Brief Reports

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Note on Euclidean Bianchi-type "cosmologies"

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An explanation using "qualitative cosmology" is given for the absence of chaotic behavior in the Euclidean analog of the Bianchi cosmologies.

The solutions to Einstein's equations in four spatial dimensions (Euclidean general relativity) are important to the study of "spacetime foam" and other Euclidean quantum gravity theories.[1] Of particular interest have been Euclidean extensions of solutions in spacetime and other Euclidean solutions which represent instantons.[2]

Here we note that the Einstein equations for the spatially homogeneous Bianchi models[3] are easily expressed in four spatial dimensions using the $3+1$ formalism of Geroch[4] and Smarr.[5] These equations are given below. The notation $\oplus$ describes a sign which is as shown in the Euclidean equations and reversed in the spacetime equations.

If the Euclidean Einstein equations for the metric g_{ab} $(a,b=0,1,2,3)$ are projected onto a spacelike hypersurface Σ with normal n_a such that $n_a n^a = \oplus 1$, we obtain *in vacuo* for lapse α and zero shift the dynamical equations

$$\mathscr{L}_{\alpha n} h_{ab} = -2\alpha\kappa_{ab} \quad , \tag{1}$$

$$\mathscr{L}_{\alpha n} \kappa_{ab} = -\alpha(2\kappa_{ca}\kappa^c_b - \kappa\kappa_{ab} \oplus \mathscr{R}_{ab}) \oplus D_a D_b \alpha \quad , \tag{2}$$

and the constraints

$$\mathscr{R} \ominus \kappa^2 \oplus \kappa_{ab}\kappa^{ab} = 0 \tag{3}$$

and

$$D_a \kappa^a_b - D_b \kappa = 0 \quad . \tag{4}$$

In Eqs. (1)–(4), $\kappa_{ab} = -h^c_a h^d_b \nabla_c n_d$ is the extrinsic curvature, h_{ab} the induced metric on Σ, ∇_a the covariant derivative for g_{ab}, D_a the covariant derivative for h_{ab}, κ the trace of κ_{ab}, $\mathscr{R}_{ab}$ the Ricci tensor formed from D_a, h_{ab}, and $\mathscr{R}$ the scalar curvature $h_{ab}\mathscr{R}^{ab}$. (Repeated indices are summed.) The tensors h_{ab}, κ_{ab}, and $\mathscr{R}_{ab}$, and the operator D_a are orthogonal to n_a. Indices may be raised and lowered with h^{ab} defined by $h^{ac}h_{cb} = \delta^a_b$, $h^{ac}h_{ac} = 3$. $\mathscr{L}$ denotes a Lie derivative.

We wish to consider the solutions to Eqs. (1)–(4) which, if n^a were timelike, would be spatially homogeneous cosmologies, with the spatial metric h_{ab} one of the Bianchi types. For Σ-homogeneous h_{ab}, the momentum constraints (4) vanish identically. As in the spacetime case,[3] the four-dimensional Euclidean solution (corresponding to the spacetime dynamical solution) may be obtained from the Hamiltonian constraint (3).

This equation may be written in the "Hamiltonian cosmology" form given by Misner[6] and Ryan.[7] Ryan has shown[8] that for most Bianchi types a qualitative understanding of the dynamics may be obtained by visualizing a system point described by the minisuperspace variables (β_+,β_-) moving in an expanding potential well described by $V(\beta_+,\beta_-,\Omega)$, where

$$h\mathscr{R} = -\mathscr{R}_0{}^4 e^{-4\Omega}(V-1) \quad . \tag{5}$$

The variables β_+, β_- describe the anisotropy, and Ω, the logarithm of the spatial volume, is an intrinsic time in the spacetime case; h is the determinant of h_{ab}. The potential is portrayed graphically for all Bianchi types in Ref. 3.

Recently, Barrow has shown[9] that Bianchi-type-IX (spacetime) cosmologies exhibit chaotic behavior. Briefly, this may be described as two solutions, i.e., trajectories in the β_+-β_- plane, with infinitesimally different initial data becoming exponentially far from each other a finite Ω later. Barrow has pointed out that chaotic behavior is not present in the corresponding Euclidean models as described by Eq. (3).[10]

We wish to explain the absence of chaotic behavior in the Euclidean models by appealing to Eqs. (3) and (5). Since $\mathscr{R}(h_{ab})$ is intrinsic to Σ, it does not change upon passing from the spacetime to the Euclidean solution. The sign change in the Hamiltonian constraint causes the derivation of the Arnowitt-Deser-Misner[11] Hamiltonian H as in Ref. 3 to yield

$$H^2 = p_+{}^2 + p_-{}^2 \oplus 24\pi^2 h\mathscr{R} \quad , \tag{6}$$

rather than the opposite of the circled sign. Although the dynamical interpretation is invalid in the Euclidean problem, Eq. (6) is obtained if $p_\Omega{}^2$ is arbitrarily chosen to be H^2.

At least in Σ's of the class-A[3] Bianchi types I, II, VI$_{-1}$, VII$_0$, VIII, and IX, the system is described at each Ω by a point in the β_+-β_- plane. In spacetime, the potential V may be approximated to have infinitely steep walls with the system point inside. The system point is effectively free (Kasner) far from the walls. Approximate "bounce laws" may be obtained to describe the interaction of the system point with the potential walls.[7] In type IX, for example, the Kasner exponents change during this interaction. The chaotic behavior described by Barrow arises from the qualitative change of an expanding to contracting direction which occurs when the system point enters the "corners" of the potential.

If the same analysis is repeated for the Euclidean case described by Eq. (6), the behavior must be understood in terms of an inverted potential. For type IX, Eq. (6) becomes[3]

$$H^2 = p_+^2 + p_-^2 - 8\pi^2 \mathcal{R}_0^4 e^{-4\Omega}\{2e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-)-1] + e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)\}. \quad (7)$$

The shape of the equipotentials is unchanged in passing to the Euclidean case. The potential minimum at the origin has, however, become a maximum. The walls decrease steeply to $-\infty$ as $|\beta_\pm|$ increase. For convenience, let a single (triangular) equipotential[3] represent for some fixed Ω and H^2 (with $H^2 < 0$ allowed) the curve described by Eq. (7) with $p_+ = p_- = 0$. Since the region inside the curve for the same fixed Ω corresponds to greater H^2, it is forbidden to the system point. Thus the system point is either above or outside the potential (now a barrier) in the Euclidean case. Even if $H^2 < 0$ is possible, there is no Euclidean analog of the system point repeatedly bouncing off the corners of the potential. The presence of at most one bounce for the system point in the Euclidean case means that the separation between trajectories with infinitesimally different initial points will be linear rather than exponential in Ω. In the Euclidean case there is no qualitative change in the dynamics analogous to the change from expanding to contracting directions in the spacetime case which occurs when the system point enters the "corners" of the potential. It is this qualitative change of dynamics in the spacetime type-IX cosmologies which produces the chaos defined by loss of information about initial conditions. Because the system point must remain outside the corners of the potential, qualitative dynamical changes which cause the chaotic behavior of spacetime type-IX cosmologies cannot occur in the Euclidean analog solutions. This does not preclude interesting behavior when the additional restrictions needed for instanton solutions are imposed.[2]

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[1]See, for example, S. W. Hawking, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge Univ., Cambridge, 1979).

[2]T. Eguchi and A. J. Hanson, Ann. Phys. (N.Y.) 120, 82 (1979), and references therein; T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. 66, 213 (1980), Appendix D.

[3]See M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton Univ., Princeton, 1975).

[4]R. Geroch, J. Math. Phys. 13, 956 (1972).

[5]L. Smarr, Ph.D. thesis, University of Texas, 1974.

[6]C. W. Misner, Phys. Rev. 186, 1319 (1969).

[7]M. P. Ryan, Jr., Ann. Phys. (N.Y.) 65, 506 (1971).

[8]M. P. Ryan, Jr., J. Math. Phys. 15, 812 (1974), and Ref. 7.

[9]J. D. Barrow, Phys. Rev. Lett. 46, 963 (1981); Phys. Rep. 85, 1 (1982).

[10]J. D. Barrow (private communication).

[11]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).