Finiteness of broken $N=4$ super Yang-Mills theory

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Using a light-cone gauge formulation for $N=4$ extended supersymmetry, it is shown that an explicit breaking of the supersymmetry by the addition of mass terms does not disturb off-shell finiteness to any order provided the sum of fermion masses equals the sum of scalar masses and appropriate cubic interactions between scalars are included.

I. INTRODUCTION

Calculations have shown that the $N=4$ extended supersymmetric Yang-Mills[1] theory is ultraviolet finite up to the three-loop order.[2] Suggestive arguments due to Ferrara and Zumino,[3] Sohnius and West,[4] and Stelle[5] indicate finiteness to all orders. A proof of this (based on $N=2$ superfields) has been advanced by Howe, Stelle, and Townsend.[6] Recently, Mandelstam[7] invented a light-cone gauge formulation of the $N=4$ theory and used it to argue finiteness to all orders: In this formulation the off-shell Green's functions appear to be ultraviolet finite (although it is not yet clear that the light-cone gauge does not introduce pathologies in the form of unphysical singularities, operator ordering ambiguities, etc.).

Such a remarkable theory would therefore seem to merit further investigation. One of the main obstacles to its being used as a realistic model is of course the absence of any scale. There are no masses in the supersymmetric Lagrangian and there would appear to be no dimensional transmutation. It is a truly scale-invariant theory. How can this scale invariance be broken? One possibility would be simply to assume that one or more of the scalar fields in the system develop vacuum expectation values. The vacuum is mestastable, however, and the minimization of the effective potential does not determine these values: They have to be regarded as constants of integration. Although some of the gauge symmetry would be broken in such a vacuum, the resulting spectrum would contain many massless states. The infrared behavior would be very singular, so making the theory difficult to interpret.

Our proposal is to break explicitly the scale invariance, the supersymmetry, and the global SU(4) which the theory possesses, by adding mass terms to the original Lagrangian. We shall argue that this can be done without disturbing the ultraviolet behavior. It is not evident, of course, that breaking supersymmetry softly by means of mass terms can be achieved without reintroducing at least logarithmic divergences into the theory. And in fact we do find that it is necessary to include trilinear interactions among the scalars, together with the mass terms, if finiteness is to be preserved. (Similar trilinear interactions appear in massive $N=1$ supersymmetric theories.)

In order to demonstrate finiteness we shall use a light-cone gauge formulation of the $N=4$ theory due to Brink, Lindgren, and Nilsson (BLN).[8] (This formulation is presumably equivalent to Mandelstam's, but we find that it is easier to work with.) The strategy is straightforward. First, to the conventionally formulated, $N=4$ supersymmetric Lagrangian, in component notation, we add a fermion mass term. We then transform the component Lagrangian to the light-cone gauge, a step which involves elimination of the dependent (or nonpropagating) components. We then attempt to express the resulting Lagrangian in terms of the BLN superfields. At this point it becomes clear that the light-cone gauge Lagrangian must be augmented by appropriate scalar mass terms and trilinear interactions. Although the mass-dependent terms explicitly break the $N=4$ supersymmetry, the superfield notation is important and useful because it permits a relatively simple analysis of the ultraviolet convergence of the broken theory.

The paper is organized as follows. In Sec. II, to establish notation, the symmetric theory in the light-cone gauge is reviewed together with the superfields of Brink *et al.* Section III is devoted to an explanation of the finiteness arguments for the symmetric theory. We believe that Mandelstam's idea can be expressed more clearly, and with some improvements, by means of the BLN superfields. Mass terms are introduced in Sec. IV where, also, they are expressed in superfield notation together with the light-cone gauge interactions which they generate. Scalar mass terms and trilinear interactions are introduced at this stage. It is shown by treating the mass terms and mass-dependent interactions as insertions that the ultraviolet convergence of Green's functions is preserved. There are seven masses in the theory (four for fermions and three for complex scalars) with one relation [sum of scalar masses equals sum of fermion masses, formula (4.7)] among them. Finally, it is observed that the potential for the scalar fields in this theory now has a stable minimum. This means that the infrared problem is reduced to the effects of the (unbroken) gauge symmetry—a phenomenon familiar in QCD.

II. LIGHT-CONE GAUGE FORMULATION OF $N=4$ SUPER YANG-MILLS THEORY

The Lagrangian, in four-dimensional space-time, for the unbroken $N=4$ theory,[1] can be expressed in the form

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}{}^2-\bar{\psi}^{\alpha}i\nabla\!\!\!/\,\psi_\alpha+\tfrac{1}{4}\nabla_\mu\bar{H}^{\alpha\beta}\nabla_\mu H_{\alpha\beta}-\frac{g}{\sqrt{2}}(\bar{H}^{\alpha\beta}\cdot\psi_\alpha^T\times C^{-1}\psi_\beta+\text{H.c.})-\frac{g^2}{16}\bar{H}^{\alpha\beta}\times\bar{H}^{\gamma\delta}\cdot H_{\alpha\beta}\times H_{\gamma\delta}\ , \tag{2.1}$$

where the Yang-Mills vector A_μ, the chiral spinors ψ_α, and the scalars $H_{\alpha\beta}$ are all in the adjoint representation of the gauge group. Gauge-covariant derivatives are defined such that

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+gA_\mu\times A_\nu\ ,$$
$$\nabla_\mu\psi_\alpha=\partial_\mu\psi_\alpha+gA_\mu\times\psi_\alpha\ ,$$
$$\nabla_\mu H_{\alpha\beta}=\partial_\mu H_{\alpha\beta}+gA_\mu\times H_{\alpha\beta}\ .$$

The indices α and β, take the values $1,\ldots,4$ corresponding to a global SU(4) symmetry with respect to which A_μ is a singlet, ψ_α a quartet, and $H_{\alpha\beta}$ a real sextet,

$$H_{\alpha\beta}=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\bar{H}^{\gamma\delta}\ , \tag{2.2}$$

where $\bar{H}^{\gamma\delta}=(H_{\gamma\delta})^*$. The fields A, ψ, and H span a single multiplet of the $N=4$ extended supersymmetry, and the Lagrangian (2.1) involves one parameter g which is dimensionless.

The theory appears to be ultraviolet finite in the sense that all gauge-invariant quantities computed from it are ultraviolet convergent. Mandelstam[4] has given an argument whereby this convergence extends to off-shell Green's functions in a particular type of gauge, the light-cone gauge. We now review some of the features of this gauge.

With the coordinates

$$x^+=\frac{1}{\sqrt{2}}(x^0+x^3),\quad x^-=\frac{1}{\sqrt{2}}(x^0-x^3)\ ,$$
$$v=\frac{1}{\sqrt{2}}(x^1+ix^2),\quad \bar{v}=\frac{1}{\sqrt{2}}(x^1-ix^2) \tag{2.3}$$

one can associate the covariant components

$$A_+=\frac{1}{\sqrt{2}}(A_0+A_3),\quad A_-=\frac{1}{\sqrt{2}}(A_0-A_3)\ ,$$
$$A_v=\frac{1}{\sqrt{2}}(A_1-iA_2),\quad A_{\bar{v}}=\frac{1}{\sqrt{2}}(A_1+iA_2)\ , \tag{2.4}$$

and similarly for the components of the gradient operator ∂_μ. In these coordinates the line element takes the form

$$dx^\mu dx_\mu=2dx^+dx^- -2\,dv\,d\bar{v}\ . \tag{2.5}$$

The light-cone gauge is defined by the condition $A_0=A_3$, or

$$A_-=0\ . \tag{2.6}$$

We shall regard x^+ as the "time" coordinate so that those equations of motion which do not involve ∂_+ will be treated as *constraints*. It is possible to formally solve such constraints, thereby eliminating "unphysical" components of the system. The first component to eliminate is A_+. One can do this by solving the equation

$$J_-=\nabla_\mu F_{\mu-}$$
$$=\partial_-F_{+-}-\nabla_v F_{\bar{v}-}-\nabla_{\bar{v}}F_{v-}\ ,$$

where, since $A_-=0$, we have $F_{+-}=-\partial_-A_+$. Thus

$$A_+=\frac{1}{\partial_-{}^2}(\nabla_v\partial_-A_{\bar{v}}+\nabla_{\bar{v}}\partial_-A_v-J_-)\ . \tag{2.7}$$

The current J_- can itself be expressed in terms of H and the physical parts of ψ. To separate the unphysical fermionic components, choose a basis which diagonalizes γ_5 and write

$$\psi_\alpha = 2^{1/4}\begin{bmatrix}\zeta_\alpha\\ \chi_\alpha\\ 0\\ 0\end{bmatrix} .$$

One easily finds the constraint part of the Dirac equation and solves it to obtain

$$\zeta_\alpha = \frac{1}{i\partial_-}(i\nabla_v \chi_\alpha + gH_{\alpha\beta}\times\bar{\chi}^\beta) . \tag{2.8}$$

It is then possible to give the current J_- of (2.7) in terms of physical components

$$J_- = -2ig\bar{\chi}^\alpha \times \chi_\alpha - \frac{g}{2}\bar{H}^{\alpha\beta}\times\partial_- H_{\alpha\beta} , \tag{2.9}$$

where $\bar{\chi}^\alpha = (\chi_\alpha)^*$ and $\bar{H}^{\alpha\beta} = (H_{\alpha\beta})^*$.

The unphysical fields A_+ and ζ_α can now be eliminated from the Lagrangian (2.1). The Lagrangian for A_v, χ_α, and $H_{\alpha\beta}$ which results is rather complicated. Its explicit form has been given in the paper of Brink, Lindgren, and Nilsson.[8] We shall not reproduce it here in component form, because these authors have succeeded in developing a superfield version which is much more compact [see (2.18) below] and which we now motivate.

The light-cone gauge condition (2.6) breaks the Lorentz group SO(1,3) to the subgroup SO(1,1)×SO(2), defined by

$$\left.\begin{array}{l} x^\pm \to e^{\pm\alpha}x^\pm, \quad -\infty \le \alpha < \infty \\ v \to e^{i\beta}v \\ \bar{v} \to e^{-i\beta}\bar{v} \end{array}\right\} \quad 0 \le \beta \le 2\pi . \tag{2.10}$$

However, and this is the main point, it is possible to erect an extended supersymmetry on this subgroup. Introduce the anticommuting coordinates θ_α and their complex conjugates $\bar{\theta}^\alpha$, $\alpha = 1,2,\ldots,N$, such that, under SO(1,1)×SO(2),

$$\begin{aligned} \theta_\alpha &\to e^{(\alpha - i\beta)/2}\theta_\alpha , \\ \bar{\theta}^\alpha &\to e^{(\alpha + i\beta)/2}\bar{\theta}^\alpha . \end{aligned} \tag{2.11}$$

Define the supertranslations

$$\begin{aligned} \theta_\alpha &\to \theta_\alpha + \epsilon_\alpha , \\ \bar{\theta}^\alpha &\to \bar{\theta}^\alpha + \bar{\epsilon}^\alpha , \\ x^- &\to x^- + \frac{i}{2}(\bar{\theta}\epsilon - \bar{\epsilon}\theta) + \frac{i}{2}\bar{\epsilon}\epsilon \end{aligned} \tag{2.12}$$

with x^+, v, and $\bar{v}$ invariant. One can define superfields in the usual way and explore the consequences of this supersymmetry. In particular, with the Lagrangian for $N=4$ super Yang-Mills, in the light-cone gauge, expressed in terms of such superfields one can exploit the properties of superfield Feynman rules to show that the Green's functions are finite.

It was shown by Brink *et al.* that *all* the physical components of the $N=4$ theory could be accommodated in one self-conjugate multiplet of the symmetry (2.10)—(2.12). The multiplet is realized by a single "chiral-type" superfield,

$$\phi = \exp(-\tfrac{1}{2}\bar{\theta}\theta i\partial_-)\left[\frac{1}{i\partial_-}A_v(x) + \frac{1}{i\partial_-}\theta_\alpha\bar{\chi}^\alpha(x) + \frac{i}{2}\theta_\alpha\theta_\beta\bar{H}^{\alpha\beta}(x) - \frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta}\theta_\alpha\theta_\beta\theta_\gamma\chi_\delta(x) + \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta i\partial_- A_{\bar{v}}(x)\right] . \tag{2.13}$$

It is chiral in the sense that $\bar{\theta}$ appears only in the combination

$$z = x^- - \frac{i}{2}\bar{\theta}\theta ,$$

which transforms under the action (2.12) according to

$$z \to z - i\bar{\epsilon}\theta .$$

This transformation does not involve $\bar{\theta}$ explicitly.

It is useful to define the differential operator

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{i}{2}\theta_\alpha\partial_- , \tag{2.14}$$

where $\partial/\partial\bar{\theta}^\alpha$ is interpreted as a left derivative. The conjugate operator is

$$\bar{D}^\alpha = \frac{\partial}{\partial\theta_\alpha} - \frac{i}{2}\bar{\theta}^\alpha\partial_- , \tag{2.14'}$$

where $\partial/\partial\theta_\alpha$ is a right derivative. This means

$$\bar{D}^\alpha\bar{F}(x,\theta,\bar{\theta}) = \overline{D_\alpha F}(x,\theta,\bar{\theta}) ,$$

where the bar denotes complex conjugation, together with a reversal of the order of anticommuting quantities. With these definitions one can establish the algebra

$$\{D_\alpha,D_\beta\}=0,\ \ \{\bar{D}^\alpha,\bar{D}^\beta\}=0\ ,$$
$$\{D_\alpha,\bar{D}^\beta\}=-\delta_\alpha^\beta i\partial_-\ . \tag{2.15}$$

The superfield (2.13) is annihilated by D_α,

$$D_\alpha\phi=0\ . \tag{2.16}$$

It also satisfies a reality condition

$$\phi=\frac{1}{\partial_-{}^2}D_1D_2D_3D_4\bar{\phi}\ . \tag{2.17}$$

Both of the properties (2.16) and (2.17) are important and useful in treating the ultraviolet-convergence question.

Finally, it was shown by Brink *et al.* that the light-cone version of the Lagrangian (2.1) can be expressed in the form

$$\mathscr{L}=\mathscr{L}_2+\mathscr{L}_3+\mathscr{L}_4\ ,$$

where

$$\mathscr{L}_2=\int d^4\theta\, d^4\bar{\theta}\tfrac{1}{2}\bar{\phi}\cdot\frac{\Box}{\partial_-{}^2}\phi\ ,$$

$$\mathscr{L}_3=\int d^4\theta\, d^4\bar{\theta}\tfrac{2}{3}g\left[\bar{\phi}\cdot\frac{1}{i\partial_-}(\phi\times\partial_{\bar{v}}\phi)+\text{H.c.}\right]\ ,$$

$$\mathscr{L}_4=\int d^4\theta\, d^4\bar{\theta}\left[-\frac{g^2}{2}\right]\left[\frac{1}{\partial_-}(\phi\times\partial_-\phi)\cdot\frac{1}{\partial_-}(\bar{\phi}\times\partial_-\bar{\phi})+\tfrac{1}{2}\phi\times\bar{\phi}\cdot\phi\times\bar{\phi}\right]\ , \tag{2.18}$$

and the integrals over θ and $\bar{\theta}$ are normalized (our definitions differ in detail from those of Ref. 8) such that

$$\int d\theta_1=0,\ \ \int d\theta_1\theta_1=-1\ ,$$
$$\int d\bar{\theta}^1\bar{\theta}^1=1,\ldots\ .$$

It is not difficult to verify that (2.18) is an invariant with respect to the residual symmetry of the light-cone gauge expressed in formulas (2.10)–(2.12). In particular, we have

$$\phi\to e^{\alpha-i\beta}\phi,\ \ \bar{\phi}\to e^{\alpha+i\beta}\bar{\phi}\ ,$$
$$d^4\theta\to e^{-2\alpha+2i\beta}d^4\theta,\ \ d^4\bar{\theta}\to e^{-2\alpha-2i\beta}d^4\bar{\theta}\ , \tag{2.19}$$
$$\partial_\pm\to e^{\pm\alpha}\partial_\pm,\partial_v\to e^{-i\beta}\partial_v,\ \ \partial_{\bar{v}}\to e^{i\beta}\partial_{\bar{v}}$$

under (2.10) and (2.11).

To compute amplitudes corresponding to the Lagrangian (2.18), one needs the free propagator for the real chiral superfield ϕ. One can show that it is given by

$$\langle T\phi(x_1,\theta_1)\phi(x_2,\theta_2)\rangle=\frac{\hbar}{i}\frac{(D)^4}{\Box}\delta_{12}\ , \tag{2.20}$$

where

$$\delta_{12}=\delta(x_1-x_2)\delta(\theta_1-\theta_2)\delta(\bar{\theta}_1-\bar{\theta}_2)\ .$$

Equivalent forms are

$$\langle T\phi(x_1,\theta_1)\bar{\phi}(x_2,\theta_2)\rangle=\frac{\hbar}{i}\frac{(D_1)^4(\bar{D}_1)^4}{\partial_{1-}{}^2\Box_1}\delta_{12}=\frac{\hbar}{i}\frac{(\bar{D}_2)^4(D_2)^4}{\partial_{2-}{}^2\Box_2}\delta_{12}\ . \tag{2.21}$$

III. OFF-SHELL CONVERGENCE

A demonstration of the ultraviolet convergence of off-shell amplitudes in the unbroken $N=4$ theory is easily arranged. The first point to observe is that the BLN superfield (2.13) has canonical dimension equal to zero. The same is true of the coupling constant g and the superspace measure $d\mu=d^4x\times d^4\theta d^4\bar{\theta}$. This suggests that the necessarily dimensionless coefficient functions which would arise in a functional Taylor expansion of the effective action $\Gamma(\phi)$ should all be logarithmically divergent, at the worst. That is, on writing

$$\Gamma(\phi)=\sum_E\int d\mu_1\cdots d\mu_E\Gamma(\mu_1,\ldots,\mu_E;g)\times\phi(\mu_1)\cdots\phi(\mu_E)\ , \tag{3.1}$$

where $\mu=(x,\theta,\bar{\theta})$, we know that the functions $\Gamma(\mu_1,\ldots,\mu_E;g)$ are dimensionless. They are associated with connected, one-line irreducible supergraphs with E external lines, and it is not difficult to set up Feynman rules for them. If naive power counting can be trusted—and we shall assume that it can—then only logarithmic divergences are to be expected. However, a more detailed examination of the three- and four-point vertices to which the external lines are attached shows that the true situation is better than this. The main point of Mandelstam's argument is that the external-line wave functions $\phi(\mu_j)$ necessarily have at least one derivative, D_α, $\bar{D}^\alpha$, or ∂_μ, acting on each of them. In other words, the dimensionless coefficient functions $\Gamma(\mu_1,\ldots,\mu_E)$ are in fact given as derivatives of functions with *negative* dimensionality and so are superficially convergent. This means that there can be no ultraviolet infinities at any order. The proof depends on the detailed structure of the vertices implied by (2.18). We consider the three- and four-point vertices in turn.

The three-point vertex is associated with the action term

$$S_3=\tfrac{2}{3}g\int d^4x\,d^4\theta\,d^4\bar{\theta}\left[-\frac{1}{i\partial_-}\bar{\phi}\cdot\phi\times\partial_{\bar{v}}\phi+\text{H.c.}\right]$$

$$=\tfrac{2}{3}ig\int d^4x\,d^4\theta\,\partial_-\phi\cdot\phi\times\partial_{\bar{v}}\phi+\text{H.c.} \tag{3.2}$$

The reduction to chiral form is achieved here by means of the reality condition $D_1D_2D_3D_4\bar{\phi}=\partial_-{}^2\phi$. Consider an infinitesimal variation of ϕ and write

$$\delta S_3=\int d^4x\,d^4\theta\,\delta\phi\cdot J_3(x,\theta)+\text{H.c.}\ ,$$

where J_3 is chiral (but not real),

$$J_3=-\tfrac{2}{3}ig[\partial_-(\phi\times\partial_{\bar{v}}\phi)+\partial_-\phi\times\partial_{\bar{v}}\phi+\partial_{\bar{v}}(\partial_-\phi\times\phi)]$$
$$=-ig[\partial_-(\phi\times\partial_{\bar{v}}\phi)+\partial_{\bar{v}}(\partial_-\phi\times\phi)]\ . \tag{3.3}$$

The latter identity results from the antisymmetry of the vector product. Now, an external line which attaches to a three-point vertex couples to the operator J_3. This effective coupling therefore takes the form

$$\int d^4x\,d^4\theta\,\phi^{\text{ext}}(x,\theta)\cdot J_3(x,\theta)=ig\int d^4x\,d^4\theta(\partial_-\phi^{\text{ext}}\cdot\phi\times\partial_{\bar{v}}\phi+\partial_{\bar{v}}\phi^{\text{ext}}\cdot\partial_-\phi\times\phi)\ , \tag{3.4}$$

and one sees that one power of ∂_- or ∂_v is associated with each such external line.

At four-point vertices the situation is more complicated. From the action term

$$S_4=-\frac{g^2}{2}\int d^4x\,d^4\theta\,d^4\bar{\theta}\left[\frac{1}{\partial_-}(\phi\times\partial_-\phi)\cdot\frac{1}{\partial_-}(\bar{\phi}\times\partial_-\bar{\phi})+\tfrac{1}{2}\phi\times\bar{\phi}\cdot\phi\times\bar{\phi}\right] \tag{3.5}$$

one obtains the variation

$$\delta S_4=\int d^4x\,d^4\theta\,d^4\bar{\theta}(\delta\phi\cdot u+\delta\bar{\phi}\cdot\bar{u})\ ,$$

where

$$u=\frac{g^2}{2}\left[\partial_-\phi\times\frac{1}{\partial_-{}^2}(\bar{\phi}\times\partial_-\bar{\phi})+\partial_-\left[\phi\times\frac{1}{\partial_-}(\bar{\phi}\times\partial_-\bar{\phi})\right]-\bar{\phi}\times(\phi\times\bar{\phi})\right]\ . \tag{3.6}$$

However, the variations $\delta\phi$ and $\delta\bar{\phi}$ are not independent. In order to get δS_4 into its most useful form, write

$$\int d^4x\,d^4\theta\,d^4\bar{\theta}\,\delta\bar{\phi}\cdot\bar{u}=\int d^4x\,d^4\theta\,d^4\bar{\theta}\left[\frac{(\bar{D})^4}{\partial_-{}^2}\delta\phi\right]\cdot\bar{u}$$
$$=\int d^4x\,d^4\theta\,d^4\bar{\theta}\,\delta\phi\cdot\frac{(\bar{D})^4}{\partial_-{}^2}\bar{u}$$
$$=\int d^4x\,d^4\theta\,\delta\phi\cdot\frac{(D)^4(\bar{D})^4}{\partial_-{}^2}\bar{u}\ .$$

It follows that the effective coupling of an external line to a four-point vertex can be expressed in the form

$$\int d^4x\,d^4\theta\,\phi^{\text{ext}}(x,\theta)\cdot J_4(x,\theta)+\text{H.c.}\ , \tag{3.7}$$

where the chiral operator J_4 is given by

$$J_4=\frac{(D)^4(\bar{D})^4}{\partial_-{}^2}\bar{u}\ . \tag{3.8}$$

This appears to be singular. In order to show that this singularity is harmless we reverse the order of $(D)^4$ and $(\bar{D})^4$, using the anticommutator $\{D,\bar{D}\}=-i\partial_-$. Schematically,

$$J_4=\frac{(\bar{D})^4(D)^4}{\partial_-{}^2}\bar{u}+\frac{(\bar{D})^3(D)^3}{i\partial_-}\bar{u}+\cdots\ , \tag{3.9}$$

where the terms indicated by dots are regular at $\partial_-=0$. Inserting (3.9) into (3.7) and integrating by parts, with respect to θ, gives

$$\int d^4x\,d^4\theta\,\phi^{\text{ext}}\cdot\left[\frac{(\bar{D})^4(D)^4}{\partial_-{}^2}\bar{u}+\frac{(\bar{D})^3(D)^3}{i\partial_-}\bar{u}+\cdots\right]$$
$$=\int d^4x\,d^4\theta[\phi^{\text{ext}}\cdot(D)^4\bar{u}+D\phi^{\text{ext}}\cdot(D)^3\bar{u}+\cdots]\ . \tag{3.10}$$

We have used the fact that the external-line wave function must satisfy the reality condition

$(\bar{D})^4\phi^{\rm ext}=\partial_-{}^2\bar{\phi}^{\rm ext}$. In (3.10) the second term exhibits one D acting on the external wave function. Successive terms, not shown explicitly, would have more powers of D (or ∂_-). Only the first term is potentially dangerous in that it might lead to logarithmic divergences. To show that this does not happen, we must examine the structure of $(D)^4\bar{u}$. From (3.6),

$$\begin{aligned}\frac{2}{g^2}(D)^4\bar{u}&=(D)^4\left[\partial_-\bar{\phi}\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)+\partial_-\left[\bar{\phi}\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)\right]-\phi\times(\bar{\phi}\times\phi)\right]\\&=\partial_-{}^3\phi\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)+\partial_-\left[\partial_-{}^2\phi\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)\right]-\phi\times(\partial_-{}^2\phi\times\phi)\\&=\partial_-\left[2\partial_-{}^2\phi\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)\right]-\partial_-{}^2\phi\times\frac{1}{\partial_-}(\phi\times\partial_-\phi)-\phi\times(\partial_-{}^2\phi\times\phi)\\&=\partial_-\left[2\partial_-{}^2\phi\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)-\partial_-\phi\times\frac{1}{\partial_-}(\phi\times\partial_-\phi)\right]+\partial_-\phi\times(\phi\times\partial_-\phi)-\phi\times(\partial_-{}^2\phi\times\phi)\\&=\partial_-\left[2\partial_-{}^2\phi\times\frac{1}{\partial_-{}^2}(\phi\times\partial_-\phi)-\partial_-\phi\times\frac{1}{\partial_-}(\phi\times\partial_-\phi)+\phi\times(\phi\times\partial_-\phi)\right].\end{aligned}$$

Hence one power of ∂_- comes out to act on $\bar{\phi}^{\rm ext}$ in (3.10). The claim is therefore proved: Every external line is associated with at least one derivative, ∂_μ or D. All vertex functions are superficially convergent and the theory must be ultraviolet finite.

IV. MASS TERMS

Our purpose is to break the extended supersymmetry by introducing masses for the spinor and scalar components. Of these the more problematical are the spinor masses because, in the light-cone gauge, they induce new interaction terms. (It will turn out that new scalar interactions are implied as well, but their form will be fixed by the finiteness requirement.) In order to discover the new fermionic interactions it is necessary only to introduce the mass term in Lorentz-invariant component form, and then to eliminate unphysical components as in Sec. II. The new interactions can then be seen as necessary for on-shell Lorentz invariance.

To the Lagrangian (2.1) we add the mass term

$$-\tfrac{1}{2}M^{\alpha\beta}\psi_\alpha^T\cdot C^{-1}\psi_\beta+\text{H.c.}\,, \tag{4.1}$$

where $M^{\alpha\beta}$ is a symmetric matrix. It then follows that the equation of constraint (2.8) must be replaced by

$$\zeta_\alpha=\frac{1}{i\partial_-}\left[i\nabla_v\chi_\alpha-\frac{1}{\sqrt{2}}\bar{M}_{\alpha\beta}\bar{\chi}^\beta+gH_{\alpha\beta}\times\bar{\chi}^\beta\right]. \tag{4.2}$$

When the unphysical components ζ_α are eliminated from (2.1) the following mass-dependent terms appear:

$$M^{\alpha\gamma}\bar{M}_{\gamma\beta}\bar{\chi}^\beta\cdot\frac{1}{i\partial_-}\chi_\alpha+\left[\sqrt{2}g\left[\frac{1}{i\partial_-}\chi_\alpha\right]\cdot M^{\alpha\beta}\left[H_{\beta\gamma}\times\bar{\chi}^\gamma+iA_v\times\chi_B\right]+\text{H.c.}\right]. \tag{4.3}$$

To show that these terms are compatible with ultraviolet convergence, it is natural to look for a superfield expression of them so that the BLN formalism, with these additions, can be used as in Sec. III. Our strategy is then to examine the component structure of some terms which are bilinear and trilinear in ϕ and which contain explicit θ dependence. These terms, which explicitly break the $N=4$ supersymmetry [and the global SU(4)] will be chosen so as to include fermionic pieces like (4.3). In addition, we must ensure that no vector mass terms are implied and that the scalar masses and interactions are compatible with a stable, non-negative potential. If the form (4.3) is respected, and if there is no vector mass term, then we can be sure that we are dealing with a Lorentz-invariant and gauge-invariant theory. The finiteness question requires the use of light-cone gauges but, once this has been settled, there should be no obstacle to using any other kind of gauge.

Consider the bilinear term

$$\int d^4x\, d^4\theta\, d^4\bar\theta\, \bar\theta^1\theta_1\bar\theta^2\theta_2 D_1\bar\phi\cdot\frac{1}{i\partial_-}\bar D^1\phi=\int d^4x\left[\bar\chi^1\cdot\frac{1}{i\partial_-}\chi_1+\bar\chi_2\cdot\frac{1}{i\partial_-}\chi_2-\bar H^{31}\cdot H_{31}-\bar H^{41}\cdot H_{41}\right]. \tag{4.4}$$

Notice that it makes no contribution to the vector mass and that it is symmetric under the interchange $1\leftrightarrow 2$ (the scalar components are of course self-dual, $H_{41}=\bar H^{32}$, $H_{31}=-\bar H^{42}$, etc.). This suggests that a suitable candidate for the mass terms would be

$$S_{M2}=\int d^4x\, d^4\theta\, d^4\bar\theta \sum_{\alpha,\beta}\tfrac{1}{2}a_{\alpha\beta}\bar\theta^\alpha\theta_\alpha\bar\theta^\beta\theta_\beta D_\beta\bar\phi\cdot\frac{1}{i\partial_-}\bar D^\beta\phi\,, \tag{4.5}$$

where $a_{\alpha\beta}$ is a symmetric matrix with vanishing diagonal elements. It yields the following masses:

$$\begin{aligned}
&M^2(A)=0\,,\\
&M^2(\chi_1)=a_{12}+a_{13}+a_{14}\,,\\
&M^2(\chi_2)=a_{21}+a_{23}+a_{24}\,,\\
&M^3(\chi_3)=a_{31}+a_{32}+a_{34}\,,\\
&M^2(\chi_4)=a_{41}+a_{42}+a_{43}\,,\\
&M^2(H_{14})=a_{12}+a_{13}+a_{42}+a_{43}\,,\\
&M^2(H_{24})=a_{23}+a_{21}+a_{43}+a_{41}\,,\\
&M^2(H_{34})=a_{31}+a_{32}+a_{41}+a_{42}\,.
\end{aligned} \tag{4.6}$$

The six parameters $a_{\alpha\beta}$ are independent. Among the eight masses we have, in addition to $M^2(A)=0$, the relation[9]

$$M^2(\chi_1)+M^2(\chi_2)+M^2(\chi_3)+M^2(\chi_4)=M^2(H_{14})+M^2(H_{24})+M^2(H_{34})\,. \tag{4.7}$$

More general mass terms could be imagined, but we have not pursued the matter further.

To obtain the interaction terms of (4.3) consider the trilinear expression

$$\tfrac{1}{3}\int d^4x\, d^4\theta\, d^4\bar\theta\, \theta_4 D_4\bar\phi\cdot\phi\times\frac{1}{\partial_-}\bar\phi=-\int d^4x\left[\left[\frac{1}{i\partial_-}\chi_4\right]\cdot(H_{4a}\times\bar\chi^a+iA_v\times\chi_4)-H_{14}\cdot H_{24}\times H_{34}\right]. \tag{4.8}$$

Since our fermion mass matrix is diagonal, according to (4.6), we can take

$$M^{11}=M_1(a)=(a_{12}+a_{13}+a_{14})^{1/2}\,, \tag{4.9}$$

etc. It then becomes clear that the interaction part of (4.3) is contained in the trilinear term

$$S_{M3}=-\frac{\sqrt2}{3}g\int d^4x\, d^4\theta\, d^4\bar\theta\sum_\alpha M_\alpha(a)\theta_\alpha D_\alpha\bar\phi\cdot\phi\times\frac{1}{\partial_-}\bar\phi+\text{H.c.}\,, \tag{4.10}$$

which must be added, together with (4.5), to the BLN action (2.18). Note that (4.10) includes interactions among the scalar fields as shown in (4.8).

To summarize, the total action is given by

$$\begin{aligned}
S=\int d^4x\, d^4\theta\, d^4\bar\theta\Bigg[&\tfrac{1}{2}\bar\phi\cdot\frac{\Box}{\partial_-{}^2}\phi+\tfrac{2}{3}g\left[\bar\phi\cdot\frac{1}{i\partial_-}(\phi\times\partial_{\bar v}\phi)+\text{H.c.}\right]\\
&-\frac{g^2}{2}\left[\frac{1}{\partial_-}(\phi\times\partial_-\phi)\cdot\frac{1}{\partial_-}(\bar\phi\times\partial_-\bar\phi)+\tfrac{1}{2}\phi\times\bar\phi\cdot\phi\times\bar\phi\right]\\
&+\sum_{\alpha\beta}\tfrac{1}{2}a_{\alpha\beta}\bar\theta^\alpha\theta_\alpha\bar\theta^\beta\theta_\beta D_\beta\bar\phi\cdot\frac{1}{i\partial_-}\bar D^\beta\phi-\left[\sum_\alpha\frac{\sqrt2}{3}gM_\alpha(a)\theta_\alpha D_\alpha\bar\phi\cdot\phi\frac{1}{\partial_-}\bar\phi+\text{H.c.}\right]\Bigg]\,,
\end{aligned} \tag{4.11}$$

where $a_{\alpha\beta}$ is a real, symmetric matrix and $M_\alpha(a)$ is given by (4.9). The expression (4.11) is the form taken in the light-cone gauge by the manifestly gauge-invariant and Lorentz-invariant action

$$\int d^4x \left[-\tfrac{1}{4}F_{\mu\nu}{}^2 - \bar{\psi}^\alpha \cdot i\nabla\!\!\!/\,\psi_\alpha + \tfrac{1}{4}\nabla_\mu \bar{H}^{\alpha\beta}\cdot\nabla_\mu H_{\alpha\beta} - \frac{g}{\sqrt{2}}(\bar{H}^{\alpha\beta}\cdot\psi_\alpha^T \times C^{-1}\psi_\beta + \text{H.c.}) - \tfrac{1}{2}\sum_\alpha M_\alpha(a)(\psi_\alpha^T\cdot C^{-1}\psi_\alpha + \text{H.c.}) - V(h) \right], \tag{4.12}$$

where $V(H)$ is given by

$$V(H) = M^2(H_{14})\,|\,H_{14}\,|^2 + M^2(H_{24})\,|\,H_{24}\,|^2 + M^2(H_{34})\,|\,H_{34}\,|^2 - \frac{\sqrt{2}}{3!}g\sum_\alpha M_\alpha(a)\epsilon^{\alpha\beta\gamma\delta}H_{\beta\alpha}\cdot H_{\gamma\alpha}H_{\delta\alpha} + \text{H.c.} + \frac{g^2}{16}\bar{H}^{\alpha\beta}\times\bar{H}^{\gamma\delta}\cdot H_{\alpha\beta}\times H_{\gamma\delta} . \tag{4.13}$$

It is not immediately clear that the potential (4.13) is an acceptable one. In view of the well-known property of the original symmetric theory, that its (purely quartic) potential vanishes in certain directions (in the space of $H_{\alpha\beta}$), one might suspect that the presence of cubic terms in (4.13) would cause it to be unbounded below. To settle this doubt it is useful to express V in terms of three independent complex scalars,

$$\phi_1 = H_{14},\quad \phi_2 = H_{24},\quad \phi_3 = H_{34} . \tag{4.14}$$

The potential now reads

$$\begin{aligned} V = {} & M_{14}{}^2\,|\,\phi_1\,|^2 + M_{24}{}^2\,|\,\phi_2\,|^2 + M_{34}{}^2\,|\,\phi_3\,|^2 \\ & + \sqrt{2}g(M_1\phi_1^*\cdot\phi_2\times\phi_3 + M_2\phi_1\cdot\phi_2^*\times\phi_3 + M_3\phi_1\cdot\phi_2 \times\phi_3^* + M_4\phi_1^*\cdot\phi_2^*\times\phi_3^* + \text{H.c.}) \\ & + g^2\sum_{i,j}(\phi_i\times\phi_j\cdot\phi_i^*\times\phi_j^* + \phi_i\times\phi_j^*\cdot\phi_i^*\times\phi_j) . \end{aligned} \tag{4.15}$$

The quartic term vanishes if and only if

$$\phi_i\times\phi_j = 0,\quad \phi_i\times\phi_j^* = 0,\quad i,j = 1,2,3 ; \tag{4.16}$$

otherwise it is positive. However, if (4.16) is satisfied, it is clear that the cubic terms in (4.15) also vanish so that the potential reduces to its quadratic part in these directions. Stability is therefore ensured if the three masses $M_{14}{}^2$, $M_{24}{}^2$, and $M_{34}{}^2$ are positive. This would seem to rule out the possibility of spontaneously breaking the gauge symmetry, but radiative corrections could alter the picture.[10]

Finally, we come to the question of finiteness. As in Sec. III, we use the dimensional argument. Integrals which are ultraviolet convergent by virtue of the structure of vertices to which external lines are attached will clearly remain convergent when masses are introduced into the propagators. However, there are new mass-dependent vertices to be considered. These are characterized by the last term in (4.11)

$$-\frac{\sqrt{2}}{3}g\int d^4x\, d^4\theta\, d^4\bar{\theta}\sum_\alpha M_\alpha(a)\theta_\alpha D_\alpha\bar{\phi}\cdot\phi\times\frac{1}{\partial_-}\bar{\phi} . \tag{4.17}$$

With this interaction it can happen that $1/\partial_-$ acts on an external-line wave function, thereby raising the superficial degree of divergence of the associated graph. Such a potentially dangerous contribution can be rearranged as follows:

$$-\frac{\sqrt{2}}{3}g\int d^4x\, d^4\theta\, d^4\bar{\theta}\sum_\alpha M_\alpha(a)\theta_\alpha D_\alpha\bar{\phi}\cdot\phi\times\frac{1}{\partial_-}\bar{\phi}^{\text{ext}} = \frac{\sqrt{2}}{3}g\int d^4x\, d^4\theta\, d^4\bar{\theta}\sum_\alpha\left[\frac{M_\alpha(a)D_\alpha}{\partial_-}\bar{\phi}^{\text{ext}}\right]\cdot\theta_\alpha\bar{\phi}\times\phi \tag{4.18}$$

on integrating by parts with respect to $\bar{\theta}^\alpha$. Now the operator $M_\alpha D_\alpha/\partial_-$ which acts on the external wave function has dimension $+\frac{1}{2}$. This means that the vertex is not dangerous: The convergence arguments of Sec. III therefore apply also to the broken theory. Off-shell amplitudes are ultraviolet convergent in the light-cone gauge.

The Lagrangian (4.11) has no unbroken supersymmetry: All four θ's appear explicitly. It is evident though that $N=1$ supersymmetry can be recovered by removing some of the mass-dependent terms. Thus, for example, the explicit dependence on θ_1

and $\bar{\theta}^1$ is removed by taking

$$a_{12}=a_{13}=a_{14}=0 .$$

There results an $N=1$ supersymmetric[11] theory with the following multiplets:

$$(A,\chi_1),\ \ M^2=0 ,$$

$$(\chi_2,H_{34}),\ \ M^2=a_{23}+a_{24} ,$$

$$(\chi_3,H_{24}),\ \ M^2=a_{32}+a_{34} ,$$

$$(\chi_4,H_{14}),\ \ M^2=a_{42}+a_{43} .$$

By eliminating θ_2 as well, i.e., taking

$$a_{23}=a_{24}=0 ,$$

one obtains $N=2$ theory with the multiplets

$$(A,\chi_1,\chi_2,H_{34}),\ \ M^2=0 ,$$

$$(\chi_3,\chi_4,H_{14},H_{24}),\ \ M^2=a_{34} .$$

V. DISCUSSION

We conclude that, barring technical difficulties with the light-cone gauge, the explicitly broken $N=4$ super Yang-Mills theory is ultraviolet finite. The technical questions are not trivial, of course. The presence of a nonlocal operator $1/\partial_-$ could perhaps give rise to unacceptable singularities in physical amplitudes. We have nothing to say about this. It can also be argued that operator ordering problems might render higher-loop calculations ambiguous. But if we suppose that all such problems can be overcome, then we appear to have a nontrivial four-dimensional theory which is free of ultraviolet singularities.

The infrared behavior, though simplified by the introduction of mass terms, is not trivial. Since our potential has a stable minimum which preserves the local symmetry, the theory has massless non-Abelian vectors, like QCD. Perhaps the usual confining mechanism would operate here as well. However, it must be emphasized that we do not have a "running" coupling constant, in the usual sense. The bare coupling constant of this theory should be thought of as an observable quantity since one must be able to express physical amplitudes in terms of it, in a cutoff-independent way. (It may be worth pointing out that the particular type of regularization employed in any calculation is not important now since there will never be a need to separate divergent parts and cancel them with counterterms.) Of course it is possible, and perhaps desirable, to replace the bare coupling constant with some "effective" parameter which relates more directly to observable quantities. The magnitude of such an effective parameter would of course depend on the energy scale E, used in defining it. However, it would presumably not vary as $(\ln E)^{-1/2}$.

Can such a theory be fundamental? At present there are too many arbitrary parameters, and we have no prescription for choosing them. It is tempting to speculate that the mass terms are on the Planck scale and that an appropriate choice of them would cause this theory to generate massless helicity-2 bound states, the graviton, and that the Einstein theory (or even a composite supergravity theory) will emerge as an effective field theory of the low-energy sector. (For example, one might hope that calculations of the Adler-Zee[12,13] type could be carried out here. That is, it might be possible to compute G_N unambiguously in terms of M.)

To conclude, the $N=4$ extended supersymmetry theory softly broken by the addition of mass-containing terms now appears even more of a remarkable mathematical construction. This is perhaps the only known ultraviolet-finite field theory with inbuilt mass scales. To explore its implications is a challenge that should not be overlooked.

After this paper had been prepared for publication we received a report by Brink, Lindgren, and Nilsson[14] which gives an alternative proof of finiteness, using their formalism, when no masses are included.

Note added. We have futher examined the question of spontaneous internal-symmetry breaking referred to at the end of Sec. IV.

We find that the potential (4.15) does have a nontrivial minimum at which local symmetries are broken. To see this in simple terms we take the six parameters $a_{\alpha\beta}$ to be equal so that

$$\tfrac{1}{3}{M_1}^2=\tfrac{1}{3}{M_2}^2=\tfrac{1}{3}{M_3}^2=\tfrac{1}{3}{M_4}^2=\tfrac{1}{4}{M_{14}}^2=\tfrac{1}{4}{M_{24}}^2=\tfrac{1}{4}{M_{34}}^2=a .$$

For the case of gauge SU(2) it is then easy to show that the point

$$\phi_1=\begin{bmatrix}u\\0\\0\end{bmatrix},\ \ \phi_2=\begin{bmatrix}0\\u\\0\end{bmatrix},\ \ \phi_3=\begin{bmatrix}0\\0\\u\end{bmatrix}$$

with $u=-(\sqrt{3}+1)\sqrt{a}/g$, is a local minimum.

Since the three vectors are mutually orthogonal, it is clear that the gauge symmetry is completely broken. It would therefore seem that this model has no massless states and is thus both infrared and ultraviolet finite.

With larger gauge symmetries the range of possible symmetry-breaking patterns is larger and needs to be examined.

Another simple case can be analyzed without difficulty. Taking $a_{12}=a_{13}=a_{14}=0$ so that $N=1$ supersymmetry is preserved, the potential reduces to the form

$$V=|M_{14}\phi_1^*+\sqrt{2}g\phi_2\times\phi_3|^2+|M_{24}\phi_2+\sqrt{2}g\phi_3\times\phi_1^*|^2$$
$$+|M_{34}\phi_3+\sqrt{2}g\phi_1^*\times\phi_2|^2+\frac{g^2}{2}|\phi_1^*\times\phi_1+\phi_2^*\times\phi_2+\phi_3^*\times\phi_3|^2 .$$

This potential has an absolute minimum ($V=0$) at

$$\phi_1=\begin{bmatrix}u_1\\0\\0\end{bmatrix},\quad \phi_2=\begin{bmatrix}0\\u_2\\0\end{bmatrix},\quad \phi_3=\begin{bmatrix}0\\0\\u_3\end{bmatrix},$$

where

$$u_1=-\frac{(M_{24}M_{34})^{1/2}}{g},\; u_2=-\frac{(M_{34}M_{14})^{1/2}}{g},$$
$$u_3=-\frac{(M_{14}M_{24})^{1/2}}{g}.$$

The local SU(2) is again completely broken and the spectrum is free of massless states.

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