Quantum-chromodynamic potential model for light and heavy quarkonia

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Our earlier treatment for the investigation of the $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ spectra is extended to other quarkonia with u, d, s, c, and b quarks. We find that it is possible to account fully for the observed ground-state hyperfine splittings of light as well as heavy quarkonia provided that the perturbative quantum-chromodynamic potential is supplemented not only by the nonperturbative confining potential but also by a nonperturbative annihilation potential which rapidly decreases with increase in the quark mass. We also give our predictions for the experimentally interesting $b\bar{u}$, $b\bar{s}$, and $b\bar{c}$ systems with unknown mass splittings.

I. INTRODUCTION

In a recent paper[1] we investigated the spectra of heavy quarkonia with the use of a perturbative quantum-chromodynamic potential supplemented by a phenomenological long-range confining potential, and our results were found to be in excellent agreement with experiments. It would be interesting to see to what extent the same quantum-chromodynamic potential is applicable to quarkonia with light quarks u, d, and s. The use of a nonrelativistic potential for a light-quark—antiquark system is obviously questionable. For instance, while for a heavy quark it is possible to take into account the relativistic correction to the kinetic energy by including the term $-\vec{p}^{\,4}/8m^3$ in the Hamiltonian, this term becomes too large for a light quark and no longer provides the correct relativistic correction. However, the resulting error in the kinetic energy can be absorbed for the ground state in the potential constant C. Thus, our treatment may be reasonable for hyperfine splittings in the ground states of lighter quarkonia, but may prove unsuitable for their excited states.[2]

With the above viewpoint, we shall extend our earlier treatment to various quarkonia with u, d, s, c, and b quarks. We shall show that it is possible to account fully for the observed ground-state hyperfine splittings of light as well as heavy quarkonia provided that the perturbative quantum-chromodynamic potential is supplemented not only by the nonperturbative confining potential but also by a nonperturbative annihilation potential which rapidly decreases with increase in the quark mass. We shall also give our predictions for the experimentally interesting $b\bar{u}$, $b\bar{s}$, and $b\bar{c}$ systems with unknown mass splittings.

We shall follow the same calculational procedure as used in Ref. 1, and while varying potential parameters so as to bring about agreement between our theoretical results and known experimental values, we shall ensure that quantum-chromodynamic transformation relations are respected. Thus, α_s will transform as

$$\alpha_s' = \frac{\alpha_s}{1+\beta_0(\alpha_s/4\pi)\ln(\mu'^2/\mu^2)} , \tag{1.1}$$

where $\beta_0 = 11 - \frac{2}{3}n_f$, and n_f is the number of light-quark flavors. The transformation relation for the quark mass m will be

$$m' = m\left[\frac{\alpha_s'}{\alpha_s}\right]^{\gamma_0/2\beta_0} , \tag{1.2}$$

where we have inferred from the general result for γ_m, given in Ref. 3, that for our quark masses and ranges for values of μ, the effective values of γ_0 can be taken as $\frac{16}{3}$ for the s quark, 4 for the c quark, and $\frac{4}{3}$ for the b quark.

Our results for mesons with and without light quarks will be presented in Secs. II and III.

II. MESONS WITH LIGHT QUARKS

The quark contents of mesons with light quarks u, d, and s are shown in Table I, where we have assumed that ϕ and ω are ideally mixed with

$$\theta = \tan^{-1}(1/\sqrt{2}) \approx 35° ,$$

while η and η' have the mixing angle

$$\theta = \tan^{-1}(1/\sqrt{2}) - \pi/4 \approx -10° .$$

TABLE I. Quark contents of mesons with light quarks.

Meson	Quark content	Meson	Quark content
π^-,ρ^-	$d\bar{u}$	D^0,D^{*0}	$c\bar{u}$
π^0,ρ^0	$(1/\sqrt{2})(u\bar{u}-d\bar{d})$	D^+,D^{*+}	$c\bar{d}$
K^-,K^{*-}	$s\bar{u}$	B^-,B^{*-}	$b\bar{u}$
$\bar{K}^0,\bar{K}^{*0}$	$s\bar{d}$	B^0,B^{*0}	$b\bar{d}$
η	$\frac{1}{2}(u\bar{u}+d\bar{d}-\sqrt{2}s\bar{s})$	F_c^+,F_c^{*+}	$c\bar{s}$
η'	$\frac{1}{2}(u\bar{u}+d\bar{d}+\sqrt{2}s\bar{s})$	F_b^0,F_b^{*0}	$b\bar{s}$
ω	$(1/\sqrt{2})(u\bar{u}+d\bar{d})$		
ϕ	$-s\bar{s}$		

These mixing angles for vector and pseudoscalar mesons are theoretically simple and in agreement with imprecise experimental results.

Table II gives our values for hyperfine splittings for those mesons in Table I which are not affected by the annihilation channel. The quark-antiquark potentials used by us are given in Appendix A. For mesons containing the u or d quark, the potential parameters are

$$\mu=0.916\text{ GeV},\quad \alpha_s=0.6573,$$
$$m_u=m_d=0.331\text{ GeV},\quad m_s=0.533\text{ GeV}, \tag{2.1}$$
$$m_c=1.346\text{ GeV},\quad m_b=5.081\text{ GeV},$$

while for F and F^* mesons, which do not contain the u or d quark but contain the s quark,

$$\mu'=1.131\text{ GeV},\quad \alpha_s'=0.5484,$$
$$m_s'=0.505\text{ GeV},\quad m_c'=1.293\text{ GeV}, \tag{2.2}$$
$$m_b'=5.013\text{ GeV}.$$

The values of α_s and quark masses in (2.1) and (2.2) are related to each other as well as to those for $c\bar{c}$ and $b\bar{b}$ in Ref. 1 through the quantum-chromodynamic transformation relations (1.1) and (1.2), while the values of μ are related through the empirical relation[4]

$$\mu^2/m_u \approx \mu'^2/m_s'. \tag{2.3}$$

The parameter A always retains the value

$$A=0.177\text{ GeV}^2, \tag{2.4}$$

while C can be adjusted for each quarkonium such that the theoretical mass of the ground state agrees with the experimental value.

The results in Table II are in agreement with available experimental data.[5] Since the B and B^* mesons are currently of much experimental interest,[6,7] it should also be noted that our value of 23 MeV for the hyperfine splitting of B^* and B is much smaller than those predicted by earlier authors.[8] If we accept the experimental value[7] $m_B \approx 5.272$ GeV, we obtain $m_{B^*} \approx 5.295$ GeV, which leads to the conclusion that the $B^*\bar{B}$ threshold is close to the $\Upsilon(4S)$ resonance of the $b\bar{b}$ system.

The potentials in Appendix A are unable to account for the masses or the mixing angles of those mesons in Table I which are affected by the annihilation channel. To overcome this difficulty, we shall assume that self-conjugate quarkonia acquire an additional coupling among themselves through annihilation channel which is nonperturbative in nature and appreciable only for the light mesons.[9] We shall, therefore, replace the perturbative annihilation potential (A6) by a phenomenological potential $\overline{\mathscr{V}}$, which represents the annihilation-channel coupling among $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$. Moreover, in view of the near equality of the u- and d-quark masses, we shall take

$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid u\bar{u}\rangle = \langle u\bar{u} \mid \overline{\mathscr{V}} \mid d\bar{d}\rangle = \langle dd \mid \overline{\mathscr{V}} \mid d\bar{d}\rangle. \tag{2.5}$$

According to the quark contents of mesons shown in Table I, π^0 and ρ^0 are not affected by the coupling $\overline{\mathscr{V}}$. Moreover, separating our Hamiltonian as $\mathscr{H}=\overline{\mathscr{H}}+\overline{\mathscr{V}}$, and substituting the quark contents of η and η' into

$$\langle \eta \mid \overline{\mathscr{H}}+\overline{\mathscr{V}} \mid \eta\rangle_P = M_\eta,$$
$$\langle \eta' \mid \overline{\mathscr{H}}+\overline{\mathscr{V}} \mid \eta'\rangle_P = M_{\eta'},$$
$$\langle \eta \mid \overline{\mathscr{H}}+\overline{\mathscr{V}} \mid \eta'\rangle_P = 0,$$

we obtain for the pseudoscalar states

$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid u\bar{u}\rangle_P = \tfrac{1}{4}(M_{\eta'}+M_\eta)-\tfrac{1}{2}M_\pi,$$
$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid s\bar{s}\rangle_P = \frac{1}{2\sqrt{2}}(M_{\eta'}-M_\eta), \tag{2.6}$$
$$\langle s\bar{s} \mid \overline{\mathscr{V}} \mid s\bar{s}\rangle_P = \tfrac{1}{2}(M_{\eta'}+M_\eta)-\langle s\bar{s} \mid \overline{\mathscr{H}} \mid s\bar{s}\rangle_P.$$

With the use of the potential (A5) and the parameters (2.2) and (2.4), the hyperfine splitting for $s\bar{s}$ is found to be 333 MeV, so that

$$\langle s\bar{s} \mid \overline{\mathscr{H}} \mid s\bar{s}\rangle_P \approx M_\phi - 333\text{ MeV}.$$

This result together with the experimental values[5] of M_π, M_η, $M_{\eta'}$, and M_ϕ, upon substitution into (2.6), gives

$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid u\bar{u}\rangle_P \approx 308\text{ MeV},$$
$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid s\bar{s}\rangle_P \approx 145\text{ MeV}, \tag{2.7}$$
$$\langle s\bar{s} \mid \overline{\mathscr{V}} \mid s\bar{s}\rangle_P \approx 67\text{ MeV},$$

which shows that for pseudoscalar states $\langle \overline{\mathscr{V}} \rangle$ rapidly decreases with increase in the quark mass, and may be ignored for heavy mesons.

A similar analysis for ω and ϕ with the use of their quark contents and experimental masses[5] indicates that

$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid u\bar{u}\rangle_V = \tfrac{1}{2}(M_\omega - M_\rho) \approx 7\text{ MeV},$$
$$\langle u\bar{u} \mid \overline{\mathscr{V}} \mid s\bar{s}\rangle_V \approx 0, \tag{2.8}$$
$$\langle s\bar{s} \mid \overline{\mathscr{V}} \mid s\bar{s}\rangle_V \approx 0.$$

It is not surprising that expectation values of the nonperturbative annihilation potential are much larger for pseudoscalar states than for vector states. Indeed, according to (A6), the annihilation potential to fourth order contributes only to the pseudoscalar states.

III. HEAVY-QUARK MESONS

The treatment of $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ systems with the use of our quantum-chromodynamic potential was fully discussed in our earlier paper,[1] where our results were found to be in excellent agreement with available experimental data. In addition, it should be pointed out that our results for $b\bar{b}$ indicated that the fine-structure splittings of the $2\,^3P$ states are 14 and 20 MeV, while the center of gravity of the $2\,^3P$ states lies 97 MeV below the $3\,^3S_1$ state. These predictions are also in excellent agreement with the experimental findings.[10]

Finally, Table III gives our results for the $b\bar{c}$ spectrum, where it is necessary to take into account the spin-orbit mixing terms in (A2) and (A3), which convert the pure states ^3J_J and ^1J_J into the mixed states $^3J_J'$ and $^1J_J'$ as explained in Appendix B. The potential parameters here are

TABLE II. Hyperfine splittings for mesons with light quarks.

Meson pair	Hyperfine splitting (MeV)	Meson pair	Hyperfine splitting (MeV)
(ρ,π)	632	(B^*,B)	23
(K^*,K)	400	(F_c^*,F_c)	135
(D^*,D)	141	(F_b^*,F_b)	25

$$\mu=1.880\text{ GeV},\quad \alpha_s=0.3920\,,$$
$$m_c=1.200\text{ GeV},\quad m_b=4.890\text{ GeV}\,, \tag{3.1}$$
$$A=0.177\text{ GeV}^2\,,$$

which correspond[4] to those for the $c\bar{c}$ system in Ref. 1 with the addition of the transformed b-quark mass. The results in Table III differ considerably from those of other authors,[11,12] and in particular our value of 33 MeV for hyperfine splitting in the ground state is much smaller than the earlier predictions.

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APPENDIX A: QUARK-ANTIQUARK POTENTIALS

Our Hamiltonian for the quark-antiquark system is

$$\mathscr{H}=m_1+m_2+\vec{p}^{\,2}/2m_1+\vec{p}^{\,2}/2m_2-\vec{p}^{\,4}/8m_1^{\,3}-\vec{p}^{\,4}/8m_2^{\,3}+\mathscr{V}_p(\vec{r})+\mathscr{V}_c(\vec{r})\,, \tag{A1}$$

where $\mathscr{V}_p(\vec{r})$ and $\mathscr{V}_c(\vec{r})$ are the perturbative[13] and confining potentials.

For a quark and an antiquark of different flavors,

$$\begin{aligned}
\mathscr{V}_p(\vec{r})=&-\frac{4\alpha_s}{3r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E)\right]+\frac{4\alpha_s}{3m_1m_2r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E)\right]\nabla^2\\
&+\frac{2\pi\alpha_s}{3}\frac{(m_1+m_2)^2}{m_1^{\,2}m_2^{\,2}}\left[\left[1-\frac{3\alpha_s}{2\pi}\right]\delta(\vec{r})-\frac{\alpha_s}{24\pi^2}(33-2n_f)\nabla^2\left[\frac{\ln\mu r+\gamma_E}{r}\right]\right]-\frac{\alpha_s^{\,2}}{9r^2}\frac{9(m_1+m_2)^2-8m_1m_2}{m_1m_2(m_1+m_2)}\\
&+\frac{32\pi\alpha_s}{9m_1m_2}\vec{S}_1\cdot\vec{S}_2\left[\left[1-\frac{19\alpha_s}{6\pi}\right]\delta(\vec{r})-\frac{\alpha_s}{8\pi}\left[8\frac{m_1-m_2}{m_1+m_2}+\frac{m_1+m_2}{m_1-m_2}\right]\ln\frac{m_2}{m_1}\delta(\vec{r})\right.\\
&\left.\qquad-\frac{\alpha_s}{24\pi^2}(33-2n_f)\nabla^2\left[\frac{\ln\mu r+\gamma_E}{r}\right]+\frac{21\alpha_s}{16\pi^2}\nabla^2\left[\frac{\ln(m_1m_2)^{1/2}r+\gamma_E}{r}\right]\right]\\
&+\frac{4\alpha_s}{m_1m_2}\frac{\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\frac{1}{3}\vec{S}_1\cdot\vec{S}_2}{r^3}\left[1+\frac{4\alpha_s}{3\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E-\tfrac{4}{3})-\frac{3\alpha_s}{\pi}[\ln(m_1m_2)^{1/2}r+\gamma_E-\tfrac{4}{3}]\right]\\
&+\frac{\alpha_s}{3m_1^{\,2}m_2^{\,2}}\frac{\vec{L}\cdot\vec{S}}{r^3}\left[[(m_1+m_2)^2+2m_1m_2]\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E-1)\right]\right.\\
&\left.\qquad+\frac{\alpha_s}{2\pi}(m_1+m_2)^2\{\tfrac{8}{3}-6[\ln(m_1m_2)^{1/2}r+\gamma_E-1]\}-\frac{3\alpha_s}{2\pi}(m_1^{\,2}-m_2^{\,2})\ln\frac{m_2}{m_1}\right]\\
&-\frac{\alpha_s}{3m_1^{\,2}m_2^{\,2}}\frac{\vec{L}\cdot(\vec{S}_1-\vec{S}_2)}{r^3}\left[(m_1^{\,2}-m_2^{\,2})\left[1-\frac{\alpha_s}{6\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E-1)-\frac{3\alpha_s}{\pi}[\ln(m_1m_2)^{1/2}r+\gamma_E-1]\right]\right.\\
&\left.\qquad-\frac{3\alpha_s}{2\pi}(m_1+m_2)^2\ln\frac{m_2}{m_1}\right]\,,
\end{aligned} \tag{A2}$$

and

$$\mathscr{V}_c(\vec{r})=Ar-A\frac{m_1^{\,2}+m_2^{\,2}}{4m_1^{\,2}m_2^{\,2}}\frac{\vec{L}\cdot\vec{S}}{r}+A\frac{m_1^{\,2}-m_2^{\,2}}{4m_1^{\,2}m_2^{\,2}}\frac{\vec{L}\cdot(\vec{S}_1-\vec{S}_2)}{r}+C\,, \tag{A3}$$

where the confining potential corresponds to a scalar-exchange interaction in the quasistatic approximation.

For a quark and an antiquark of the same flavor, the perturbative potential consists of a direct potential $\mathscr{V}'_p(\vec{r})$ and an annihilation potential $\mathscr{V}''_p(\vec{r})$, so that

$$\mathscr{V}_p(\vec{r})=\mathscr{V}'_p(\vec{r})+\mathscr{V}''_p(\vec{r}) \tag{A4}$$

with

$$\begin{aligned}\mathscr{V}'_p(\vec{r})=&-\frac{4\alpha_s}{3r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E)\right]+\frac{4\alpha_s}{3m^2 r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E)\right]\nabla^2\\
&+\frac{8\pi\alpha_s}{3m^2}\left[\left[1-\frac{3\alpha_s}{2\pi}\right]\delta(\vec{r})-\frac{\alpha_s}{24\pi^2}(33-2n_f)\nabla^2\left[\frac{\ln\mu r+\gamma_E}{r}\right]\right]-\frac{14\alpha_s^2}{9mr^2}\\
&+\frac{32\pi\alpha_s}{9m^2}\vec{S}_1\cdot\vec{S}_2\left[\left[1-\frac{35\alpha_s}{12\pi}\right]\delta(\vec{r})-\frac{\alpha_s}{24\pi^2}(33-2n_f)\nabla^2\left[\frac{\ln\mu r+\gamma_E}{r}\right]+\frac{21\alpha_s}{16\pi^2}\nabla^2\left[\frac{\ln mr+\gamma_E}{r}\right]\right]\\
&+\frac{4\alpha_s}{m^2}\frac{\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r}-\frac{1}{3}\vec{S}_1\cdot\vec{S}_2}{r^3}\left[1+\frac{4\alpha_s}{3\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E-\tfrac{4}{3})-\frac{3\alpha_s}{\pi}(\ln mr+\gamma_E-\tfrac{4}{3})\right]\\
&+\frac{2\alpha_s}{m^2}\frac{\vec{L}\cdot\vec{S}}{r^3}\left[1-\frac{11\alpha_s}{18\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E-1)-\frac{2\alpha_s}{\pi}(\ln mr+\gamma_E-1)\right],\end{aligned} \tag{A5}$$

$$\mathscr{V}''_p(\vec{r})=\frac{8\alpha_s^2}{3m^2}(1-\ln 2)(\vec{S}_1\cdot\vec{S}_2-\tfrac{1}{4})\delta(\vec{r}), \tag{A6}$$

and the confining potential is

$$\mathscr{V}_c(\vec{r})=Ar-\frac{A}{2m^2}\frac{\vec{L}\cdot\vec{S}}{r}+C. \tag{A7}$$

APPENDIX B: SPIN-ORBIT MIXING INTERACTION

For quarkonia with unequal quark and antiquark masses, it is necessary to deal with spin-orbit mixing terms of the form[14] $\vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r)$.

Consider the 2×2 matrices

TABLE III. $b\bar{c}$ spectrum. The pure states ${}^3J_J$ and ${}^1J_J$ as well as the mixed states ${}^3J'_J$ and ${}^1J'_J$ are given. The ground-state energy is approximately 6.3 GeV.

State	Excitation energy (MeV)
$1\,{}^3S_1$	33
$1\,{}^1S_0$	0
$2\,{}^3S_1$	625
$2\,{}^1S_0$	602
$1\,{}^3P_2$	543
$1\,{}^3P_1$, $1\,{}^3P'_1$	486, 484
$1\,{}^3P_0$	431
$1\,{}^1P_1$, $1\,{}^1P'_1$	513, 515
$1\,{}^3D_3$	835
$1\,{}^3D_2$, $1\,{}^3D'_2$	823, 816
$1\,{}^3D_1$	809
$1\,{}^1D_2$, $1\,{}^1D'_2$	826, 833

$$\begin{aligned}\psi'(\vec{r})&=(3/8\pi)^{1/2}\sigma_2\hat{r}\cdot\hat{\epsilon}^{(M)}R'(r),\\ \psi''(\vec{r})&=(3/16\pi)^{1/2}\vec{\sigma}\cdot(\hat{r}\times\hat{\epsilon}^{(M')})\sigma_2R''(r),\end{aligned} \tag{B1}$$

where the $\hat{\epsilon}^{(M)}$ are the unit vectors

$$\begin{aligned}&\hat{\epsilon}^{(1)}=-(1/\sqrt{2})(1,i,0),\quad \hat{\epsilon}^{(0)}=(0,0,1),\\ &\hat{\epsilon}^{(-1)}=(1/\sqrt{2})(1,-i,0).\end{aligned} \tag{B2}$$

It is possible to represent the ${}^1P_1$ and ${}^3P_1$ states by $\psi'(\vec{r})$ and $\psi''(\vec{r})$, respectively, with the understanding that

$$\vec{\sigma}^{(1)}\psi=\vec{\sigma}\psi,\quad \vec{\sigma}^{(2)}\psi=\psi\vec{\sigma}^{T}, \tag{B3}$$

and the ψ are normalized as

$$\int d\vec{r}\,\mathrm{Tr}(\psi^*\psi)=1. \tag{B4}$$

Let us evaluate the matrix element

$$\langle{}^3P_1|\vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r)|{}^1P_1\rangle=\int d\vec{r}\,\mathrm{Tr}[\psi''^*(\vec{r})\vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r)\psi'(\vec{r})],$$

which becomes, with the substitution of (B1) and the application of (B3),

$$\langle{}^3P_1|\vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r)|{}^1P_1\rangle=\frac{3}{8\sqrt{2}\pi}\int d\vec{r}\,\mathrm{Tr}[\sigma_2\vec{\sigma}\cdot(\hat{r}\times\hat{\epsilon}^{*(M')})(\vec{\sigma}\sigma_2-\sigma_2\vec{\sigma}^{T})]\cdot\vec{L}\hat{r}\cdot\hat{\epsilon}^{(M)}f(r)R''(r)R'(r).$$

The trace can be simplified by using $\sigma_2\vec{\sigma}^T\sigma_2=-\vec{\sigma}$, and readily evaluated, and thus

$$\langle {}^3P_1 \,|\, \vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r) \,|\, {}^1P_1 \rangle = \frac{3}{2\sqrt{2}\pi} \int d\vec{r}(\hat{r}\times\hat{\epsilon}^{*(M')})\cdot\vec{L}\hat{r}\cdot\hat{\epsilon}^{(M)}f(r)R''(r)R'(r) \ .$$

But,

$$(\vec{L})_i\vec{r}\cdot\hat{\epsilon}=\epsilon_{ijk}x_jp_kx_l\hat{\epsilon}_l=\epsilon_{ijk}x_j(-i\delta_{kl}+x_lp_k)\hat{\epsilon}_l = -i(\vec{r}\times\hat{\epsilon})_i+\vec{r}\cdot\hat{\epsilon}(\vec{L})_i \ ,$$

so that for the present purpose

$$(\hat{r}\times\hat{\epsilon}^{*(M')})\cdot\vec{L}\hat{r}\cdot\hat{\epsilon}^{(M)}=-i(\hat{r}\times\hat{\epsilon}^{*(M')})\cdot(\hat{r}\times\hat{\epsilon}^{(M)}) = -i(\hat{\epsilon}^{*(M')}\cdot\hat{\epsilon}^{(M)}-\hat{r}\cdot\hat{\epsilon}^{*(M')}\hat{r}\cdot\hat{\epsilon}^{(M)}) \ ,$$

and the matrix element reduces to

$$\langle {}^3P_1 \,|\, \vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r) \,|\, {}^1P_1 \rangle = -i\,2^{3/2}\delta_{MM'}\int_0^\infty dr\, r^2 f(r)R''(r)R'(r) \ , \tag{B5}$$

which leads to mixing of the ${}^3P_1$ and ${}^1P_1$ states for $M'=M$.

The above treatment can be generalized by using the representation

$$\psi({}^1J_J)=\left[\frac{(2J+1)!!}{8\pi J!}\right]^{1/2}\vec{\sigma}_2\hat{r}\cdot\hat{\eta}^{(M)}R'(r) \ ,$$
$$\psi({}^3J_J)=\left[\frac{(2J+1)!!}{8\pi(J+1)(J-1)!}\right]^{1/2}\vec{\sigma}\cdot(\hat{r}\times\hat{\eta}^{(M')})\sigma_2R''(r) \ , \tag{B6}$$

with

$$\hat{\eta}_{i_1}^{(M)}=\hat{\epsilon}_{i_1,i_2,\ldots,i_J}^{(M)}\hat{r}_{i_2}\hat{r}_{i_3}\cdots\hat{r}_{i_J} \ , \tag{B7}$$

where the polarization tensor $\hat{\epsilon}_{i_1,i_2,\ldots,i_J}^{(M)}$ is symmetrical in all its indices and traceless with respect to the contraction of any pair of indices. It then follows that

$$\langle {}^3J_J \,|\, \vec{L}\cdot(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})f(r) \,|\, {}^1J_J \rangle = -2i[J(J+1)]^{1/2}\delta_{MM'}\int_0^\infty dr\, r^2 f(r)R''(r)R'(r) \tag{B8}$$

for all values of J.

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[12]W. Buchmüller and S.-H. H. Tye, Phys. Rev. D 24, 132 (1981).

[13]We have expressed in coordinate space the perturbative quark-antiquark potentials given by S. N. Gupta and S. F. Radford [Phys. Rev. D 25, 3430 (1982)]. We have also set $N=3$, and modified the results to conform to the renormalization scheme of Ref. 3.

[14]In Appendix B, we have set $\vec{S}_1=\frac{1}{2}\vec{\sigma}^{(1)}$ and $\vec{S}_2=\frac{1}{2}\vec{\sigma}^{(2)}$ so that the $\vec{\sigma}$ matrices with superscripts 1 and 2 refer to the quark and the antiquark, while σ_2 denotes one of the components of $\vec{\sigma}$.