Fixed points of $(\phi^6)_3$ and $(\phi^4)_4$ theories

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I suggest how the presence of an ultraviolet and a nontrivial infrared-stable fixed point in $(\phi^6)_3$ theory could be related to the (presumed) absence of any nonzero fixed points in $(\phi^4)_4$ theory.

Beginning with Landau,[1] it has long been suspected that infrared-free theories such as QED do not exist as fundamental, interacting field theories. This is probably true for the simplest example of an infrared-free theory, $(\phi^4)_4$, where numerical evidence strongly indicates that the continuum limit of any interacting lattice $(\phi^4)_4$ theory is free field theory.[2] This is sometimes described as ensuing from the ghost which occurs when the renormalization-group functions, calculated to one-loop order in the infrared limit, are extrapolated to the ultraviolet regime.[1] Of course, the triviality of $(\phi^4)_4$ theory could simply be due to the detailed structure of the theory, and have nothing to do with Landau's ghost, which after all appears only when perturbative results are pushed far beyond their region of validity.

An interacting continuum field theory, with a finite renormalized coupling constant, surely exists if the bare coupling g_0 vanishes as some ultraviolet cutoff Λ diverges. This includes super-renormalizable theories, where $g_0 \sim \Lambda^{-p}$, $p>0$, and asymptotically free theories, for which $g_0 \sim 1/\ln(\Lambda)$. I suggest, however, that this does *not* include all interacting field theories. In particular, it should always be possible to construct an interacting field theory if the bare coupling approaches some fixed value, $g_0 \to g^*$, as $\Lambda \to \infty$. In the above instances, $g^*=0$, but a $g^* \neq 0$ would occur in any infrared-free theory with an ultraviolet-stable fixed point at g^*.

Large-N $(\phi^6)_3$ theory provides an example of such an infrared-free theory,[3] since it has an ultraviolet-stable fixed point $\lambda^*_{\rm UV} \sim N^{-2}$. The existence of $\lambda^*_{\rm UV}$ can be understood as the result of an effective asymptotically free theory[4] for $\lambda \sim N^{-3/2}$. From considerations of this effective theory, I was led to the conjecture that there is a nonzero infrared-stable fixed point $\lambda^*_{\rm IR} \sim N^{-3/2}$.

The existence of $\lambda^*_{\rm IR}$ has dramatic effects for $(\phi^6)_3$ theory. With a $\lambda^*_{\rm IR} \neq 0$, there would be tricritical points in three dimensions with nonclassical exponents. In this paper, I wish to address a different question: How can $(\phi^4)_4$ theory be trivial if $(\phi^6)_3$ theory is not? On simple but plausible grounds, I argue that if there is no g^* in $(\phi^4)_4$ theory (which I take to be true for all N), and there is a $\lambda^*_{\rm UV}$ in $(\phi^6)_3$ theory, then there must be a $\lambda^*_{\rm IR} > \lambda^*_{\rm UV}$ in $(\phi^6)_3$ theory. My arguments will be couched in terms of large-N $(\phi^6)_3$ theory, where the existence of $\lambda^*_{\rm UV}$ is assured. Nevertheless, as I suspect[3] that a $\lambda^*_{\rm UV}$ may exist in $(\phi^6)_3$ theory for all $N \geqslant 0$, the arguments of this paper provide support for a $\lambda^*_{\rm IR} > \lambda^*_{\rm UV}$ in this case as well. Needless to say, for small-N $(\phi^6)_3$ theory, in order to find even $\lambda^*_{\rm UV}$ only numerical simulations will suffice.

I take the β function of the large-N ϕ^6 theory in $3+\epsilon$ dimensions to be[3,4]

$$\beta(\lambda) = 2\epsilon\lambda + 3N\lambda^2 - \frac{\pi^2}{32}N^3\lambda^3 + \frac{b}{2}N^6\lambda^5 \quad . \tag{1}$$

The six-point coupling λ is defined following Ref. 3. In three dimensions, the form the β function takes is illustrated in Fig. 1. For $\epsilon=0$, there are two fixed points:

$$\lambda^*_{\rm UV} = \frac{96}{\pi^2}\frac{1}{N^2} + O(N^{-3}) \tag{2}$$

and

$$\lambda^*_{\rm IR} = \frac{\pi}{4\sqrt{b}}\frac{1}{N^{3/2}} + O(N^{-2}) \quad . \tag{3}$$

At large N, the corrections about $\lambda^*_{\rm UV}$ involve N^{-1}, while those about $\lambda^*_{\rm IR}$ involve $N^{-1/2}$. The latter is due to the unusual nature of the N^{-1} expansion—really, a $N^{-1/2}$ expansion—for the effective theory of Ref. 4.

I must emphasize that although perturbation theory in λ can be used to show $\lambda^*_{\rm UV}$ exists for large N, the existence of $\lambda^*_{\rm IR}$ is assumed. Even for large N, the presence of a $\lambda^*_{\rm IR} \sim N^{-3/2}$ must be established by nonperturbative means. The β function of Eq. (1) is meant only to be indicative of how a β function such as that of Fig. 1 would behave. In this way, while explicit computation reveals b is negative,[4] I shall take b to be positive.[5]

Consider first how $\lambda^*_{\rm UV}$ changes as ϵ increases. For $\epsilon << N^{-1}$,

$$\lambda^*_{\rm UV} = \frac{96}{\pi^2}\frac{1}{N^2} + \frac{2}{3}\frac{\epsilon}{N} + \cdots \quad . \tag{4}$$

As ϵ increases further, so $\epsilon >> N^{-1}$,

$$\lambda^*_{\rm UV} = \frac{8}{\pi}\frac{\sqrt{\epsilon}}{N^{3/2}} + \cdots \quad . \tag{5}$$

Assume for the moment that $\lambda^*_{\rm IR}$ does not exist. Then, granting merely that the β function is a smooth function of ϵ, Eq. (5) is incompatible with the triviality of $(\phi^4)_4$ theory, since for $\epsilon=1$, Eq. (5) indicates there is a $\lambda^*_{\rm UV} \sim N^{-3/2}$. Even if one started with a pure ϕ^4 theory, universality in the ultraviolet limit would imply that $\lambda^*_{\rm UV}$, and thus a nontrivial continuum theory, would be generated dynamically through the virtual effects of six-point interactions. I re-

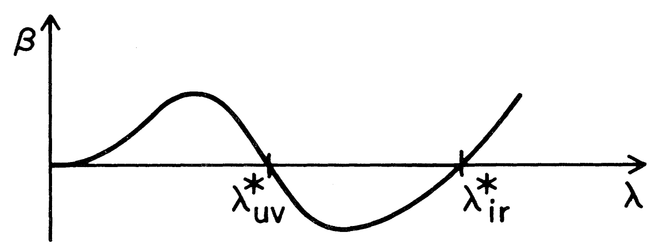

FIG. 1. The proposed β function of $(\phi^6)_3$ theory.

mark that if $\lambda_{UV}^{*} \neq 0$, for $\epsilon \leq 1$ the superficially nonrenormalizable six-point coupling will dominate the four-point coupling in the ultraviolet limit. The opposite is true in the infrared limit, where the four-point coupling prevails, with the customary tricritical phase diagram.

Consider now how λ_{IR}^{*} changes with ϵ. For $\epsilon << 1$,

$$\lambda_{IR}^{*} = \frac{\pi}{4\sqrt{b} N^{3/2}} \left[1 - \frac{512 b}{\pi^4} \epsilon + \cdots \right] . \tag{6}$$

As ϵ increases, λ_{IR}^{*} continues to decrease, while λ_{UV}^{*} increases. There is a critical dimensionality $3 + \tilde{\epsilon}_c$ where they coalesce to give a β function as in Fig. 2. With the β function of Eq. (1),

$$\tilde{\epsilon}_c = \frac{\pi^4}{4096} \frac{1}{b} . \tag{7}$$

When $\epsilon > \tilde{\epsilon}_c$, there are no nonzero fixed points whatsoever. Hence if $(\phi^4)_4$ theory is to be trivial, it must be true that

$$\tilde{\epsilon}_c < 1 . \tag{8}$$

From Eq. (7), I find this restricts b to be > 0.024.

The simplicity of my arguments should not belie the fact that without λ_{IR}^{*}, the only way to avoid a λ_{UV}^{*} in four dimensions would be to postulate some mysterious catastrophe which befalls the $3 + \epsilon$ expansion of the ϕ^6 theory. Not only is there no precedent for this in any other ϵ expansion, but the coalescence of λ_{UV}^{*} and λ_{IR}^{*} in $3 + \tilde{\epsilon}_c$ dimensions mirrors the same phenomena in $3 - \epsilon_c$ dimensions.[3] The only difference between $\tilde{\epsilon}_c$ and ϵ_c is that while $\epsilon_c \sim N^{-1}$ was calculable for large N, $\tilde{\epsilon}_c \sim N^0$ is not. Similarly, while the presence of $\lambda_{IR}^{*} > \lambda_{UV}^{*}$ in three dimensions is necessary to ensure that there are nonclassical tricritical points in two dimensions[4] (when $\epsilon_c < 1$), λ_{IR}^{*} is also needed if $(\phi^4)_4$ theory is to be trivial (with $\tilde{\epsilon}_c < 1$). Thus, though the evidence for λ_{IR}^{*} is solely indirect, it is telling.

What of theories in four dimensions? I stress that just because $(\phi^4)_4$ theory is apparently trivial does not mean that *any* theory with fundamental scalars is necessarily so. A counterexample is provided by asymptotically free theories of scalars coupled to non-Abelian gauge fields.[6]

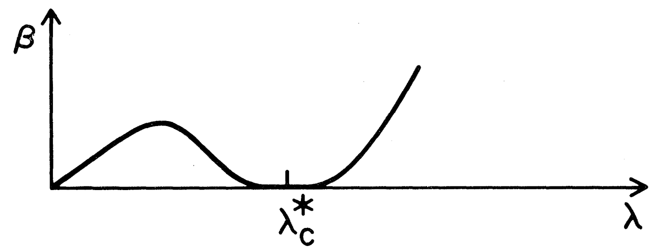

FIG. 2. The β function in $3 + \tilde{\epsilon}_c$ dimensions.

The question of infrared-free theories in four dimensions is much more difficult to answer. For such a theory to exist as an interacting, continuum field theory, there must be a nontrivial ultraviolet-stable fixed point g_{UV}^{*} for all infrared-free couplings. For definiteness, consider some number of scalars or fermions coupled to non-Abelian gauge fields, which I characterize by a single (gauge) coupling g. Following the example[4] of $(\phi^6)_3$ theory, if there is a $g_{UV}^{*} \neq 0$, there will be an effective asymptotically free theory in strong coupling, $g >> g_{UV}^{*}$. As the coupling g in the effective theory will remain dimensionless, by the gauge principle non-Abelian gauge fields can only enter in the usual fashion. This is not true for scalars or fermions, where no symmetry principle restricts their (canonical) mass dimension. Allowing in particular for higher-derivative actions, it is then easy to construct asymptotically free theories of scalars[7] and fermions[8] in four dimensions. These might be relevant as effective theories for $g >> g_{UV}^{*}$.[9]

Nevertheless, my general arguments merely suggest that interacting infrared-free theories may exist in four dimensions,[10] not that they in fact do. In the end, as for small-N $(\phi^6)_3$ theory, a definitive answer must await the difficult numerical simulations necessary to model these theories.

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[1] L. D. Landau, in *Niels Bohr and the Development of Physics,* edited by W. Pauli (Pergamon, Oxford, 1955).

[2] Recent Monte Carlo simulations include C. Aragao de Carvalho, S. Caracciolo, and J. Fröhlich, Nucl. Phys. B215 [FS7], 209 (1983). See also R. D. Pisarski, Phys. Rev. Lett. 48, 574 (1982), Refs. 14–16; Phys. Rev. D 26, 3543 (1982), Ref. 10; and, for the O(N) $(\phi^4)_4$ theory at large N, W. A. Bardeen and M. Moshe, this issue, Phys. Rev. D 28, 1372 (1983). Mathematically rigorous treatments of $(\phi^4)_4$ not in the above include M. Aizenman, Commun. Math. Phys. 86, 1 (1982); A. D. Sokal, Ann. Inst. Henri Poincaré 37, 317 (1982); D. C. Brydges, J. Frölich, and A. D. Sokal, University of Virginia reports, 1983 (unpublished).

[3] R. D. Pisarski, Phys. Rev. Lett. 48, 574 (1982).

[4] R. D. Pisarski, Phys. Rev. D 26, 3543 (1982).

[5] To be precise, for large N, λ_{IR}^{*} is determined not only by the terms $\sim N^3 \lambda^3$ and $\sim N^6 \lambda^5$ in the β function, but all terms $\sim (N^3 \lambda^2)^n \lambda$, $n = 1, 2, \ldots$. For the purposes of my heuristic discussion, I represent the effect of this infinite series by the single term in Eq. (1) $\sim N^6 \lambda^5$, with $b > 0$.

[6] D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).

[7] Consider an N-component scalar field $\vec{\sigma}$ with the Lagrangian

$$L = \frac{1}{2\tilde{g}} [(\partial^2 \vec{\sigma})^2 + m^2 (\partial_\mu \vec{\sigma})^2] ,$$

subject to the constraint $\vec{\sigma}^2 = 1$. This theory is asymptotically free in $\tilde{g}$ in four dimensions [E. Brezin, J. Zinn-Justin, and J. C. Le Guillou, J. Phys. A 9, L119 (1976)]. This theory is only meant to serve as an effective theory in Euclidean space-time, since the mass spectrum consists of states with complex mass. As an effective theory, its coupling $\tilde{g}$ is presumably a function of the original gauge coupling g (for $g >> g_{UV}^{*}$) and the scalar self-coupling.

[8] Consider a theory of fermion fields ψ with the Lagrangian

$$L = \bar{\psi} i \partial^2 \partial\!\!\!/ \psi + \cdots + f^2 (\bar{\psi}\psi)^4 ,$$

where the ellipsis denotes couplings with dimensions of mass. Like four-fermion interactions in two dimensions [D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974)], this theory is asymptotically free in the eight-point coupling f^2 in four space-time dimensions. The same caveats as in Ref. 7 apply.

[9] Implicitly, my reasoning is valid only if g_{UV}^{*} is a simple zero of

$\beta(g)$. If applied to theories with Abelian gauge fields, it would rule out a $g_{UV}^* \neq 0$. This is because in the effective theory for $g >> g_{UV}^*$, by themselves non-Abelian gauge fields will always contribute to asymptotic freedom. In contrast, whatever matter couples to an Abelian field, whether higher derivative or not, gives an Abelian gauge coupling which inevitably is infrared free. As shown by Adler [S. L. Adler, Phys. Rev. D 5, 3021 (1972); 7, 1948(E) (1973)], however, any g_{UV}^* in fermionic QED must be an essential zero. I do not know what form the effective asymptotically free theory might take in this instance.

[10]The phenomenological virtues of extended hypercolor theories with a $g_{UV}^* \neq 0$ were noted originally by B. Holdom, Phys. Rev. D 24, 1441 (1981).