corresponding to the above coordinate transformations are

$$\lambda=-\mu=\text{constant} \quad (5.12)$$

and

$$\lambda=-k^2\int\frac{dt}{a^2}\equiv\lambda_0 \ , \quad \mu=k^2\int\frac{dt}{a^2}-3\frac{\dot{a}}{a}\equiv\mu_0 \ . \quad (5.13)$$

A coupled set of two equations, each first order in time, is obtained for ξ and ζ; we will omit the details here.

In our opinion this approach has three disadvantages as compared to the method of this paper and Ref. 6: It is only applicable to perfect fluids as matter sources; it is only partially gauge invariant; and finally the matter perturbations do not enter the equations for gravitational perturbations only as source terms.

B. Press and Vishniac

Press and Vishniac[5] start in the synchronous gauge, and stay in it through thick and thin. For the case of perfect fluid sources, a set of four coupled first-order differential equations is derived for the variables h, $\dot{h}$, θ, and δ, where the relation to the notation of this paper is

$$h=3A+\nabla^2B \ , \quad (5.14)$$

$$\theta=av_{,jj} \ . \quad (5.15)$$

We can derive the equation for h by adding (2.34) and (2.39)

$$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=-8\pi G(a^2\delta_{ij}T^{ij(1)}-T_0^{0(1)}) \ . \quad (5.16)$$

In the case of a perfect fluid (5.16) becomes

$$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=-8\pi G(3P\Pi_L+\rho\delta) \ . \quad (5.17)$$

The equations of motion for θ and δ follow from energy-momentum conservation:

$$\dot{\delta}+(1+w)(\theta+\tfrac{1}{2}\dot{h})=3\frac{\dot{a}}{a}[\delta(w-c_s{}^2)-\epsilon w] \ , \quad (5.18)$$

$$\dot{\theta}+\frac{\dot{a}}{a}\theta(2-3c_s{}^2)=-\frac{c_s{}^2}{a^2(1+w)}\delta_{,jj}-\frac{w}{a^2(1+w)}\epsilon_{,jj} \ , \quad (5.19)$$

with

$$\epsilon\equiv 3w\Pi_L-c_s{}^2\delta \ . \quad (5.20)$$

Equations (5.17)–(5.19) can be written as a set of four first-order differential equations. Hence there are 4 degrees of freedom. Two of them however correspond to coordinate transformations preserving synchronous gauge.

Press and Vishniac idealize (5.17)–(5.19) to the case where the equation of state (hence the coefficients of the equations) are approximately constant in time. There is then a unique decomposition of the coupled system into four eigenmodes. Two correspond to coordinate transformations, while the remaining two are the physical degrees of freedom, i.e., are the gauge-variant representations of gauge-independent modes.

The main disadvantage of this approach is that the treatment of the gauge degrees of freedom is not entirely general. Also, in performing approximations to find the eigenmodes of the system, there is the danger of dropping small gauge-invariant terms in favor of large gauge artifacts. (The paper of Frieman and Will[19] unfortunately falls prey to this danger.)

C. Bardeen

Bardeen[6] generalizes and improves the earlier work of Hawking[7] and Olson,[8] and derives a completely gauge-invariant analysis of small cosmological perturbations. This paper is based on Bardeen's work; however the derivations in Ref. 6 are significantly different from those given here: The main idea of that approach is to use the energy-momentum conservation equations to derive explicitly gauge-invariant equations of motion for gauge-invariant matter perturbations. Relating metric and matter perturbations, equations of motion for gauge-invariant metric potentials are finally obtained.

As in Ref. 2, perturbations are decomposed into scalar harmonics. Using the definitions (5.2) and (5.4), Bardeen writes

$$g_{00}^{(1)}=-A_B(t)Q(\vec{x}) \ , \quad (5.21)$$

$$g_{0i}^{(1)}=-B_B(t)Q_i(\vec{x}) \ , \quad (5.22)$$

$$g_{ij}^{(1)}=2H_L(t)g_{ij}^{(0)}(t)Q(\vec{x})+2H_T(t)P_{ij}(\vec{x}) \ , \quad (5.23)$$

with P_{ij} as defined in Eq. (5.2), and

$$Q_i(\vec{x})\equiv-\frac{1}{k}Q_{,i}(\vec{x}) \ . \quad (5.24)$$

(The subscripts on A_B and B_B are to avoid confusion with the notation A and B of this paper.) Comparing (5.23) and (2.5) we can relate to the notation of this paper:

$$A(t,\vec{x})=[2H_L(t)+\tfrac{2}{3}H_T(t)]Q(\vec{x}) \ , \quad (5.25)$$

$$B(t,\vec{x})=\frac{2H_T(t)}{k^2}Q(\vec{x}) \ . \quad (5.26)$$

Gauge-invariant matter perturbations are

$$v_s=v-\tfrac{1}{2}a^4(a^{-2}B)_{,t} \ , \quad (5.27)$$

$$\epsilon_m=\delta+3(1+w)\frac{\dot{a}}{a}(v-a^2F) \ . \quad (5.28)$$

While ϵ_m and v_s are mathematically gauge invariant, their physical interpretation is most clear in certain gauges. The quantity ϵ_m is the energy density perturbation in comoving gauge, while v_s gives the time dependence of the rate of shear of the perturbation.

By clever manipulations of the energy-momentum conservation equations Bardeen derives two first-order differential equations for ϵ_m and v_s. These are then combined to yield a second-order differential equation for ϵ_m. Since

$$2\frac{\nabla^2\phi_H}{a^2}=-\rho\epsilon_m \tag{5.29}$$

the equation of motion for ϵ_m is equivalent to the gauge-invariant equation for ϕ_H derived in this paper.

It should be evident that our formalism is no more than a modest extension of Ref. 6. In our opinion, the derivation of the equations of motion by a direct variational principle instead of by manipulations of energy-momentum conservation equations is more direct. Our derivation expresses the source terms in a form that is immediately applicable to completely general matter sources, while Bardeen focuses on the case of a fluid with shear stresses. Also, as we will now see, it is conceptually useful to think of the metric potential ϕ_H as generating the slightly perturbed geometrical "arena" in which arbitrary matter physics can be evolved according to its usual coordinate-dependent equations, rather than to require that arbitrary matter fields be cast into a gauge-invariant form like ϵ_m.

VI. HOW TO PROCEED

In a companion paper[17] we will apply the formalism of this paper to compute density fluctuations at late times from any physical mechanism which produces initial perturbations in the early (GUT-inflationary phase) universe. For that application, or equally for any other such application, we summarize here the results of this paper.

Suppose that we are given the equations of motion for a set of matter fields in an arbitrary metric. These might be derived from varying the universal Lagrangian (1.1) with respect to the matter fields that appear in the term $\mathscr{L}_M$, or else they might derive from flat-space equations by the "comma-goes-to-semicolon rule" (principle of equivalence).

The first step is to write out the given equations of motion for the particular metric consisting of a FRW background and a synchronous-gauge perturbation, namely [Eqs. (2.4) and (2.5)]

$$g_{\mu\nu}=\begin{bmatrix}-1 & 0\\ 0 & a^2(t)\{[1+A(t,\vec{x})]\delta_{ij}+B(t,\vec{x})_{,ij}\}\end{bmatrix}. \tag{6.1}$$

The resulting equations of motion are the matter equations that must be evolved as a function of space and time. We will tell you in a minute what to plug in for A and B in these equations.

The next step is to obtain formulas for the stress energy tensor $T^{\mu\nu}$ of the matter fields, either from the universal Lagrangian [via Eq. (2.9)], or else, for a theory of matter that is not Lagrangian based, by other means. From $T^{\mu\nu}$, one immediately obtains formulas for the decomposition of the matter fields into two spatially homogeneous variables, ρ and P [Eqs. (2.17) and (2.21)], and three spatially inhomogeneous source terms $\mathscr{P}_1,\mathscr{P}_2,\mathscr{P}_3$ [Eqs. (2.35), (2.36), and (2.41)] as follows:

$$\rho\equiv-\langle T_0^0\rangle\,,\quad P\equiv\tfrac{1}{3}\langle T_i^i\rangle\,, \tag{6.2}$$

$$T_\mu^\nu=\mathrm{diag}(-\rho,P,P,P)+T_\mu^{\nu(1)}\,, \tag{6.3}$$

$$\begin{aligned}&\mathscr{P}_1\equiv\nabla^{-2}a^2T^{ij(1)}_{,ij}\,,\\ &\mathscr{P}_2\equiv\nabla^{-2}a^2(\delta_{ij}T^{ij(1)}-3\nabla^{-2}T^{ij(1)}_{,ij})\,,\\ &\mathscr{P}_3\equiv T^{0i(1)}_{,i}\,.\end{aligned} \tag{6.4}$$

The gravitational perturbation is characterized by a single scalar variable ϕ_H. On the initial time surface ϕ_H and its first time derivative are set by the initial-value equations [Eq. (3.32)]

$$\phi_H=4\pi Ga^2\nabla^{-2}(T_0^{0(1)}-3\dot{a}a\nabla^{-2}T^{k(1)}_{0,k})\,, \tag{6.5}$$

$$\dot{\phi}_H=4\pi G[a^2\nabla^{-2}(T_0^{0(1)}-3\dot{a}a\nabla^{-2}T^{k(1)}_{0,k})]_{,t}\,. \tag{6.6}$$

Subsequently, ϕ_H is evolved by its dynamical equation, in which the matter terms appear as source terms [Eq. (3.30)],

$$\ddot{\phi}_H+(4+3c_s^2)H\dot{\phi}_H+8\pi G\rho(c_s^2-w)\phi_H=4\pi G[-\mathscr{P}_1-3c_s^2Ha^2\nabla^{-2}\mathscr{P}_3+a^2H\dot{\mathscr{P}}_2+8\pi G(\tfrac{2}{3}\rho-P+\rho c_s^2)a^2\mathscr{P}_2]\,. \tag{6.7}$$

The equivalent Eq. (3.35) can alternatively be used.

Finally, we need to reconstruct A and B from ϕ_H, to feed back into the matter equations and thus close the set. On the initial time surface where (6.5) and (6.6) are applied, A and B are arbitrary functions of the three spatial coordinates. Their choice determines the gauge in which the matter equations are to be evolved. Choosing both equal to zero is a perfectly good choice. Subsequently, A and B must be evolved by the quadrature equations

$$A(t,\vec{x})=-8\pi G\int^t a^2\nabla^{-2}\mathscr{P}_3dt \tag{6.8}$$

[which is just (3.23) rewritten], and

$$B(t,\vec{x})=\int^t\frac{A-2\phi_H}{a\dot{a}}dt \tag{6.9}$$

[which is (3.7)]. It should be emphasized again that A and B are necessary only to provide the coordinate frame which may be needed to evolve the matter equations; they are not necessary to the evolution of the gravitational perturbations. On any time surface you can change A and B aribtrarily to new values A',B' (e.g., reset them to zero) by making a gauge transformation on your matter variables [Eq. (3.3)]

$$\begin{aligned}&t'=t+f^0(t,\vec{x})\,,\\ &x_i'=x_i+f_{,i}(t,\vec{x})\,,\end{aligned} \tag{6.10}$$

and the corresponding change in A and B [Eq. (3.5)]

$$A' = A + 2\frac{\dot{a}}{a} f^0 ,$$
$$B' = B + 2a^{-2} f , \tag{6.11}$$

where f and f^0 are any desired functions of the three spatial coordinates [cf. Eq. (3.5)].

The quantity $\phi_H(t)$ can be directly interpreted as the magnitude of the physical density perturbations. The density perturbations at horizon crossing for some wave number k, i.e., for time t_f given by

$$H^{-1}(t_f) = k^{-1} a(t_f) \tag{6.12}$$

are of particular interest for the theory of galaxy formation. From (3.32) and using (4.27) and (2.22) we have

$$\phi_H(t) = \frac{3}{2}\frac{H^2 a^2}{k^2}[\delta + 3H\Psi(1+w)] , \tag{6.13}$$

where δ is the fractional density perturbation and ψ is the velocity potential defined following equation (4.27) above. In comoving coordinates

$$\phi_H(t_f) = \tfrac{3}{2}\delta . \tag{6.14}$$

Thus $\phi_H(t_f)$ is equal to the energy density fluctuation in comoving coordinates at the time of horizon crossing (up to the trivial factor $\frac{3}{2}$).

In our next paper,[17] we will proceed with the application of these equations.

ACKNOWLEDGMENTS

For helpful discussions, we thank Doug Eardley, Mike Turner, Alan Guth, Paul Steinhardt, and Piet Hut. This work was supported in part by National Science Foundation Grant No. PHY-80-07351.

APPENDIX: GAUGE TRANSFORMATION OF MATTER PERTURBATIONS

The goal is to derive the transformation laws (3.10)–(3.12) of the matter inhomogeneities $\mathscr{P}_1$–$\mathscr{P}_3$ under gauge transformations (3.3).

Obviously the tensor transformations are independent of the particular form for matter. It is therefore sufficient to consider a fluid with general shear stresses.

First by (3.3), for gauge transformations preserving the synchronous gauge to first order in the perturbations,

$$v_i' = \frac{dx_i'}{dt'} = \frac{dx_i'}{dt} = v_i + \dot{f}_{,i} . \tag{A1}$$

Since the total energy density ρ_T is a scalar

$$\rho_T(t') = \rho(t')[1+\delta] = \rho(t) + \dot{\rho}(t) f^0 + \rho(t)\delta$$
$$= \rho(t)[1+\delta'] \tag{A2}$$

(again to linear order in the small quantities). Thus

$$\delta' = \delta + \frac{\dot{\rho}}{\rho} f^0 = \delta - 3(1+w) H f^0 \tag{A3}$$

(using 3.20 in the final step). Since the total pressure

$$P_T(t) = P(t)[1+\Pi_L] \tag{A4}$$

is a scalar, an analogous argument yields

$$\Pi_L' = \Pi_L - 3(1+w) c_s^2 H f^0 w^{-1} . \tag{A5}$$

Finally since

$$T^{ij} = \frac{\delta\sqrt{-g}\,\mathscr{L}_M}{\delta g_{ij}} = (a^{-2} P \Pi_T)_{,ij} \quad (i \neq j) \tag{A6}$$

it follows that Π_T must be a scalar.

Using (4.28), (4.29), and (2.41) we now immediately obtain the transformation properties of the matter inhomogeneities $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$:

$$\mathscr{P}_1' = \mathscr{P}_1 - 3(1+w)\rho c_s^2 H f^0 , \tag{A7}$$

$$\mathscr{P}_2' = \mathscr{P}_2 , \tag{A8}$$

$$\mathscr{P}_3' = \mathscr{P}_3 + (1+w)\rho\nabla^2(a^{-2} f)_{,t}$$
$$= \mathscr{P}_3 + (1+w)\rho a^{-2}\nabla^2 f^0 . \tag{A9}$$

The final identity is true for transformations preserving the synchronous gauge and follows from (3.5).

[1]E. M. Lifschitz, J. Phys. (Moscow) 10, 116 (1946).
[2]E. M. Lifschitz and I. M. Khalatnikov, Adv. Phys. 12, 185 (1963).
[3]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[4]P. J. E. Peebles, *Cosmology: The Physics of Large Scale Structure* (Princeton University Press, Princeton, 1980).
[5]W. H. Press and E. T. Vishniac, Astrophys. J. 239, 1 (1980).
[6]J. M. Bardeen, Phys. Rev. D 22, 1882 (1980).
[7]S. W. Hawking, Astrophys. J. 145, 544 (1966).
[8]D. W. Olson, Phys. Rev. D 14, 327 (1976).
[9]A. Guth, Phys. Rev. D 23, 347 (1981).
[10]A. D. Linde, Phys. Lett. 108B, 389 (1982).
[11]A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
[12]A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).
[13]S. W. Hawking, Phys. Lett. 115B, 295 (1982).
[14]A. A. Starobinsky, Phys. Lett. 117B, 175 (1982).
[15]J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D 28, 679 (1983).
[16]E. Mottola, Phys. Rev. D 27, 2294 (1983).
[17]R. Brandenberger and R. Kahn, Harvard University report, 1983 (unpublished).
[18]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation, An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).
[19]J. A. Frieman and C. M. Will, Astrophys. J. 259, 437 (1982).