Exact solution of the infrared problem

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A simple but rigorous solution of the infrared problem is obtained. The basis of this solution is a factorization of the operator corresponding to the Feynman coordinate space plus all electromagnetic corrections to it into a product of two operators. The first is a unitary operator that represents precisely the contribution corresponding to classical electromagnetic theory. The second is a residual operator that is free of infrared problems. This factorization is exact: no soft-photon approximation, or any other approximation, is used. Both the unitary operator and the residual operator are expressed in simple forms amenable to rigorous mathematical analysis. The central technical result of this work, namely the exact yet simple organization of all contributions corresponding to classical physics into unitary factors, may have other important uses.

I. INTRODUCTION

The well-known "infrared catastrophe" in quantum field theory consists of the following fact: the electromagnetic corrections to the S matrix are represented by integrals whose contributions from very soft photons often diverge. A way around this difficulty was indicated by Block and Nordsieck,[1] who showed, in some simple cases, that these infrared-divergent contributions cancel out of the expressions for the observable probabilities, provided the nonobservability of very soft photons is taken into account. The Block-Nordsieck observation has been generalized in a series of works that have culminated in the central work in this field, the paper of Yennie, Frautschi, and Suura[2] (YFS). These authors gave lengthy arguments to support their contention that all of the infrared-divergent contributions to the S matrix can be collected into exponential factors that cancel out of the expressions for observable probabilities. However, at the end of a technical appendix to their paper YFS listed some of the difficulties with their arguments, and concluded that a rigorous proof of their conjecture would probably be prohibitively complicated. The difficulties with the YFS arguments are particularly serious when the S matrix is evaluated at a singularity.

The YFS infrared separation was used by Chung[3] to define an infrared-finite S matrix: infrared finiteness was (presumably) achieved by incorporating the YFS infrared factor into coherent initial or final states. This infrared-finite S matrix was examined by Storrow,[4] Kibble,[5] and Zwanziger,[6] who found that the pole singularity normally associated with a charged stable particle was converted by the effects of soft photons to a nonpole form.

Such a change in the character of S-matrix singularities could be as catastrophic as the infrared divergence itself, for the character of singularities in momentum space determines asymptotic behavior in coordinate space.[7,8] In particular, the pole singularity normally associated with stable particles is the unique momentum-space singularity that gives the inverse-cube-law falloff in spacetime that physically characterizes stable particles. Consequently, any modification of the pole character of singularities associated with charged particles would jeopardize the ability of the theory to accommodate stable charged particles. This problem is the electrodynamic analog of the chromodynamic problem of confinement.

This apparent disruption of the stability of charged particles has serious consequences. It causes the apparent breakdown[5,6] of the usual reduction formulas, which arise directly from the factorization property of the pole singularities normally associated with stable particles. Morever, it upsets the connection between relativistic quantum theory and the experimentally measured quantities. For the basis of this connection is, again, the factorization property of the pole singularities normally associated with stable particles.

The difficulties arise from a breakdown of the YFS arguments at singularities. One important YFS assumption is that $(e^{ikx}-1)$ is of order k. For finite x this is true. But singularities are controlled by asymptotic limits in which x has passed to infinity. Thus the assumption is not valid at singularities.

The purpose of this paper is to show how the infrared problem can be solved exactly, with all terms retained and compactly represented, by making essential use of coordinate space. That coordinate space should be needed is not surprising. It was recognized from the outset[1] that the infrared problem is essentially that of separating out the contributions corresponding to an appropriate classical electromagnetic radiation field. But classical fields are described in coordinate space, and so are their sources. Moreover, by saying in coordinate space one avoids integrations over the asymptotic spacetime regions that are, from the coordinate-space point of view, the source of the infrared problem.

Examination of Storrow's calculation reveals clearly the specific difficulty with the momentum-space approach. To represent an appropriate classical contribution Storrow, following Chung, introduces a coherent state that corresponds (for small k) to the classical electromagnetic field radiated by a classical charged particle whose initial and final velocities correspond to the momenta of the initial and final charge-particle states of the scattering matrix. In momentum space no particular coordinate point is favored. Thus the point of intersection of the initial and final classical trajectories is placed arbitrarily at the origin: certain factors e^{ikx} are replaced by unity. This replacement is perhaps justifiable in certain situations, but certainly not on the singularity surface $p^2=m^2$ if the nature of the singularity on this surface is the point at issue. For this momentum-space singularity arises, via Fourier transformation, from those asymptotic coordinate-space regions that correspond to the physical possibility of a one-charged-particle-exchange process in spacetime. It is the asymptotic rate of falloff of this spacetime process that determines the nature of the singularity.[7,8] The electromagnetic field radiated by this classical one-charge-particle-exchange process has two parts: it consists of the (bremsstrahlung) radiation associated with the two separate deflections of the charged particle. These two deflections occur in two different spacetime regions. The overlap between this physically relevant radiation field and the one used by Storrow, in which both source regions are placed together at the origin, vanishes in the asymptotic limit that determines the nature of the singularity.

To deal adequately with this situation it is necessary to represent the sources of the electromagnetic field in coordinate space. Then one can introduce those classical fields, and the corresponding coherent states, whose spacetime source regions are at—or at least near—the spacetime points where the charged-particle deflections occur. The radiation field must be tied in this way to the locations of the particle deflections if one wishes to calculate the physical rate of falloff.

These considerations physically motivate the use of coordinate space. But what allows the problem to be neatly solved is the exceedingly simple way in which the classical and nonclassical contributions separate in coordinate space.

Consider first a process involving no external charged particles. Let D be a Feynman diagram involving n neutral external particles, no initial or final charged particles, no photons, and at least one external particle incident on every vertex. Let $F^D(x)=F^D(x_1,\ldots,x_n)$ be the Feynman coordinate-space function corresponding to D. Let $L(x)$ be the spacetime polygon(s) formed from the charged lines of D. The vertices of $L(x)$ are placed at the points specified by x. Let $\hat{F}^D_{\text{op}}(x)$ be the (x-dependent) operator in photon space that represents the sum of $F^D(x)$ plus all corrections to it represented by diagrams D' consisting of D plus any number of photon lines, each connected at one or both ends somewhere into the set of charged lines of D. Then one principal result is that $\hat{F}^D_{\text{op}}(x)$ can be expressed as follows:

$$\hat{F}^D_{\text{op}}(x)=U(L(x))\tilde{F}^D_{\text{opr}}(x)\ . \tag{1.1}$$

Here $U(L(x))$ is a simple well-defined unitary operator in the space of photons. Acting on the photon vacuum it creates the coherent state that corresponds to the classical electromagnetic field radiated by the charged particles moving (in the manner of Feynman) around the spacetime polygon(s) $L(x)$.

The operator $\tilde{F}^D_{\text{opr}}(x)$ is a residual operator that is free of infrared problems. It is a sum of terms corresponding to D and the various diagrams D'. Each term can be transformed into momentum space with no infrared divergence.

The basic formula (1.1) is obtained by separating each photon interaction $(-ie\gamma_\mu)$ into its "classical" and "quantum" parts by means of formula (2.3) of Sec. II. The unitary operator $U(L(x))$ represents the contributions of all "classical" photons. These are the photons that couple into the lines of D only via classical couplings. The remaining photons are called quantum photons. They have a quantum coupling into a charged line of D on at least one end. Their contributions, together with the original function $F^D(x)$, give $\tilde{F}^D_{\text{opr}}(x)$.

Taking the photon momentum-space matrix elements and performing the Fourier transform $x\to q$ one obtains the momentum-space function $\langle k'\,|\,\tilde{F}^D_{\text{opr}}(q)\,|\,k''\rangle$. In momentum space the quantum coupling takes a very simple form. To exhibit this form let

$$G_\mu(p,k)=\frac{\not{p}+m}{p^2-m^2+i0}(-ie\gamma_\mu)\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i0}\ . \tag{1.2}$$

This function represents part of the original Feynman momentum-space function. Replacement of the original coupling $(-ie\gamma_\mu)$ by the quantum coupling replaces this function $G_\mu(p,k)$ by

$$G_\mu(p,k)-\int_0^1 G_\mu(p+\alpha k,0)d\alpha\ . \tag{1.3}$$

This has one more power of k than $G_\mu(p,k)$. This extra power of k eliminates the infrared divergences.[9]

The plan of the paper is as follows. The basic formula (1.1) is derived in Sec. II. It is a simple consequence of the Ward identity. Some general features of this formula are described in Sec. III. The main point is that the connection to physics involves transition amplitudes, and these are expressed by folding the coordinate-space function $\hat{F}_{\text{opr}}(x)$ directly into the coordinate-space wave functions of the external particles of D. Thus one never introduces the Fourier transform of the function $\hat{F}_{\text{op}}(x)$. The operator $U(L(x))$ is given a simple, closed form in coordinate space, and is not tranformed to momentum space. The function $\tilde{F}^D_{\text{opr}}$, on the other hand, can be computed in momentum space, and then transformed into coordinate space.

The contribution to $\tilde{F}^D_{\text{opr}}$ that arise from diagrams $D'\neq D$ are discussed in Ref. 9. This paper deals mainly with $U(L(x))F^D(x)$. It is concerned with the contributions of the classical photons, which are the ones associated with the infrared divergences.

In Sec. IV the simple closed-loop triangle diagram D of Fig. 1 is considered. It is shown that when the function $U(L(x))F^D(x)$ is folded into the external-particle wave functions, in order to obtain physical scattering amplitudes, the charged-particle loops are effectively confined to finite spacetime regions, and that, consequently, there are no infrared divergences in these closed-loop amplitudes. This provides a rigorous starting point: these closed-loop amplitudes are finite and well defined without infrared cutoff or fictitious photon mass.

In Sec. V the coordinate-space procedure for obtaining amplitudes with charged initial and final lines is discussed in general terms. The procedure starts with processes in which all charged particles are confined to closed loops. Then the wave packets of the external particles are shifted to infinity in a way such that certain partial processes are shifted to infinity. If the photons were not massless then the dominant asymptotic form in this limit would factorize into a product of separate factors. These factors can be identified as the scattering amplitudes for the separate subprocesses, once appropriate geometric falloff factors are extracted. The program here is to show, with the aid of the basic formula, that this factorization result continues to hold also in the presence of interactions to all orders with massless photons, and that the geometric falloff factors are exactly the same as for the case with no massless particles. This type of falloff corresponds to pole singularities, and to the fact that the charged particles propagate over macroscopic distances like stable particles. What must be shown, then, is that the dominant asymptotic term has exactly this factorized form, with the precise rate of falloff that corresponds to stable charged particles, and that the residual factors are finite. These residual factors define the scattering amplitudes for processes with charged-particle external lines.

Section VI describes the mathematical details of the canonical connection between the notion of a stable physical particle, as characterized by macroscopic spacetime behavior, and the pole singularity $(p^2-m^2+i0)^{-1}$. This connection has been mentioned repeatedly in the Introduction, and is basic to the present work.

The main results are in Sec. VII. The aim is to show that the spacetime behavior that is normally associated with the pole singularity, and that characterizes stable physical particles, is not disrupted by the classical photons and that, consequently, the amplitudes associated with processes involving charged initial and final particles can be extracted from the asymptotic limits of amplitudes fo processes in which all charged particles are confined to closed loops. Specifically, one begins with a transition amplitude $A(X)=A(X_1,X_2,X_3)$ associated with diagram D of Fig. 1, in which the charged particle is confined to a closed loop. The coordinate-space wave functions of the external particles effectively confine the three vertices at x_1, x_2, and x_3 to finite neighborhoods of X_1, X_2 and X_3. A scaling $X_i \rightarrow \lambda X_i$ is then introduced: the external-particle wave functions are shifted to infinity as $\lambda \rightarrow \infty$. The two external-particle wave functions associated with each individual vertex i are translated together by the amount $(\lambda-1)X_i$.

In the absence of photons the limit

$$\lim_{\lambda\rightarrow\infty} (\lambda^{9/2})A(\lambda X)=cA_1A_2A_3 \ , \tag{1.4}$$

with an appropriate constant c, defines the amplitudes A_1, A_2, and A_3 associated with the three vertices of D.

To show that this limit exists and factorizes also in the presence of the classical photons one may separate $U(L(x))$ into factors $U_\Omega(L(x))$ and $U^\Omega(L(x))$ that act nontrivially on the photon states constructed from photons whose momentum-energy vector k lies either inside or outside a small neighborhood Ω of the point $k=0$, respectively. Then

$$\begin{aligned} U(L(x)) &= U_\Omega(L(x))U^\Omega(L(x)) \\ &= U_\Omega(L(\lambda X))U^\Omega(L(x)) \\ &\quad + U_\Omega(L(\lambda X))[U_\Omega{}^{-1}(L(\lambda X))U_\Omega(L(x)) \\ &\qquad -1]U^\Omega(L(x)) \ . \end{aligned} \tag{1.5}$$

Very soft photons are not detected. Hence for sufficiently small Ω the contribution to the probability from the leading factor $U_\Omega(L(\lambda X))$ in (1.5) occurs in the expression for the probability in the combination $U_\Omega^\dagger(L(\lambda X))U_\Omega(L(\lambda X))=1$. This means that for sufficiently small Ω the contributions to the probability arising from the first term in (1.5) alone has no contribution at all from the classical photons with k in Ω. On the other hand, the effect of the coordinate-space wave functions of the initial and final particles effectively confines $x=(x_1,x_2,x_3)$ to a neighborhood of λX. This has the consequence, proved in Appendix B, that the contributions to the probability involving the second term in (1.5) can be made an arbitrarily small fraction of the contribution from the first term of (1.5), by making Ω sufficiently small. For the norm of

$$[U_\Omega{}^{-1}(L(\lambda X))U_\Omega(L(x))-1]$$

effectively approaches zero. Thus the contributions to the transition probability from the classical photons with k in Ω can be made arbitrarily small by making Ω sufficiently small.

Because the contributions of *quantum* photons with k in Ω becomes vanishing small with Ω, almost the entire contribution to the probability from photons with k in a sufficiently small Ω comes from the single final state $U_\Omega(L(\lambda X))\,|\,\text{vac}\rangle$. This is physically reasonable: this is the coherent state that corresponds to the classical electromagnetic field radiated by a charged particle traveling (in the manner of Feynman) around $L(\lambda X)$. If one wishes to deal with coherent-state amplitudes that give the bulk of the contribution to the probability then one should use this state as the basic coherent state from which the other states are constructed. The infrared finiteness of these amplitudes is assured by essentially the same argument that ensures the infrared finiteness of the probabilities.

The question of factorization must be examined. The factorization of the contributions arising from the factor $\widetilde{F}_{\text{opr}}$ alone is assured by its infrared finiteness. The factorization of the part of $\lim \lambda^{9/2}A(\lambda X)$ arising from the classical-photon factor $U(L(x))$ must be proved.

The factor $U(L(x))$ has the form

$$U(L(x))=\exp[\langle a^*\cdot J(L(x))\rangle]\exp[\langle -J^*(L(x))\cdot a\rangle]\exp[-\tfrac{1}{2}\int iJ^*(L(x),k)\cdot J(L(x),k)d^4k(2\pi)^{-4}(k^2+i0)^{-1}] \ , \tag{1.6}$$

where

$$J_\mu(L(x),k) = -ie\int_{L(x)} dx'_\mu e^{ikx'}$$
$$= -e\sum_{i=1}^{3} \frac{z_{i\mu}}{z_i\cdot k}(e^{ikx_i} - e^{ikx_{i-1}})$$
$$= -e\sum_{i=1}^{3} e^{ikx_i}\left[\frac{z_{i\mu}}{z_i\cdot k} - \frac{z_{i+1,\mu}}{z_{i+1}\cdot k}\right] . \quad (1.7)$$

In the third line of (1.7) the current operator is expressed as a sum of contributions associated with the three vertices. Thus $U(L(x))$ can be expressed as a product of three factors, one associated with each of the three vertices, times a factor containing the cross terms. To prove factorization it is necessary to show that contributions arising from the cross terms fall off faster than $\lambda^{-9/2}$. Since the factor $\widetilde{F}_{\rm opr}$ already gives a factor $\lambda^{-9/2}$ it is necessary only to exhibit some additional falloff of the cross terms.

Falloff of the cross terms is exhibited first in a context in which one ignores the contribution from photons with k in some region Ω chosen small enough so that the ignored contributions give negligible relative contribution to the transition probabilities. But the more important factorization result deals the amplitude $A^c(\lambda X)$ obtained by introducing the appropriate coherent state, so that the amplitude itself is well defined even when Ω is contracted to the point $k=0$. These amplitudes are also shown to factorize: $\lim\lambda^{9/2}A^c(\lambda X) = A_1^c A_2^c A_3^c$. The separate factors are independent of the closed-loop process from which they are extracted. They can be identified as the scattering functions for processes with charged external particles. In this asymptotic expression there are no contributions from classical photons that are emitted at one of the three vertices and absorbed at another: all such cross terms drop out of the asymptotic limit.

Explicit closed expressions are derived for the full classical photon contributions to each function A_i^c, both in the special case corresponding to diagram D of Fig. 1, and in the general case. These contributions arise from the fact that the coordinate-space variables corresponding to the vertices of the Feynman diagram representing subprocess i will, in general, not all lie exactly at the point λX_i used in the definition of the coherent state associated with this subprocess. These expressions, together with the expressions for the quantum-photon contributions derived in paper II, give compact infrared-finite expressions for the scattering amplitudes of processes with initial and final charged particles evaluated away from singularities. Thus the method described here, though developed to deal with the delicate situations that arise at singularities, provides a simple resolution of the infrared problem also away from singularities.

II. THE BASIC FORMULA

Consider first the coordinate-space Feynman amplitude corresponding to a strong-interaction diagram D. Suppose the internal lines correspond to a charged, spin-$\frac{1}{2}$ particle closed loop. The Feynman amplitude then has the form

$$F^D(x_1,\ldots,x_n) \equiv F^D(x)$$
$$= \mathrm{Tr}\prod_{i=1}^{n}(V_i iS_F(x_i,x_{i-1})) , \quad (2.1)$$

where $x_0 = x_n$, the V_i are strong-interaction vertex parts, and

$$iS_F(x_i,x_{i-1}) = i\int \frac{d^4p_i}{(2\pi)^4}\frac{e^{-ip_i(x_i-x_{i-1})}}{\not p_i - m + i0} . \quad (2.2)$$

Associated with this function there is a spacetime closed loop $L(x) = L(x_i,\ldots,x_n)$, which is the n-sided spacetime polygon with cyclically ordered vertices located at the cyclically ordered set of points $x = (x_i,\ldots,x_n)$.

The electromagnetic corrections to the function $F^D(x)$ are now considered. A typical correction will be represented by a Feynman diagram having many photon lines incident on each of the n internal line segments of D. The photon coupling at any vertex that lies on the portion of the charged line of D that runs between x_{i-1} and x_i is now separated into its classical and quantum parts by the equation

$$-ie\gamma_\mu = C^i_\mu(k_j,z_i) + Q^i_\mu(k_j,z_i) , \quad (2.3)$$

where e is the e.m. (electromagnetic) coupling constant and

$$C^i_\mu(k_j,z_i) = -iez_{i\mu}\not k_j(z_i\cdot k_j)^{-1} . \quad (2.4)$$

Here

$$z_i = x_i - x_{i-1} , \quad (2.5)$$

and k_j is the momentum-energy of the associated photon.

Consider now the part of the Feynman diagram D corresponding to the original line segment i, which runs from x_{i-1} to x_i. Suppose m_i external photons with quantum couplings $Q^i_{\mu_j}(k_j,z_i)$ $(j=a,b,\ldots)$, are connected in the order $(a,b,\ldots)$ into this line segment i. There is a new coordinate variable $x_j, j\in(a,b,\ldots)$, for each inserted photon. Integration over these new coordinate variables x_j yields a function of x_i and x_{i-1}, and of the momenta k_j and spin indices ν_j of the m_i photons. For example, if $m_i = 2$ then this function is

$$G(x_i,x_{i-1};k_a,\nu_a,k_b,\nu_b) = \int \frac{d^4p_i}{(2\pi)^4}e^{-ip_ix_i + i(p_i+k_a+k_b)x_{i-1}}$$
$$\times \frac{i}{\not p_i - m}Q^i_{\nu_a}(k_a,z_i)\frac{i}{\not p_i + \not k_a - m}Q^i_{\nu_b}(k_b,z_i)\frac{i}{\not p_i + \not k_a + \not k_b - m} . \quad (2.6)$$

This function with the variables k_a, k_b, ν_a, and ν_b associated with the two photons a and b suppressed will be represented by the symbol $G^{(2)}(x_i,x_{i-1})$.

For arbitrary m_i the function $G^{(m_i)}(x_i,x_{i-1})$ is the natural generalization of the expression in (2.6) to the case where the ordered set $(a,b,\ldots)$ has m_i elements.

Consider next the function $G^{(m_i)}(x_i,x_{i-1})$ and the corrections to it associated with the classical coupling into the line segment i of D of a photon with momentum-energy k and spin index μ. This line already contains m_i coupling of Q type. The classical coupling can be inserted into any one of the m_i+1 segments into which line segment i is separated by these m_i coupling of Q type. The sum of the Feynman functions corresponding to these m_i+1 different possible insertions of this classical coupling $C^i_\mu(k_j,z_i)$ into line segment i is

$$\sum_{s=1}^{m_i+1} G^{(m_i)}_{\mu,s}(x_i,x_{i-1},k) \equiv G^{(m_i)}_\mu(x_i,x_{i-1},k) = G^{(m_i)}(x_i,x_{i-1})\left[\frac{-ez_{i\mu}}{k\cdot z_i}(e^{ikx_i}-e^{ikx_{i-1}})\right], \tag{2.7}$$

where $k\cdot z = k^\mu z_\mu = kz$, etc., and the variables associated with the photon *quantum* interactions are still suppressed. This result (2.7) is a simple consequence of the Ward identity

$$\frac{i}{\not p - m}(i\not k)\frac{i}{\not p+\not k-m} = \frac{i}{\not p+\not k-m} - \frac{i}{\not p-m}\ . \tag{2.8}$$

Equation (2.7) can also be expressed in the more compact form

$$G^{(m_i)}_\mu(x_i,x_{i-1},k) = G^{(m_i)}(x_i,x_{i-1})(-ie)\int_{x_{i-1}}^{x_i} dx_\mu e^{ikx}\ . \tag{2.9}$$

Consider next any Feynman diagram D' obtained by attaching into each line segment i of D a set of m_i photon lines. Each photon line of D' is required to begin or end on a Q-type vertex lying on one of the n segments of D. The Feynman function corresponding to D' can be expressed as

$$F^{D'}(x) = \mathrm{Tr}\prod_{i=1}^n V_i G^{(m_i)}(x_i,x_{i-1})\ , \tag{2.10}$$

where the momentum-energy variables (k_j,ν_j) associated with the photons of D' are suppressed.

A photon line with classical coupling may now be inserted into any one of the m_i+1 segments of any one of the n original line segments of D. The sum of the Feynman functions corresponding to all of these ways of inserting the classical coupling is, by virtue of (2.9), simply

$$\sum_s F^{D',s}_{\mu_1}(x,k_1) = F^{D'}_{\mu_1}(x,k_1) = F^{D'}(x)(-ie)\int_{L(x)} dx_{\mu_1} e^{ik_1x} \equiv F^{D'}(x) J_{\mu_1}(L(x),k_1)\ . \tag{2.11}$$

That is, the sum of the Feynman functions corresponding to all ways of classically coupling a photon of momentum-energy k_1 and vector component μ_1 into the closed loop $L(x)$ of D' is simply the product of the original function $F^{D'}(x)$ with $(-ie)$ times the line integral of $e^{ik_1x}dx_{\mu_1}$ around the n-sided spacetime polygon $L(x)$.

Let the total number of photon couplings in D' in the above calculation be $m=\sum m_i$. Then the sum over s on the left-hand side of (2.11) is a sum over $m+n$ terms, each of which is represented by a diagram with $m+n+1$ intervals. A second photon, of momentum k_2 and spin component μ_2, can be classically coupled into this collection in $(m+n)(m+n+1)$ different ways. The sum of the Feynman functions corresponding to all of these $(m+n)(m+n+1)$ ways of classically coupling the second photon is

$$\sum_s F^{D',s}_{\mu_1\mu_2}(x,k_1,k_2) = F^{D'}_{\mu_1\mu_2}(x,k_1,k_2) = F^{D'}(x)(-ie)^2\int_{L(x)} dx'_{1\mu_1} e^{ik_1x'_1}\int_{L(x)} dx'_{2\mu_2} e^{ik_2x'_2}\ . \tag{2.12}$$

More generally, the sum of the Feynman functions corresponding to all possible ways of classically coupling a set of N photons into any fixed diagram D' that is constructed from D by the addition of photon lines that couple into the loop $L(x)$ of D is

$$F^{D'}_{\mu_1\cdots\mu_N}(x,k_1,\ldots,k_N) = F^{D'}(x)(-ie)^N\prod_{i=1}^N\int_{L(x)} dx'_{i\mu_i} e^{ik_ix'_i} \equiv F^{D'}(x)\prod_{i=1}^N J_{\mu_i}(L(x),k_i)\ . \tag{2.13}$$

This result follows directly from the Ward identity (2.8).

Suppose now a photon is emitted with classical coupling from some point on the fermion closed loop in D' and is absorbed with classical coupling on some other point on this loop. Summing over all possible line segments of D' upon which the two ends of the photon line can begin and end, and dividing by two to compensate for a double counting, one obtains the contribution to the Feynman function

$$\Delta F^{D'}(x) \equiv F^{D'}(x)\left[-\frac{e^2}{2}\right]\int_{L(x)} dx'_\mu \int_{L(x)} dx''_\nu \int \frac{d^4k}{(2\pi)^4} i\frac{e^{-ik(x'-x'')}}{k^2+i\epsilon}(-g^{\mu\nu}) \equiv F^{D'}(x)\left[\frac{-e^2}{2}\right]\int_{L(x)}\int_{L(x)} dx'\cdot dx'' iD_F(x'-x'')\ , \tag{2.14}$$

where D_F is the scalar part of the Feynman photon propagator. Its real part, which comes from the principal-value part of $D_F(k)=-(k^2+i\epsilon)^{-1}$, is

$$\mathrm{Re}D_F(x'-x'')=\frac{1}{4\pi}\delta((x'-x'')^2)\ . \tag{2.15}$$

This gives a "Coulomb" contribution $\Delta_C F^{D'}$ to $\Delta F^{D'}$ that is $F^{D'}(x)$ times

$$i\Phi(L(x))=\frac{i(-ie)^2}{8\pi}\int_{L(x)}\int_{L(x)}dx'\cdot dx''\delta((x'-x'')^2)\ . \tag{2.16}$$

The factor $\Phi(L(x))$ is the classical action corresponding to the motion of the charged particles along the spacetime paths defined by the polygon $L(x)$.

The contribution from the effect of m such photons, is just $F^{D'}(x)[i\Phi(L(x))]^m/m!$, where the factor $(m!)^{-1}$ compensates for multiple overcounting. Thus the sum of $F^{D'}$ and all these Coulomb corrections to it is just

$$F_c^{D'}(x)=F^{D'}(x)\exp[i\Phi(L(x))]\ . \tag{2.17}$$

Thus if a classical photon is defined to be a photon that couples into L only via the classical interaction then the net effect of all of the virtual classical photons is simply to multiply the original function $F^{D'}(x)$ by the Coulomb phase factor $\exp[i\Phi(L(x))]$ associated with the polygon $L(x)$.

The real (as opposed to virtual) classical photons correspond to the term $\pi\delta(k^2)$ in $i\Delta_F(k)=i(k^2+i\epsilon)^{-1}$. The real classical photons that are both emitted and absorbed on the closed loop $L(x)$ give a contribution to (2.14) of the form

$$\begin{aligned}\Delta_R F^{D'}(x)&=F^{D'}(x)\exp\left[-\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}2\pi\delta^+(k^2)J_\mu^*(L(x),k)(-g^{\mu\nu})J_\nu(L(x),k)\right]\\&\equiv F^{D'}(x)\exp[-\tfrac{1}{2}\langle J^*(L(x))\cdot J(L(x))\rangle]\ ,\end{aligned} \tag{2.18}$$

where

$$\delta^+(k^2)=\theta(k_0)\delta(k^2) \tag{2.19}$$

and

$$\begin{aligned}J_\mu(L(x),k)&=-ie\int_{L(x)}dx'_\mu e^{ikx'}\\&=-J_\mu^*(L(x),-k)\\&=\bar{J}_\mu(L(x),-k)\ .\end{aligned} \tag{2.20}$$

In the final line of (2.18) a bracket notation similar to Kibble's is introduced.

Real photons with classical couplings can also be emitted and absorbed from the charged-fermion loop. It is convenient to consider the S matrix to be an operator in the space of the external photons. The photon emitted by the classical photon coupling to the closed loop $L(x)$ is created by the operator

$$\begin{aligned}a^*(L(x))&=\int\frac{d^4k}{(2\pi)^4}2\pi\delta^+(k^2)a_\mu^*(k)(-g^{\mu\nu})J_\nu(L(x),k)\\&\equiv\langle a^*\cdot J(L(x))\rangle\ .\end{aligned} \tag{2.21}$$

If M such photons are created then the operator that creates the final state is $\langle a^*\cdot J(L)\rangle^M(M!)^{-1}$, where the factor $(M!)^{-1}$ compensates for an overcounting of Feynman diagrams. Thus the operator that creates the full set of final photon states generated by the classical coupling to the fermion closed loop L is

$$C(L)=\exp[\langle a^*\cdot J(L)\rangle]\ . \tag{2.22}$$

Similarly, the operator that annihilates the set of initial photons that are absorbed by the classical coupling to the closed loop L is

$$A(L)=\exp[-\langle J^*(L)\cdot a\rangle]\ . \tag{2.23}$$

The full Feynman operator function corresponding to $F^D(x)$ plus all electromagnetic corrections associated with Feynman diagrams that have no charged lines other than the loop $L(x)$ is, therefore,

$$\begin{aligned}\hat{F}^D_{\mathrm{op}}(x)&=e^{\langle a^*\cdot J(L(x))\rangle}\tilde{F}^D_{\mathrm{op}}(x)e^{-\langle a\cdot J^*(L(x))\rangle}\\&\quad\times e^{i\Phi(L(x))-\langle J^*(L(x))\cdot J(L(x))\rangle/2}\ .\end{aligned} \tag{2.24a}$$

Here $\tilde{F}^D_{\mathrm{op}}(x)=\sum F^{D'}_{\mathrm{op}}(x)$ is the sum of photon-space operators $F^{D'}_{\mathrm{op}}(x)$ that corresponds to the set of all Feynman diagrams D' that can be constructed by connecting onto the n internal line segments of D some combination of photon lines, with, however, the condition that each photon line must be coupled at one end or the other into some internal line segment i of D with a quantum coupling $Q^i_\mu(k_j,z_i)$. The operator $F^{D'}_{\mathrm{op}}(x)$ corresponding to D' is constructed from the corresponding Feynman function $F^{D'}(x,k_1,\ldots,k_m)$ by the formula

$$\begin{aligned}F^{D'}_{\mathrm{op}}(x)=\int:\prod_{j=1}^m\frac{d^4k_j}{(2\pi)^4}(2\pi)\delta(k_j{}^2)\bar{a}(k_j):\\\times F^{D'}(x,k_1,\ldots,k_m)\ ,\end{aligned} \tag{2.24b}$$

where $\bar{a}(k_j)=a(-k_j)=a^\dagger(k_j)$ creates a photon of momentum-energy k_j if $k_j^0>0$, and the two colons imply a Wick normal-ordering of the product of operator $\bar{a}(k_j)$ that they enclose.

As our interest is in infrared rather than ultraviolet problems we shall restrict all k integrations by $\theta(2K-|k^0|)\theta(K-|\vec{k}|)$, where K is some very large number. This cutoff factor will, for example, replace the factor $\delta((x_1-x_2)^2)$ that arises from (2.14), and that occurs in (2.15), by its nonultraviolet part, and will render all quantities occurring in the above formula (2.24) well defined.

Let $\hat{F}^{D0}_{\rm op}(L(x))$ be the part of the operator $\hat{F}^{D}_{\rm op}(L(x))$ of (2.24) that comes from the original part $F^D(x)$ of the operator $\tilde{F}^{D}_{\rm op}(x)$. Introducing, for any function $f(k)$, the notation $\bar{f}(k)=f(-k)$ one obtains from formula (2.24)

$$\hat{F}^{D0}_{\rm op}(x)=F^D(x)\exp[\langle \bar{a}\cdot J(L(x))\rangle]\exp[\langle \bar{J}(L(x))\cdot a\rangle] \times\exp[\tfrac{1}{2}\langle \bar{J}(L(x))\cdot J(L(x))\rangle]\exp[i\Phi(L(x))] = F^D(x)U(L(x)) \,. \tag{2.25}$$

Consider next the part $\hat{F}^{D1}_{\rm op}[\psi_1,\ldots,\psi_N]$ of $\tilde{F}^{D}_{\rm op}[\psi_1,\ldots,\psi_N]$ in (2.24) that comes from the part of $\tilde{F}^{D}_{\rm op}(x)$ that corresponds to diagrams D' having exactly one quantum coupling. The sum of the terms $F^{D'}_{\rm op}(x)$ of (2.24b) over all diagrams D' having a single quantum coupling to an external photon line (and no other photon coupling) is

$$\sum{}' F^{D'}_{\rm op}(x)=\sum{}'\int\frac{d^4k}{(2\pi)^4}2\pi\delta(k^2)\bar{a}(k)F^{D'}(x,k) \equiv\langle \bar{a}\cdot Q\rangle+\langle \bar{Q}\cdot a\rangle \,, \tag{2.26}$$

where the first and second terms on the right-hand side of (4.5) correspond to the first and second terms in

$$2\pi\delta(k^2)=2\pi\delta^+(k)+2\pi\delta^-(k) \,, \tag{2.27}$$

respectively.

The operator $\tilde{F}^{D1}_{\rm op}(x)$ arising from the sum of $F^{D'}_{\rm op}(x)$ over all D' having exactly one quantum coupling is then

$$\tilde{F}^{D1}_{\rm op}(x)=\langle \bar{a}\cdot Q\rangle+\langle \bar{Q}\cdot a\rangle+\tfrac{1}{2}\langle \bar{J}\cdot Q\rangle +\tfrac{1}{2}\langle \bar{Q}\cdot J\rangle+i\langle \bar{J}\cdot Q\rangle_{\rm PV} \,, \tag{2.28}$$

where the last three terms come from the diagrams D' that have a photon line with one quantum coupling to $L(x)$ and one classical coupling to $L(x)$, and

$$\langle \bar{J}\cdot Q\rangle_{\rm PV}={\rm PV}\int\frac{d^4k}{(2\pi)^4}\frac{\bar{J}_\mu(k)(-g^{\mu\nu})Q_\nu(k)}{k^2} \,, \tag{2.29}$$

where PV stands for principal value.

The basic formula (2.24) can be written in the slightly more convenient form

$$\hat{F}^{D}_{\rm op}(x)=\exp(\langle \bar{a}\cdot J\rangle)\tilde{F}^{D}_{\rm op}(x)\exp(\langle \bar{J}\cdot a\rangle) \times\exp(\tfrac{1}{2}\langle \bar{J}\cdot J\rangle+I\Phi) \,, \tag{2.30}$$

where $J=J(L(x))$ and $\Phi=\Phi(L(x))$. The term $\langle \bar{Q}\cdot a\rangle$ in (2.28) commutes through $\exp(\langle \bar{J}\cdot a\rangle)$, but $\langle \bar{a}\cdot Q\rangle$ does not:

$$[\exp(\langle \bar{J},a\rangle),\langle \bar{a},Q\rangle]=\langle \bar{J},Q\rangle\exp(\langle \bar{J},a\rangle) \,. \tag{2.31}$$

Thus the part of $\hat{F}^{D}_{\rm op}(x)$ coming from $\tilde{F}^{D1}_{\rm op}(x)$ is

$$\hat{F}^{D1}_{\rm op}(x)=\exp(\langle \bar{a}\cdot J\rangle)\exp(\langle \bar{J}\cdot a\rangle) \times\exp\left[\tfrac{1}{2}\langle \bar{J}\cdot J\rangle+i\Phi\right] \times[\tilde{F}^{D1}_{\rm op}(x)-\langle \bar{J}\cdot Q\rangle] =U(L(x))(\langle \bar{a}\cdot Q\rangle+\langle \bar{Q}\cdot a\rangle -\tfrac{1}{2}\langle \bar{J}\cdot Q\rangle+\tfrac{1}{2}\langle \bar{Q}\cdot J\rangle+i\langle \bar{J}\cdot Q\rangle_{\rm PV}) \,. \tag{2.32}$$

Note that the sign of the contribution associated with the emission of a real (as opposed to virtual) photon from a quantum coupling to $L(x)$, and its subsequent absorption by the classical coupling to $L(x)$, has been reversed. This reversal of sign is represented by the following change of the Feynman denominator associated with the propagation of the $Q-C$ photon:

$$k^2+i\epsilon\to(k^0+i\epsilon)^2-|\vec{k}|^2 \,. \tag{2.33}$$

Here k is the momentum-energy of the photon emitted by the quantum coupling and absorbed by the classical coupling. Thus (5.11) can be written in the form

$$\hat{F}^{D1}_{\rm op}(x)=U(L(x))\tilde{F}^{D1}_{\rm opr}(x) \,, \tag{2.34}$$

where the subscript r stands for the retarded character of the propagator in

$$\tilde{F}^{D1}_{\rm opr}(x)=\langle \bar{a}Q(L(x))\rangle+\langle \bar{Q}(L(x))\cdot a\rangle +i\int\frac{dk^4}{(2\pi)^4}\frac{\bar{J}_\mu(L(x),k)(-g^{\mu\nu})Q_\nu(L(x),k)}{(k^0+i\epsilon)^2-|\vec{k}|^2} \,. \tag{2.35}$$

This result can be extended immediately to the contributions to $\tilde{F}^{D}_{\rm op}(x)$ with arbitrary numbers of quantum couplings. One obtains

$$\hat{F}^{D}_{\rm op}(x)=U(L(x))\tilde{F}^{D}_{\rm opr}(x) \,, \tag{2.36}$$

where $\tilde{F}^{D}_{\rm opr}(x)$ is the same as the $\tilde{F}^{D}_{\rm op}(x)$ in (2.24b) except that each $F^{D'}(x_1,k_1,\ldots,k_m)$ is replaced by $F^{D'}_r(x,k_1,\ldots,k_m)$, which is calculated from the Feynman rules modified by the change in denominator shown in (5.13) and (5.14) for each photon line that links a quantum coupling to $L(x)$ to a classical coupling $L(x)$. This is our basic formula.

III. FEATURES OF THE BASIC FORMULA

In this section some general features of the basic formula (2.36) are discussed.

A. Isolation of infrared problems

A principal result of this work, and the paper that follows,[9] is that the infrared problems are confined to the operator $U(L(x))$ that appears in (2.36): the residual effects involving quantum couplings produce no infrared divergences.

B. Connection of physics

For clarity of presentation the strong-interaction diagram D will often be taken to be the simple one illustrated in Fig. 1.

The quantity $\hat{F}^D_{\text{op}}(x)$ given in (2.36) is an operator in the photon space. It is connected to physics via the transition operator $T^D_{\text{op}}[\psi_1,\ldots,\psi_N]$, which is obtained by folding into $\hat{F}^D_{\text{op}}(x)$ the wave functions $\psi_j(x_j)$ of the initial and final particles of the strong-interaction process represented by diagram D. If j specifies a *final* particle then $\psi_j(x_j)$ is the complex conjugate of the usual wave function of this particle. Thus

$$T^D_{\text{op}}[\psi_1,\ldots,\psi_N]=\int\prod_{i=1}^{n}d^4x_i\prod_{j=1}^{N}\psi_j(x_{i(j)})\hat{F}^D_{\text{op}}(x)\ , \tag{3.1}$$

where $i(j)$ is the label of the vertex i upon which external line j of D is incident.

C. Connection to classical physics

The operator $U(L(x))$ in (2.36) is closely connected to classical phyics. The phase $\Phi(L(x))$ is the contribution to the classical action from the motion, in the manner of Feynman, of a classical charged particle around the closed spacetime loop $L(x)$. The other three exponential factors combine to give a unitary operator which, when acting on the photon vacuum, creates a coherent photon state. This coherent state is the one associated with the classical electromagnetic field radiated by a charged particle moving around the closed spacetime loop $L(x)$. These results follow from Kibble's formula (15), in Ref. 5, first citation.

D. Exactness of basic formula

Formula (2.36) is exact. No soft-photon approximation—or any other approximation—has been used to reorganize the photon contributions into the form (2.36), in which the infrared problems are confined to exponentials related to classical physics.

IV. SMALLNESS OF THE SOFT-PHOTON CONTRIBUTIONS IN CERTAIN SIMPLE SITUATIONS

The transition operator $T^D_{\text{op}}[\psi_1,\ldots,\psi_N]$ is calculated by folding the initial and final wave functions $\psi_j(x_j)$ into the operator $\hat{F}^D_{\text{op}}(x)$ of (2.36). The detailed properties of the contributions to $\tilde{F}^D_{\text{op}}(x)$ that come from the diagrams

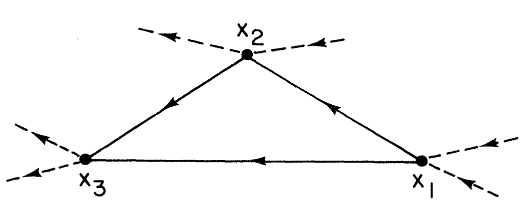

FIG. 1. A simple strong-interaction diagram D. The dotted external lines represent neutral particles. The solid triangle corresponds to $L(x)=L(x_1,x_2,x_3)$.

$D'\neq D$ will be examined later, in paper II. Thus we shall concentrate here on the part $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_N]$ of $T^D_{\text{op}}[\psi_1,\ldots,\psi_N]$ that arises from the part $F^D(x)$ of $\tilde{F}^D_{\text{op}}(x)$. Because all the contributions to $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_N]$ have very simple forms it is easy to obtain rigorous bounds on the magnitude of various specified contributions to it.

We shall suppose that the $\psi_j(p)$ are infinitely differentiable functions of compact support. Then for each external particle j there will be a "dominant region," in which $|\psi_j(x)|$ can be appreciable, and a "tail region," in which $|\psi_j(x)|$ is very small and falling off faster than any inverse power of the spatial distance from the dominant region. (See Ref. 7 for discussions of these properties.)

In calculating the transition amplitude the coordinate-space wave function $\psi_j(x_j)$ is evaluated at the point $x_j=x_{i(j)}$, where $i(j)$ is the vertex of D upon which external line j of D is incident. Consider, for definiteness, the diagram D of Fig. 1, and the corresponding transition amplitude $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]$.

Suppose the supports of the six wave functions in $\vec{p}_i/p_i{}^0$ space are disjoint. Then the dominant regions associated with the six wave functions will be asymptotically disjoint. In particular, the maximum of the absolute value of the product of any two wave functions in the region lying outside a ball of Euclidean radius R centered at the origin will fall off faster than any power of R^{-1}. Consequently the contribution to $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]$ from very soft photons is negligible.

To see this let $\Omega(b)$ be the k-space region

$$\Omega(b)\equiv\{k;\,|k^0|\leq 2b,\,|\vec{k}|\leq b\}\ . \tag{4.1}$$

And let $U_\Omega(L(x))$ be the operator $U(L(x))$ with all k integrations restricted to the region $\Omega(b)$. The difference between $U_\Omega(L(x))$ and the value it would have if there were no contributions at all from $k\in\Omega$ photons is $U_\Omega(L(x))-1$. Hence the contribution to $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]$ from the $k\in\Omega$ photon is

$$\begin{aligned}T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]_\Omega&\\ \equiv\int dx_1dx_2dx_3\,&\psi_1(x_1)\psi_2(x_1)\\ &\times\psi_3(x_2)\psi_4(x_2)\psi_5(x_3)\psi_6(x_3)\\ &\times[U_\Omega(L(x_1,x_2,x_3))-1]F^D(x_1,x_2,x_3)\ .\end{aligned} \tag{4.2}$$

Let $\mathscr{R}(R)$ represent the x-space region

$$\mathscr{R}(R)\equiv\{x;\,|x_i|_{\text{Eucl}}\leq R,\,i\in\{1,2,3\}\}\ . \tag{4.3}$$

And define $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]_{\Omega,\mathscr{R}}$ and $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]^{\mathscr{R}}_\Omega$ to be the parts of $T^{D0}_{\text{op}}[\psi_1,\ldots,\psi_6]_\Omega$ arising from the integration regions $x\in\mathscr{R}$ and $x\notin\mathscr{R}$ respectively.

The unitary operator $U_\Omega(L(x))$ has unit norm. Hence for every b the norm of $U_{\Omega(b)}(L(x))-1$ satisfies

$$|U_{\Omega(b)}(L(x))-1|\leq 2\ . \tag{4.4}$$

The ultraviolet cutoff ensures that the functions $|S_F(x_i-x_{i-1})|$ are bounded. Hence $|F^D(x)|$ is bounded:

$$|F^D(x)|\leq C\ . \tag{4.5}$$

These two bounds, and the faster than any power of R^{-1} falloff of the maximum of the absolute value of the product of any two wave functions ensures that the norm of

$$T_{\rm op}^{D0}[\psi_1, \ldots, \psi_6]_{\Omega(b)}^{\mathscr{R}(R)}$$

falls off faster than any power of R^{-1}. Hence for any $\epsilon > 0$, however small, there is an $R = R(\epsilon)$ such that for all b

$$| T_{\rm op}^{D0}[\psi_1, \ldots, \psi_6]_{\Omega(b)}^{\mathscr{R}(R(\epsilon))} | < \epsilon/2 \; . \tag{4.6}$$

Consider next the remaining part $T_{\rm op}^{D0}[\psi_1, \ldots, \psi_6]_{\Omega(b),\mathscr{R}(R(\epsilon))}$. Take $b \ll R(\epsilon)^{-1}$. Then the exponential factor $\exp(ikx')$ in (2.20) is close to unity, and its integral around the closed loop $L(x)$ enjoys a bound of the form

$$| J_\mu(L(x),k) | < ckR^2 \; . \tag{4.7}$$

Insertion of this bound into (2.14), with the k^0 contour distorted into a semicircle of radius $2b$, gives for the absolute value of $e^2/2$ times the integral (2.14) a bound

$$c'(bR)^4 \ll 1 \; , \tag{4.8}$$

where c' is some constant. Exponentiation preserves essentially this bound: for sufficiently small b

$$| \langle 0 | U_{\Omega(b)}(L(x)) - 1 | 0 \rangle | < 2c''(bR)^4 \; . \tag{4.9}$$

Here $|0\rangle$ is the photon vacuum. The boundedness of $F^D(x_1,x_2,x_3)$ then ensures that for some sufficiently small

$$b = b(\epsilon, R(\epsilon)) = b(\epsilon) > 0$$

the following bound holds:

$$| \langle 0 | T_{\rm op}^{D0}[\psi_1, \ldots, \psi_6]_{\Omega(b(\epsilon)),\mathscr{R}(R(\epsilon))} | 0 \rangle | < \epsilon/2 \; . \tag{4.10}$$

This result, combined with (4.6), shows that for $\epsilon > 0$, however small, there is a $b(\epsilon)$ such that

$$| \langle 0 | T_{\rm op}^{D0}[\psi_1, \ldots, \psi_6]_{\Omega(b(\epsilon))} | 0 \rangle | < \epsilon \; . \tag{4.11}$$

In other words, the contribution to the transition amplitude $T_{\rm op}^{D0}[\psi_1, \ldots, \psi_6]$ from the very soft photons $k \in \Omega(b)$ can be made arbitrarily small by choosing b sufficiently small.

V. DISCUSSION OF INFRARED DIVERGENCES

True infrared divergences do not arise if all charged particles are confined to finite spacetime closed loops. This fact is exploited in the procedure adopted above: the expressions are made free of infrared divergences, and hence amenable to rigorous mathematical analysis, by considering transition amplitudes corresponding to processes in which the charged particles are confined to closed loops, which are kept effectively finite by the damping provided by the wave functions $\psi_j(x)$ of the initial and final particles.

Infrared divergences traditionally arise in processes in which some of the initial or final particles are charged: the momenta of initial and final particles are then restricted by mass-shell constraints, which cause the singularities of certain Feynman denominators at $k=0$ to produce divergences.

One may, of course, consider all charged particles in the universe to be confined to closed loops. In a certain narrow technical sense this would solve the infrared-divergence problem: there would be no strict divergences of $T_{\rm op}^{D}[\psi_1, \ldots, \psi_n]$ for the entire universe. But this is not a physically adequate solution of the problem, for the following reason: the closed loops, though finite, will be huge, and the factors $\Phi(L(x))$ and $\langle J^*(L(x)) \cdot J(L(x)) \rangle$ both diverge logarithmically under dilation of the closed loop. Thus for loops the size of the universe these quantities are, for all practical purposes, infinite. No predictions about laboratory phenomena should depend on such numbers. The theory, to be useful, must allow the predictions about local phenomena to depend only on local specifications, not on the detailed ancient history of the particular electrons that are being used in some experiment. Some factorization is required to extract the local aspects.

Usually this factorization is achieved by means of the pole-factorization property. In the absence of massless particles one can show that if the sources of various particles are far away from a certain reaction among these particles then the only significant part of the larger process that includes also the sources comes from the residues of the poles singularities associated with the exchanged particles. The net residue is a product of separate factors, one for each source and one for the interaction. In this way the descriptions of the sources of the particles of the reaction can be effectively separated from the description of the reaction among them. Were it not for this pole-factorization property, or some similar property, the whole universe would have to be considered as a unit.

The residue of the pole is evaluated by restricting the exchanged particles to the mass shell. But a restriction of a charged particle to its mass shell brings us back to the traditional infrared divergences. Thus the procedure of starting from a universe in which all particles are confined to closed loops does not, without further analysis, solve the problem. One must establish the requisite factorization properties, which are in any case needed for a satisfactory theory of particles, and must confirm that the residues are finite. These residues will represent the amplitudes for processes with charged external particles. We now proceed to those tasks.

VI. SPACETIME POLE-FACTORIZATION PROPERTY

Suppose the initial and final momentum-energies of a many-particle reaction are related in a manner that permits a classical one-particle-exchange process of the kind shown in Fig. 2.

The Feynman rules ensure that the scattering function of the overall process will have a pole-type singularity $i2m(p^2 - m^2 + i0)^{-1}$, and that the residue of this pole is simply the product of the scattering amplitudes associated with the two subprocesses. The "discontinuity" associated with the pole is the difference of the boundary values from the upper and lower half-planes in p^2, and is therefore $2\pi\delta(p^2 - m^2)2m$ times the product of the scattering functions of the two subprocesses.

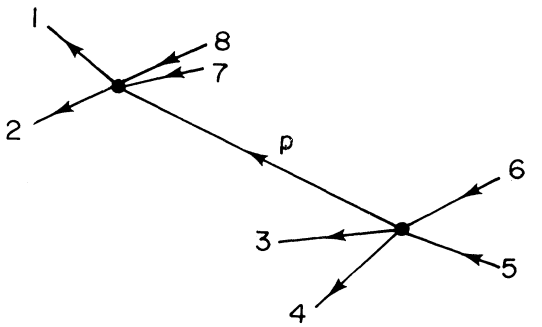

FIG. 2. A one-particle-exchange process. Momentum-energy is conserved in each of the two subprocess, and the intermediate particle momentum is denoted by p.

The pole character of this singularity and the fact that the residue factorizes in this way is crucial to the interpretation of quantum theory. It ensures that stable particles behave as stable particles should. Suppose, for example, that we fold in the wave functions of the initial and final particles of the overall reaction. Then the first (lower) interaction can be regarded as a subreaction in which a particle of mass m is produced, and the second interaction can be regarded as a subreaction in which this particle is detected. If these two subreactions are far apart then the rate at which the transition probability decreases as the two subreactions are moved further apart must be in accord with classical ideas about the flux of stable particles emerging from a source that is small in comparison to the large distance between the source and the detector.

If we take the momentum-space wave functions of the initial and final particles of the overall process to be infinitely differentiable functions of small compact support, and if the scattering functions for the two subprocesses are nonsingular in the regions defined by these small compact supports, then the scattering function $f_1(p,p_3,p_4,-p_5,-p_6)$ of the first subprocess folded into the wave functions $\bar{\phi}_3(p_3)\bar{\phi}_4(p_4)\phi_5(p_5)\phi_6(p_6)$ of this subprocess will give an infinitely differentiable and compactly supported wave function $\psi_1(p)$ of the particle produced in this first subreaction. Similarly, the scattering function $f_2(p_1,p_2,-p,-p_7,-p_8)$ of the second process folded into the wave functions $\bar{\phi}_1(p_1)\bar{\phi}_2(p_2)\phi_7(p_7)\phi(p_8)$ of this subprocess will give an infinitely differentiable and compactly supported wave function $\psi_2(-p)\equiv\bar{\psi}_2(p)$ of the particle detected at the second reaction. Thus the transition amplitude associated with the preparation of a particle represented by wave function $\psi_1(p)$, and the subsequent detection of a particle represented by (complex conjugated) wave function $\bar{\psi}_2(p)$, namely,

$$\langle\bar{\psi}_2\cdot\psi_1\rangle=\int\frac{d^4p}{(2\pi)^4}\bar{\psi}_2(p)2\pi\delta^+(p^2-m^2)2m\,\psi_1(p)\ , \tag{6.1}$$

is equal to the result of folding the wave functions $\phi_j(j=1,\ldots 6)$ of the external particles of the overall reaction into the discontinuity $2\pi\delta(p^2-m^2)2m$ of the overall scattering function.

We are interested in the dependence of this amplitude on the location of the detector. Thus we translate the wave functions $\phi_j(x_j)$ of the external particles of the second (detection) subprocess by a vector $\Delta x=\tau v$, where $v^2=1$ and $v^0>0$. This is achieved by the change

$$\phi_j(x_j)\rightarrow\phi_j^{\Delta x}(x_j)=\phi_j(x_j-\Delta x)\ .$$

This change induces the change

$$\phi_j(p_j)\rightarrow\phi^{\Delta x}(p_j)=\phi(p_j)^{ip_j\cdot\Delta x}$$

in the momentum-space functions. Then momentum-energy conservation in the second process yields the resulting change in $\bar{\psi}_2(p)$:

$$\bar{\psi}_2(p)\rightarrow\bar{\psi}_2^{\Delta x}(p)=\bar{\psi}_2(p)e^{-ip\cdot\Delta x}\ . \tag{6.2}$$

Actually, we are interested in the rate of falloff of the transition amplitude of the overall process itself as the magnitude τ of the shift Δx tends to infinity. However, if we had used in place of $(p^2-m^2+i0)^{-1}$ the boundary value $(p^2-m^2-i0)^{-1}$ then this modified transition amplitude would fall off faster than any power of τ.[7] Thus, modulo these terms that fall off faster than any power of τ we may use, in place of the actual pole form $i(p^2-m^2+i0)^{-1}$, rather the difference (or discontinuity)

$$i(p^2-m^2+i0)^{-1}-i(p^2-m^2-i0)^{-1}=2\pi\delta(p^2-m^2)\ .$$

Then, in the notation of (6.1) and (6.2), the question becomes: what is the rate of falloff of $\langle\bar{\psi}_2^{\tau v}\cdot\psi_1\rangle$ as $\tau\rightarrow\infty$?

This question is answered by the following corollary to a theorem proved in Appendix A.

Corollary A. Suppose $\bar{\psi}_2(p)\psi_1(p)$, considered as a function of the three-vector $\vec{p}$, is continuous together with its first and second derivatives, and vanishes for $|\vec{p}|\rangle R<\infty$. Then for any real v satisfying $v^2=1$ and $v^0>0$ the following limit holds:

$$\lim_{\tau\rightarrow\infty}\left[\frac{2\pi i\tau}{m}\right]^{3/2}e^{im\tau}\langle\bar{\psi}_2^{\tau v}\cdot\psi_1\rangle=\bar{\psi}_2(mv)\psi_1(mv)\ . \tag{6.3}$$

In terms of probabilities this relationship becomes

$$\lim_{\tau\rightarrow\infty}\left[\frac{2\pi\tau}{m}\right]^{3}|\langle\bar{\psi}_2^{\tau v}\cdot\psi_2\rangle|^2=|\bar{\psi}_2(mv)|^2|\psi_1(mv)|^2\ . \tag{6.4}$$

This result allows the squares of the magnitudes of the momentum-space wave functions $\psi_1(mv)$ and $\bar{\psi}_2(mv)$ to be identified as flux densities for emission and absorption of particles moving in the direction v. The factor τ^{-3} corresponds to the fact that stable particles do not disappear or materialize while moving from the source to the detector: the probabilities in the macroscopic domains have the same geometric falloff as the probabilities for classical stable particles.

If one were to increase the degree of the singularity then the falloff would become too slow. And if one were to decrease the degree of singularity then the falloff would become too fast.

The connections described above show that one cannot expect to extract reliable information about the singularity structure of a function from an approximation to it that disrupts its asymptotic behavior in coordinate space. For the asymptotic structure of transition amplitudes in coordinate space determines the analytic structure in momentum space, and vice versa.[7,8]

Storrow[4] examined the question of the effect of infrared photons on this pole singularity and concluded that the usual pole form $(p^2-m^2+i0)^{-1}$ was changed to $(p^2-m^2+i0)^{-1-\beta}$, where β was of order of the fine structure constant. Such a form would entail large deviations in the macroscopic regime from the classically expected behavior of stable particles.

VII. TRIANGLE-DIAGRAM FACTORIZATION AND AMPLITUDES FOR PROCESSES WITH CHARGED INITIAL AND FINAL PARTICLES

These pole-singularity considerations can be carried over to reactions such as the one illustrated in Fig. 1, in which a charged particle runs around a closed loop.

Let X_1, X_2, and X_3 be the vertices of a large spacetime closed loop $L(X)$. Let p_1, p_2, and p_3 be the momentum-energies of the three intermediate lines, as determined by the masses m_i of the three charged lines and the differences ΔX of the X_i. Suppose the wave functions $\psi_j^{X_i}(x)$ of the two external particles incident upon vertex i are large in a neighborhood of X_i, but have a product that falls off faster than any power of $|x-X_i|^{-1}$ as x moves away from X_i. And suppose that the scattering function for each of the three subreactions, folded into the wave functions $\psi_j^{X_i}$ of the two associated external particles, but evaluated at the momenta p_j associated with the two appropriate intermediate particles, is nonzero. This configuration defines a transition operator

$$A(\lambda X)=T_{\text{op}}^{D}[\psi_1^{\lambda X_{i(1)}},\ldots,\psi_6^{\lambda X_{i(6)}}] \quad (7.1)$$

that would be expected to have contributions corresponding to the reaction represented in Fig. 1. Indeed, if there were no infrared problem then $A(\lambda X)$ would be dominated at large λ by a term that falls off as $\lambda^{-9/2}$, and that arises from the pole singularities $(p_j^2-m_j^2+i0)^{-1}$ corresponding to the three charged lines in Fig. 1.

The diagrams D' contributing to this dominant term would be[7,8] those in the class $\mathscr{C}_D$ consisting at those D' that are separated into three disjoint diagrams by cutting three charged lines, one corresponding to each line of D. Modulo self-energy-diagram considerations the dominant $\lambda^{-9/2}$ contribution to $A(\lambda X)$ would be obtained by replacing each of the three poles $i(p_j'^2-m_j^2+i0)^{-1}$ by the corresponding mass-shell delta-functions $2\pi\delta(p_j'^2-m_j^2)$. Indeed, by factoring off $(c\lambda)^{-9/2}$, and an appropriate unitary factor that does not affect probabilities, one would obtain a limiting value that is just the product of the scattering functions for the three processes, with the ϕ_j's folded in, evaluated of the points $p_j'=p_j$. This is the triangle-diagram generalization of (6.3).

These pole-factorization results are not disrupted by the infrared photons. Equations (7.1), (3.1), and (2.36) give

$$A(\lambda X)=\int\prod_{i=1}^{3}d^4x_i\prod_{j=1}^{6}\psi^{\lambda X_{i(j)}}(x_{i(j)})U(L(x))\tilde{F}_{\text{opr}}^{D}(x)\ . \quad (7.2)$$

Let Ω be some small neighborhood of the point $k=0$. Then $U(L(x))$ can be written in the form

$$\begin{aligned}U(L(x))&=U_\Omega(L(x))U^\Omega(L(x))\\&=U_\Omega(L(\lambda X))U^\Omega(L(x))\\&\quad+U_\Omega(L(\lambda X))[U_\Omega^{-1}(L(\lambda X))U_\Omega(L(x))\\&\quad-1]U^\Omega(L(x))\ ,\end{aligned} \quad (7.3)$$

where the operators $U_\Omega(L(x))$ and $U^\Omega(L(x))$ are the operators obtained by restricting the k integrations that occur in the definition (2.25) of $U(L(x))$ to $k\in\Omega$ and $k\notin\Omega$, respectively. Then one may write

$$A(\lambda X)=A_{\text{dom}}(\lambda X)+A_{\text{rem}}(\lambda X)\ , \quad (7.4)$$

where $A_{\text{dom}}(\lambda X)$ and $A_{\text{rem}}(\lambda X)$ arise from the first and second terms in the final line of (7.3), respectively. In particular, one has

$$A_{\text{dom}}(\lambda X)=U_\Omega(L(\lambda X))A^\Omega(\lambda X)\ , \quad (7.5)$$

where

$$A^\Omega(\lambda X)=\int\prod_{i=1}^{3}d^4x_i\prod_{j=1}^{6}\psi^{\lambda X_{i(j)}}(x_{i(j)})U^\Omega(L(x))\tilde{F}_{\text{opr}}^{D}(x)\ . \quad (7.6)$$

The probability corresponding to the transition operator $A(\lambda X)$ is

$$P(\lambda X)=\text{Tr}A(\lambda X)\rho_{\text{in}}A^\dagger(\lambda X)\rho_{\text{fin}}\ , \quad (7.7)$$

where ρ_{in} and ρ_{fin} are the density operators for the initial and final photons. Final infrared photons are not detected. Thus ρ_{fin} acts as a unit operator on the infrared (i.e., $k\in\hat{\Omega}$) parts of the photon states. The noninfrared (i.e., $k\notin\hat{\Omega}$) photons play no essential role in the discussion, and can be assumed to be absent from both the initial and final states. Thus if

$$\rho_0^{\hat{\Omega}}\equiv|0^{\hat{\Omega}}\rangle\langle 0^{\hat{\Omega}}| \quad (7.8)$$

is the operator that projects all noninfrared $(k\notin\hat{\Omega})$ photon oscillator state vectors onto their ground or vacuum states, but leaves unchanged all photon oscillator states corresponding to photons with momenta $k\in\hat{\Omega}$ then one may write

$$\rho_{\text{fin}}=\rho_0^{\hat{\Omega}} \quad (7.9)$$

and

$$\rho_{\text{in}}=\rho_0^{\hat{\Omega}}\rho_{\text{in},\hat{\Omega}}\ , \quad (7.10)$$

where $\rho_{\text{in},\hat{\Omega}}$ specifies the initial condition of the infrared photons, but leaves unchanged all noninfrared parts.

Suppose Ω is contained in $\hat{\Omega}$. Then the contribution of

$A_{\rm dom}(\lambda X)$ to the probability $P(\lambda X)$ is

$$\begin{aligned}P_{\rm dom}(\lambda X) &= \mathrm{Tr}[\langle 0^{\hat\Omega}\,|\,A_{\rm dom}(\lambda X)\,|\,0^{\hat\Omega}\rangle \\ &\quad \times \rho_{{\rm in},\hat\Omega}\langle 0^{\hat\Omega}\,|\,A^{\dagger}_{\rm dom}(\lambda X)\,|\,0^{\hat\Omega}\rangle] \\ &= \mathrm{Tr}[\langle 0^{\hat\Omega}\,|\,U_{\Omega}(L(\lambda X))A^{\Omega}(\lambda X)\,|\,0^{\hat\Omega}\rangle] \\ &\quad \times \rho_{{\rm in},\hat\Omega}\langle 0^{\hat\Omega}\,|\,A^{\hat\Omega\dagger}(\lambda X)U^{\dagger}_{\Omega}(L(\lambda X))\,|\,0^{\hat\Omega}\rangle] \\ &= \mathrm{Tr}[\langle 0^{\hat\Omega}\,|\,A^{\Omega}(\lambda X)\,|\,0^{\hat\Omega}\rangle \\ &\quad \times \rho_{{\rm in},\hat\Omega}\langle 0^{\hat\Omega}\,|\,A^{\Omega}(\lambda X)\,|\,0^{\hat\Omega}\rangle]\,, \qquad (7.11)\end{aligned}$$

where the traces are in the space associated with the infrared photons, and the unitarity of $U_{\Omega}(L(\lambda X))$ has been used to obtain the last line.

Let $\Omega=\Omega(b)$ be a set of the form

$$\Omega(b)\equiv\{k:|\,k^0\,|\leq 2b,\,|\,\vec{\rm k}\,|\leq b\}\ . \qquad (7.12)$$

And suppose, as in Sec. II, that the wave functions $\psi_j(p_j)$ are infinitely differentiable with disjoint compact supports in $\vec{\rm p}_j/p_j$ space. Then it is shown in Appendix B that for some fixed Λ and for any $\epsilon>0$, however small, there is a $b(\epsilon)$ such that for any $b<b(\epsilon)$ and all $\lambda>\Lambda$ the contributions to $P(\lambda X)$ that involve $A_{\rm rem}(\lambda X)$ are less than ϵ times $P(\lambda X)$:

$$P(\lambda X)-P_{\rm dom}(\lambda X)<\epsilon P(\lambda X)\ . \qquad (7.13)$$

This smallness of the contributions from $A_{\rm rem}(\lambda X)$ arises from the fact that the faster-than-any-power falloffs of the wave functions $\psi_j^{\lambda X_i}(x)$ effectively confine x to a finite neighborhood of λX. Yet for all $|\,k\,|\ll|\,x-\lambda X\,|^{-1}$ the currents $J(L(x),k)$ and $J(L(\lambda X),k)$ are nearly equal. Consequently, the operators $U(L(x))$ and $U(L(\lambda X))$ are nearly equal, and hence the factor $[U_{\Omega}{}^{-1}(L(\lambda X))U_{\Omega}(L(x))-I]$ appearing in $A_{\rm rem}(\lambda X)$ tends effectively to zero with the size of $\Omega=\Omega(b)$.

The value of b is now taken small enough so that, to some high preordained level of accuracy, the probability $P(\lambda X)$ is adequately represented by $P_{\rm dom}(\lambda X)$. Then the remainder can be ignored: it is a negligible fraction of the whole.

Equations (7.11) and (7.6) show that the operator $U_{\Omega}(\lambda X)$ drops completely out of the calculation of $P_{\rm dom}(\lambda X)$. Thus no error at all is induced in the calculation of $P_{\rm dom}(\lambda X)$ if one replaces the operator $\widetilde F^{D}_{\rm opr}(x)$ in the basic formula (2.36) by

$$\hat F^{D\Omega}_{\rm opr}(x)\equiv U^{\Omega}(L(x))\widetilde F^{D}_{\rm opr}(x)\ . \qquad (7.14)$$

This substitution eliminates all contributions to $U(L(x))$ that arise from the photons with $k\in\Omega$. This elimination of $k\in\Omega$ contributions ensures the infrared finiteness of $P_{\rm dom}(\lambda X)$, and hence of $P(\lambda X)$ itself, provided the operator $\widetilde F^{D}_{\rm opr}(x)$ introduces no infrared divergences.

The infrared properties of $\widetilde F^{D}_{\rm opr}(x)$ are studied in paper II. An ultraviolet cutoff is imposed, and the possibility of a divergence of the sum over the infinite number of different diagram D' with quantum coupling Q is not examined. Subject to these limitations it is shown that the photon momentum-space eigenstates of the Fourier transform $\widetilde F^{D}_{\rm opr}(q)$ of $\widetilde F^{D}_{\rm opr}(x)$ are well defined and have the usual triangle-diagram singularity: the dominant contribution to the discontinuity around the triangle-diagram singularity surface is evaluated as a sum over contributions corresponding to all ways in which the diagrams D' can be cut into three disjoint parts by cutting three line segments, one corresponding to each of the three internal lines of D, and replacing the corresponding propagator $i(\not p+m)/p^2-m+i\epsilon$ by $2\pi\delta(p^2-m^2)(\not p+m)$. This restriction of charged-lines to their mass shells produces no infrared divergence.

Since the quantum photons give no infrared problems and the classical photons with $k\in\Omega$ do not enter we expect to obtain the normal factorization properties. To verify this consider first the vacuum-to-vacuum matrix element $\langle 0\,|\,\widetilde F^{D}_{\rm opr}(q)\,|\,0\rangle$. Since the singularity at the triangle-diagram singularity surface is normal the corresponding asymptotic behavior in coordinate space is also normal. Indeed, the three-particle generalization of the theorem of Appendix A ensures that if one defines

$$F(\lambda X)\equiv\int\prod_{i=1}^{3}d^4x_i\prod_{j=1}^{6}\psi^{\lambda X_{i(j)}}(x_{i(j)})\langle 0\,|\,\widetilde F^{D}_{\rm opr}(x)\,|\,0\rangle\ , \qquad (7.15)$$

then

$$\lim_{\lambda\to\infty}\prod_{j=1}^{3}\left[\left[\frac{2\pi i c_j\lambda}{m_j}\right]^{3/2}e^{im_jc_j\lambda}\right]F(\lambda X)=\prod_{i=1}^{3}F_i(\psi_{j(i)},p_i,p_{i+1})\ , \qquad (7.16)$$

where $F_i(\psi_{j(i)},p_i,p_{i+1})$ is the amplitude associated with vertex i of D. Specifically, $F_i(\psi_{j(i)},p_i,p_{i+1})$ is the scattering function for the subprocess associated with vertex i, folded into the wave functions ψ_j of the particles corresponding to the two external lines of D incident upon the vertex i, and evaluated at the momenta p_i and p_{i+1} of the charged particles associated with the two internal lines of D incident upon i. The quantities p_i and c are specified by

$$p_i=m_i(X_i-X_{i-1})/\,|\,X_i-X_{i-1}\,|_{\rm Mink} \qquad (7.17a)$$

and

$$c_i=|\,X_i-X_{i-1}\,|_{\rm Mink}\ . \qquad (7.17b)$$

The property of $\widetilde F_{\rm opr}(x)$ just described refers to its vacuum-to-vacuum matrix element. If the initial state represented by $\rho_{{\rm in},\hat\Omega}$ is the vacuum state then the operator $\widetilde F^{D}_{\rm opr}(x)$ in (7.6) that occurs in the formula (7.11) for $P_{\rm dom}(\lambda X)$ acts on the vacuum state. Then the vacuum-to-vacuum matrix element of $\widetilde F^{D}_{\rm opr}(x)$ will contribute to the probability $P_{\rm dom}(\lambda X)$ a term

$$P^0_{\rm dom}(\lambda X)=\int\prod_{i=1}^{3}(d^4x_i d^4y_i)\prod_{j=1}^{6}[\psi_j^{\lambda X_{i(j)}}(x_{i(j)})\psi_j^{\lambda X_{i(j)}*}(y_{i(j)})]\langle 0\,|\,\widetilde{F}^D_{\rm opr}(x)\,|\,0\rangle\langle 0\,|\,\widetilde{F}^{D^\dagger}_{\rm opr}(y)\,|\,0\rangle$$
$$\times\langle 0^{\hat\Omega}\,|\,U^{\hat\Omega}(L(x))\,|\,0^{\hat\Omega}\rangle\langle 0^{\hat\Omega}\,|\,U^{\hat\Omega\dagger}(L(y))\,|\,0^{\hat\Omega}\rangle$$
$$\times\sum_{n'_{\hat\Omega-\Omega}}\langle n'_{\hat\Omega-\Omega}\,|\,U_{\hat\Omega-\Omega}(L(x))\,|\,0_{\hat\Omega-\Omega}\rangle\langle 0_{\hat\Omega-\Omega}\,|\,U^\dagger_{\hat\Omega-\Omega}(L(y))\,|\,n'_{\hat\Omega-\Omega}\rangle\ . \tag{7.18}$$

The superscript $\hat\Omega$ on $U^{\hat\Omega}(L(x))$ means restriction of the integrals occurring in $U(L(x))$ to contributions from the photons with $k\notin\hat\Omega$ (i.e., to noninfrared photons) and the subscript $\hat\Omega-\Omega$ means restriction to photons with $k\in(\hat\Omega-\Omega)$ (i.e., to infrared photons that are not very soft). The sum over states $|\,n'_{\hat\Omega-\Omega}\rangle$ is a sum over all states of the oscillators corresponding to photons with $k\in(\hat\Omega-\Omega)$.

Expression (7.18) for $P^0_{\rm dom}(\lambda X)$ combines the infrared finite quantities $\langle 0\,|\,\widetilde{F}^D_{\rm opr}(x)\,|\,0\rangle$ and $\langle 0\,|\,\widetilde{F}^{D^\dagger}_{\rm opr}(y)\,|\,0\rangle$ with the unitary factors corresponding to classical photons with $k\notin\Omega$.

To establish an asymptotic factorization property for $P^0_{\rm dom}(\lambda X)$ recall first that

$$\langle 0^{\hat\Omega}\,|\,U^{\hat\Omega}(L(x))\,|\,0^{\hat\Omega}\rangle=\exp[i\Phi^{\hat\Omega}(L(x))]\exp-\tfrac{1}{2}\langle J^*(L(x))\cdot J(L(x))\rangle^{\hat\Omega}]\ , \tag{7.19}$$

where

$$\Phi^{\hat\Omega}(L(x))\equiv{\rm PV}\int\frac{d^4k}{2(2\pi)^4}\frac{J^*_\mu(L(x),k)(-g^{\mu\nu})J_\nu(L(x),k)}{k^2}\chi^{\hat\Omega}(k) \tag{7.20a}$$

and

$$\langle J^*(L(x))\cdot J(L(x))\rangle^{\hat\Omega}\equiv\int\frac{d^4k}{(2\pi)^4}J^*_\mu(L(x),k)(-g^{\mu\nu})J_\nu(L(x),k)2\pi\delta^+(k^2)\chi^{\hat\Omega}(k)\ . \tag{7.20b}$$

Here $\chi^{\hat\Omega}(k)$ is a factor that cuts out the contributions from both infrared and ultraviolet photons.

The current appearing in (7.20) is

$$\begin{aligned}J_\mu(L(x),k)&=-ie\int_{L(x)}dx'_\mu e^{ikx'}\\&=-e\sum_{i=1}^{3}\frac{z_{i\mu}}{z_i\cdot k}(e^{ikx_i}-e^{ikx_{i-1}})\\&=-e\sum_{i=1}^{3}e^{ikx_i}\left[\frac{z_{i\mu}}{z_i\cdot k}-\frac{z_{i+1,\mu}}{z_{i+1}\cdot k}\right]\\&\equiv\sum_{i=1}^{3}J_{i\mu}(x_i,z_i,z_{i+1},k)\ ,\end{aligned} \tag{7.21}$$

where $J_{i\mu}(x_i,z_i,z_{i+1},k)$ is the partial current associated with vertex i of D.

If each of the two currents in (7.20b) is decomposed into its three partial currents one obtains nine terms in all. Each of these nine terms is associated with one wiggly line in the diagram of Fig. 3.

Two of the nine terms are associated with each of the three wiggly lines that run between two different vertices, and one of the nine terms is associated with each wiggly line that begins and ends on the same vertex.

The contributions to (7.18) from the six terms in (7.20b) that correspond to interactions between different vertices fall off faster than λ^{-9}. To see this, consider first a typical contribution of this kind to (7.20b):

$$\begin{aligned}\langle J^*_{i-1}(x_{i-1})\cdot J_i(x_i)\rangle^{\hat\Omega}&\\=e^2\int&\frac{d^4k}{(2\pi)^4}e^{ik(x_i-x_{i-1})}(2\pi)\delta^+(k^2)\chi^{\hat\Omega}(k)\\&\times\left[\frac{z_{i-1\mu}}{z_{i-1}\cdot k}-\frac{z_{i\mu}}{z_i\cdot k}\right](-g^{\mu\nu})\left[\frac{z_{i\nu}}{z_i\cdot k}-\frac{z_{i+1\nu}}{z_{i+1}\cdot k}\right]\ .\end{aligned} \tag{7.22}$$

And consider first the values of (7.22) at points x in

$$\mathscr{R}(\lambda^\eta,\lambda X)\equiv\{x:|\,x-\lambda X\,|_{\rm Eucl}\le\lambda^\eta\}\ ,$$

where $0<\eta\ll 1\ll\lambda$. Since the X_i are chosen so that the differences X_i-X_{i-1} are all timelike, and satisfy $|\,X_i^0-X_{i-1}^0\,|>1$, the vectors $z_i\equiv x_i-x_{i-1}$ for points x in

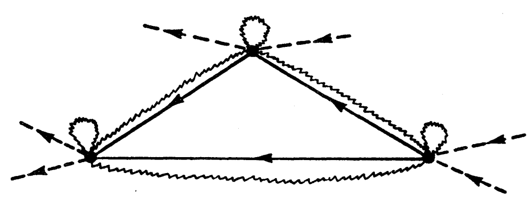

FIG. 3. A triangle diagram with wiggly lines representing the classical-photon contributions.

$\mathscr{R}(\lambda^\eta,\lambda X)$

must also be timelike. On the other hand, k is lightlike in the support of $\delta^+(k^2)$. Hence the only singularities of the integrand in (7.22) apart from those of the cutoff function $\chi^{\hat\Omega}(k)$, are those of $\delta^+(k^2)$. But then the properties of Fourier transforms[7,8] ensure that $\langle J_{i-1}^*(x_{i-1})\cdot J(x_i)\rangle^{\hat\Omega}$ falls off at least as fast as $|x_j-x_{j-1}|_{\rm Eucl}^{-1}$ in all directions except those on the light cone. And in these latter directions it is bounded.

Due to the timelike character of the differences $z_i=z_i-x_{i-1}$ for x in $\mathscr{R}(\lambda^\eta,\lambda X)$ this $|x_i-x_{i-1}|_{\rm Eucl}^{-1}$ falloff of (7.22) in timelike directions, together the bound $C\lambda^{-9+8\eta}$ on the remaining factors, entails a faster than λ^{-9} falloff of the $x\in\mathscr{R}(\lambda^\eta,\lambda X)$ contributions of $\langle J_{i-1}^*(x_{i-1})\cdot J_i(x_i)\rangle^{\hat\Omega}$ to the $P_{\rm dom}^0(\lambda X)$ defined in (7.18). On the other hand, the faster than any power of $|x-\lambda X|_{\rm Eucl}^{-1}$ falloff of the product of the wave functions in (7.18) ensures the faster than any power of λ^{-1} falloff of the contributions to the integral over x in (7.18) from points x not in $\mathscr{R}(\lambda^\eta,\lambda X)$, since the remaining factors in the integrand are bounded. Thus the full contribution to the probability $P_{\rm dom}^0(\lambda X)$ defined in (7.18) from the parts of (7.20b) that correspond to interactions between different vertices x_i falls off faster than λ^{-9}.

The three surviving terms in (7.20b) arise from the self-interaction counterparts of the integral in (7.22). These self-interaction terms, which correspond to the wiggly lines of Fig. 3 that begin and end on the same point, have x_i in place of x_{i-1} in (7.22). Hence they have no x dependence.

Consider next the integral in (7.20a). Arguments similar to those just given, and described in detail in Appendix D, show that the contributions of (7.20a) to (7.18) arising from the sum of products of factors J_i^* and J_j over $i\neq j$ fall off faster than λ^{-9}, provided the effect of the self-energy counterterm is included. The sole surviving term in the limit $\lambda\to\infty$ comes, therefore, only from the self-interaction terms involving the product of J_i^* with J_i. These terms have no x dependence. Thus the full contribution from the factor $\langle 0^{\hat\Omega}\,|\,U^{\hat\Omega}(L(x))\,|\,0^{\hat\Omega}\rangle$ to the dominant large-λ behavior of $P_{\rm dom}^0(\lambda X)$ defined by (7.18) is simply a product of three independent constants, one from each vertex of D.

The final factor in the expression (7.18) for $P_{\rm dom}^0(\lambda X)$ is a sum over the states $|n_{\hat\Omega-\Omega}\rangle$. These states can be taken to be the photon momentum eigenstates $|k_1,\ldots,k_n)_{\hat\Omega-\Omega}\rangle$. Since the photons that contribute to $U_{\hat\Omega-\Omega}(L(x))$ have k restricted to a region $\hat\Omega-\Omega$ that is bounded both from above and from below these cases can be treated by methods essentially the same as those just given: one simply treats the classical photons coupled into the three vertices of D like extra external particles. One may, for convenience, recombine the parts $k\notin\hat\Omega$ and $k\in\hat\Omega-\Omega$ and consider the matrix element

$$\langle k_1,\ldots,k_n\,|\,U^{\Omega}(L(x))\,|\,0\rangle=M^{\Omega}(kx)\;. \tag{7.23}$$

This function decomposes into a sum of terms, one for each way of coupling the set of photons $(k_1,\ldots,k_n)$ into the three vertices. Let γ be an index that runs over the various possibilities. Let α be an index that runs over the n photons, and let $i(\gamma,\alpha)$ label the vertex into which photon α couples for possibility γ. Then

$$\begin{aligned}\langle k_1,\ldots,k_n\,|\,U^{\Omega}(L(x))\,|\,0\rangle &=\sum_\gamma\langle k_1,\ldots,k_n\,|\,U^{\Omega}(L(x))\,|\,0\rangle_\gamma\\ &=\sum_\gamma M_\gamma^{\Omega}(k,x)\;.\end{aligned} \tag{7.24}$$

The x dependence of $M_\gamma^{\Omega}(k,x)$ is $\exp[i(x\cdot k)_\gamma]$, where

$$(x\cdot k)_\gamma\equiv\sum_{\alpha=1}^{n}k_\alpha x_{i(\gamma,\alpha)}\;. \tag{7.25}$$

Thus the function $M_\gamma^{\Omega}(k,x)e^{-i\lambda(X\cdot k)_\gamma}$ depends only on the differences $x_i-\lambda X_i (i=1,2,3)$. The wave functions $\psi_j^{\lambda X_{i(j)}}$ also depend only on these differences. Thus the three factors from $M_\gamma^{\Omega}(k,x)e^{-i\lambda(kX)_\gamma}$ simply modify the product of wave functions appearing in (7.16). Hence that earlier result yields immediately also

$$\lim_{\lambda\to\infty}\prod_{j=1}^{3}\left[\left[\frac{2\pi i c_i\lambda}{m_i}\right]^{3/2}e^{im_ic_i\lambda}\right]e^{-i\lambda(X\cdot k)_\gamma}\int\prod_{i=1}^{3}d^4x_i\prod_{j=1}^{6}\psi_j^{\lambda X_{i(j)}}(x_{i(j)})\,,\langle k_1,\ldots,k_n\,|\,U^{\Omega}(L(x))\,|\,0\rangle_\gamma$$
$$\times\langle 0\,|\,\tilde F_{\rm opr}^{D}(x)\,|\,0\rangle=\prod_{i=1}^{3}A_{i\gamma}^{\Omega}(\psi_{j(i)},p_i,p_{i+1};k_{\alpha(\gamma,i)}\;, \tag{7.26}$$

where

$$\alpha(\gamma,i)\equiv\{\alpha;i(\gamma,\alpha)=i\}\;, \tag{7.27a}$$

and the argument j in the last line runs over the set

$$J(i)\equiv\{j;i(j)=i\}\;. \tag{7.27b}$$

The right-hand side of Eq. (7.26) is a sum of contributions, one for each way in which any diagram D_γ' contributing to the left-hand side can be cut into three disjoint parts by cutting three charged-line segments, one corresponding to each internal line of D. The contribution on the right-hand side is obtained from the corresponding one on the left-hand side by setting $\lambda=0$ and replacing the Feynman propagator $i(\not p_i+m_i)/\,(p_i^2-m_i^2+i\epsilon)$ associated with the cut segment by $(\not p_i+m_i/2m_i)$, where

$$p_i=m_i(X_i-X_{i-1})/\,|X_i-X_{i-1}|_{\rm Mink}\;. \tag{7.28}$$

However, the Feynman diagrams on the left-hand side that contain self-energy corrections to the cut charged-line segment should be ignored, because the renormalization counterterms exactly eliminate their effects on this mass-shell line.

In constructing

$$A^{\Omega}_{i\gamma}(\psi_{j(i)},p_i,p_{i+1};k_{\alpha(\gamma,i)})$$

the quantities $v_{i\mu}/v_i\cdot k_\alpha$ and $v_{i+1\mu}/v_{i+1}\cdot k$ that arise from the classical coupling have been replaced first by

$$(X_i-X_{i-1})_\mu/(X_i-X_{i-1})\cdot k$$

and

$$(X_{i+1}-X_i)_\mu/(X_{i+1}-X_i)\cdot k\ ,$$

by omitting terms tending to zero in the limit $\lambda\to\infty$, and then, with the aid of (7.28), by $p_{i\mu}/p_i\cdot k$ and $p_{i+1\mu}/p_{i+1}\cdot k$.

Due to the exclusion from $U^{\Omega}(L(x))$ of contributions from photons with $k\in\Omega$ the value of the energy k^0_α of each final photon in $A^{\Omega}_{i\gamma}$ is greater than some fixed minimum value. Since the energy carried into and out of the subreaction i by the particles represented by the lines of D are constrained by the compact support of the wave functions $\psi_{j(i)}(p_j)$, and by the fixed values of the momenta p_i and p_{i+1}, the amplitudes

$$A^{\Omega}_{i\gamma}(\psi_{j(i)},p_i,p_{i+1};k_{\alpha(\gamma,i)})$$

must vanish if the set $\alpha(\gamma,i)$ has more than some finite number of elements. Thus the sum over final photon states needed in the calculation of

$$\lim\lambda^{-9}P_{\mathrm{dom}}(\lambda X)$$

is limited to states containing some finite number of photons.

Equation (7.26) exhibits an asymptotic factorization property of the amplitudes from which the probability $P^0_{\mathrm{dom}}(\lambda X)$ is constructed. This quantity $P^0_{\mathrm{dom}}(\lambda X)$ is the contribution to $P_{\mathrm{dom}}(\lambda X)$ from the infrared-finite matrix element $\langle 0\,|\,\widetilde{F}^{D}_{\mathrm{opr}}(x)\,|\,0\rangle$. Consider next the contribution from the matrix element $\langle k\,|\,\widetilde{F}^{D}_{\mathrm{opr}}(x)\,|\,0\rangle$. The analysis of paper II shows that the dominant singularity on the triangle-diagram surface of the Fourier transform of this function is normal. Thus the three-particle generalization of the theorem of Appendix A gives

$$\lim_{\lambda\to\infty}\prod_{i=1}^{3}\left[\left[\frac{2\pi i c_i\lambda}{m_i}\right]^{3/2}e^{im_ic_i\lambda}\right]\int\prod_{i=1}^{3}d^4x_i\psi^{\lambda X_{i(j)}}(x_{i(j)})\langle k\,|\,\widetilde{F}^{D}_{\mathrm{opr}}(x)\,|\,0\rangle=F_1(k)F_2F_3+F_1F_2(k)F_3+F_1F_2F_3(k)\ , \tag{7.29}$$

where

$$F_i\equiv F_i(\psi_{j(i)},p_i,p_{i+1}) \tag{7.30}$$

is the function occurring in (7.16), and

$$F_i(k)\equiv F_i(\psi_{j(i)},p_{j(i)},k) \tag{7.31}$$

is the amplitude for the process in which a photon of momentum-energy k is emitted by the part of the reaction at vertex i that is represented by $\widetilde{F}^{D}_{\mathrm{opr}}$.

The traditional infrared analysis suggests that an infrared divergence might arise from the coupling of the soft photon of momentum k onto the external on-mass-shell charged line of the reaction at vertex i. However, the coupling of an external photon of momentum k into $\widetilde{F}^{D}_{\mathrm{opr}}$ must be via a quantum-coupling $Q_\mu(k,z)$, which, for a coupling into the mass-shell charged line, occurs in the context

$$\begin{aligned}(\not p+m)Q_\mu(k,z)\frac{\not p+\not k+m}{(p+k)^2-m^2}&=(-ie)(\not p+m)\left[\gamma_\mu-\frac{z_\mu\not k}{z\cdot k}\right]\frac{\not p+\not k+m}{(2p\cdot k)}=(-ie)\left[(\not p+m)\left[\gamma_\mu-\frac{z_\mu\not k}{z\cdot k}\right]\frac{\not p+m}{2p\cdot k}+(\not p+m)\left[\gamma_\mu-\frac{z_\mu\not k}{z\cdot k}\right]\frac{\not k}{2p\cdot k}\right]\\&=(-ie)\left[\left[\frac{(\not p+m)(-\not p+m)}{2p\cdot k}\right]\left[\gamma_\mu-\frac{z_\mu\not k}{z\cdot k}\right]+(\not p+m)\left[2p_\mu-\frac{z_\mu(2p\cdot k)}{z\cdot k}\right]\frac{1}{2p\cdot k}+(\not p+m)\left[\gamma_\mu-\frac{z_\mu\not k}{z\cdot k}\right]\frac{\not k}{2p\cdot k}\right]\\&=(-ie)\left[(\not p+m)\gamma_\mu\frac{\not k}{2p\cdot k}\right]\ .\end{aligned} \tag{7.32}$$

The last line follows from the facts that k^2 vanishes, and that $p_\mu=mv_\mu$ is parallel to $v_\mu=z_\mu/|z|$, as prescribed by (7.28).

This result shows that the quantum coupling into the mass-shell line has one extra power of k in the numerator, relative to the usual γ_μ coupling. This extra power of k eliminates the usual infrared divergence. In fact, it is precisely this extra power of k in the quantum coupling that is the basis of the proof given in paper II that the momentum-space matrix elements of $\widetilde{F}^{D}_{\mathrm{opr}}(p)$ and their discontinuities are infrared finite.

By virtue of the infrared finiteness of $\widetilde{F}^{D}_{\mathrm{opr}}(p)$ the photons represented by it will not lead to any infrared problems. The ρ_{in} is assumed, for simplicity, to be the vacuum projector. Thus the matrix element

$$M^{\Omega}_{00}(\lambda X)=\langle 0\,|\,A^{\Omega}(\lambda X)\rho_{\mathrm{in}}A^{\Omega\dagger}(\lambda X)\,|\,0\rangle \tag{7.33}$$

will be infrared finite.

Equations (7.4)—(7.11) show that $M_{00}^{\Omega}(\lambda X)$ is a contribution $P_{\rm dom}^{00}(\lambda X)$ to $P_{\rm dom}(\lambda X)$. It has no infrared anomalies, and hence falls off at the normal λ^{-9} rate. On the other hand, the equations

$$P_{\rm dom}(\lambda X)={\rm Tr}A_{\rm dom}(\lambda X)\rho_{\rm in}A_{\rm dom}^{\dagger}(\lambda X)\rho_{\rm fin}\ , \qquad (7.34)$$

$$A_{\rm dom}(\lambda X)=U_{\Omega}(\lambda X)A^{\Omega}(\lambda X)\ , \qquad (7.35)$$

and (7.33) show that the full contribution to $P_{\rm dom}^{00}(\lambda X)=M_{00}^{\Omega}$ from final photons with $k\in\Omega$ arises exclusively from the single final coherent state $U_{\Omega}(\lambda X)\,|\,0_{\Omega}\rangle$. Similarly, the full contribution $P_{\rm dom}^{kk}(\lambda X)$ to $P_{\rm dom}(\lambda X)$ arising from the infrared-finite matrix element

$$\langle k_{\Omega}\,|\,A^{\Omega}(\lambda X)\rho_{\rm in}A^{\Omega\dagger}(\lambda X)\,|\,k_{\Omega}\rangle\ ,$$

where $|\,k_{\Omega}\rangle$ is $|\,k_1,\dots,k_{\mu}\rangle$ with all $k_i\in\Omega$, is carried exclusively by the single final coherent state $U_{\Omega}(\lambda X)\,|\,k_{\Omega}\rangle$. Thus if one wants to use final photon states that give dominant contributions to the asymptotic large-λ behavior of the probability then one cannot choose as the basis of the final $k\in\Omega$ photon space the usual momentum states $|\,k_{\Omega}\rangle=|\,(k_1,\dots,k_n)_{\Omega}\rangle$. For the use of these final states would introduce factors $\langle k_{\Omega}'\,|\,U_{\Omega}(L(\lambda X))\,|\,k_{\Omega}\rangle$ that all approach zero as $\lambda\to\infty$. The more appropriate basis for the final $k\in\Omega$ photon states is the set of coherent states $U_{\Omega}(L(\lambda X))\,|\,k_{\Omega}\rangle$: each of these carries the full contribution to $P_{\rm dom}(\lambda X)$ associated with the corresponding infrared-finite matrix element $\langle k_{\Omega}\,|\,A^{\Omega}(\lambda X)\rho_{\rm in}A^{\Omega\dagger}(\lambda X)\,|\,k_{\Omega}\rangle$. By using these coherent states one obtains for the individual final-state matrix elements the $\lambda^{-9/2}$ falloff property that corresponds to the λ^{-9} falloff property of the probabilities.

Use of these coherent states $U_{\Omega}(L(\lambda X))\,|\,k_{\Omega}\rangle$ is dictated also by physical considerations. For the unitary operator $U_{\Omega}(L(\lambda X))$ incorporates into the final photon states the quantum mechanical counterpart of the $k\in\Omega$ part of the classical electromagnetic field radiated by the closed loop $L(\lambda X)$. These classical contributions physically dominate the small k, large-λ behavior, and hence they must be incorporated into the final states if the resulting matrix elements are to have any physical significance in the limit $\lambda\to\infty$.

These coherent states $U_{\Omega}(L(\lambda X))\,|\,k_{\Omega}\rangle$ may be compared to those used to Storrow, Kibble, Zwanziger, and by Kulish and Faddeev. In the closed-loop case, where no charged particles occur initially or finally, these authors use the normal states $|\,k\rangle$. But the use of these states would, as just mentioned, give the individual matrix elements spurious damping factors that suppress the dominant large-λ behavior in coordinate space and consequently disrupt the analytic structure in momentum space.

Similarly, in the analysis of the pole-diagram singularity Storrow used coherent states that correspond to placing both scattering centers of the pole-diagram process at a common point, namely, the origin of spacetime. This choice effectively neglects effects of the factors e^{-ikx_i} in the expression (7.21) for the current. These exponential factors shift the parts of the current that correspond to separate scattering processes to the points x_i where these separate processes occur. Placing these separate contributions the origin is mathematically and physically inappropriate when the critical question is the form of a limit in which the separate subprocesses are shifted in different directions to infinity.

Storrow's neglect of the factors e^{ikx_i} stems from an analogous step made by Yennie, Frautschi, and Suura, who argue that terms containing the difference factors $(1-e^{ikx})$, acquire a convergence factor k in the infrared regime, and hence can be placed with the infrared convergent terms. This is an awkward step, since it disrupts momentum-energy conservation, and hence is more than just a shift of small terms into the residual collection. For it makes the infrared function large where it formerly vanished.

In any case this step is certainly not permissible when one is interested in the singularity structure. For in this case one must deal simultaneously with the regime

$$x\ \text{fixed},\quad k\to 0\ ,$$
$$\text{hence}\ kx\to 0\ , \qquad (7.36)$$

as well as the regime

$$k\ \text{small},\quad x\to\infty\ ,$$
$$\text{hence}\ kx\to\infty\ . \qquad (7.37)$$

One cannot keep making k smaller and smaller as x becomes larger and larger, because then the conclusions would hold only at the point $k=0$, where the Feynman functions are ill-defined. The methods developed in this paper cover simultaneously both of these two regimes.

To obtain nice factorization results for amplitudes analogous to the factorization results for probabilities established above let us consider the physically appropriate matrix elements. It is only in the very soft domain $k\in\Omega$ that the choice of final states $U_{\Omega}(L(\lambda X))\,|\,n\rangle$ is essential, but any abrupt change of representation at some arbitrary point would introduce spurious complications. Hence we use the basis $U(L(\lambda X))\,|\,(k_1,\dots,k_n)\rangle$.

The effect of this new choice of basis states is to replace the unitary operator $U^{\Omega}(L(x))$ in (7.26) by

$$U^{\dagger}(L(\lambda X))U_{\Omega}(L(\lambda X))U^{\Omega}(L(x))$$
$$=U^{\Omega\dagger}(L(\lambda X))U^{\Omega}(L(x))\ , \qquad (7.38)$$

where the operator $U_{\Omega}(L(\lambda X))$ from (7.5) and (7.11), which drops out of probabilities but contributes to matrix elements, has been reinstated.

Equation (B37) of Appendix B gives

$$U^{\Omega\dagger}(L(\lambda X))U^{\Omega}(L(x))=\exp\{\langle a^{*}\cdot[J(L(x))-J(L(\lambda X))]\rangle^{\Omega}\}\exp\{-\langle[J(L(x))-J(L(\lambda X))]^{*}\cdot a\rangle^{\Omega}\}$$
$$\times\exp\{-\tfrac{1}{2}\langle[J(L(x))-J(L(\lambda X))]^{*}\cdot[J(L(x))-J(L(\lambda X))]\rangle^{\Omega}\}$$
$$\times\exp\{-i\Phi[J(L(x)),J(L(\lambda X))]^{\Omega}\}\ , \qquad (7.39)$$

where

$$\Phi(J,J_1)^{\Omega}=\tfrac{1}{2}\langle (J+J_1)^{*}\cdot(J-J_1)\rangle_r^{\Omega} , \tag{7.40}$$

and

$$\langle A\cdot B\rangle_r^{\Omega}=\int \frac{d^4k}{(2\pi)^4}\frac{A_\mu(k)(-g^{\mu\nu})B_\mu(k)}{(k^0+i0)^2-|\vec{k}|^2}\chi^{\Omega}(k) . \tag{7.41}$$

Equation (7.26) with $U^{\Omega}(L(x))$ replaced by $U^{\Omega\dagger}(L(\lambda X))U^{\Omega}(L(x))$ is called (7.26′). Arguments essentially the same as those leading to (7.26) show that the contributions to (7.26′) from terms having a product of partial currents J_i^* and J_j with $i\neq j$ fall off faster than $\lambda^{-9/2}$, and do not contribute to the limit. What remains in the limit are three factors, one arising from each partial current J_i, $i\in\{1,2,3\}$. The asymptotic factor associated in (7.26′) with vertex i is denoted by $A_{i\gamma}^{\Omega'}(\psi_{j(i)},p_i,p_{i+1};k_{\alpha(\gamma,i)})$.

The effect of the factor $\exp[-i\tau(X\cdot k)_\gamma]$ in (7.26′) is to replace the arguments x_i in the operators that contribute to $A_{i\gamma}^{\Omega'}(\psi_{j(i)},p_i,p_{i+1},k_{\alpha(\gamma,i)})$ by $x_i-\lambda X_i$. Thus if subscript i means restriction to contributions from the partial current J_i then the classical-photon contribution to $A_{i\gamma}^{\Omega'}$ arises from the operator

$$\begin{aligned}(U^{\Omega\dagger}(L(\lambda X_i-\lambda X_i))U^{\Omega}(L(x_i-\lambda X_i)))_i&=\exp\{\langle a^*\cdot[J_i(x_i-\lambda X_i)-J_i(0)]\rangle^{\Omega}\}\exp\{-\langle [J_i(x_i-\lambda X_i)-J_i(0)]^*\cdot a\rangle^{\Omega}\}\\&\quad\times\exp\{-\tfrac{1}{2}\langle [J_i(x_i-\lambda X_i)-J_i(0)]^*[J_i(x_i-\lambda X_i)-J_i(0)]\rangle^{\Omega}\}\\&\quad\times\exp\{-\frac{i}{2}\langle [J_i(x_i-\lambda X_i)+J_i(0)]^*[J_i(x_i-\lambda X_i)-J_i(0)]\rangle_r^{\Omega}\}\\&=U^{\Omega\dagger}(J_i(0))U^{\Omega}(J_i(x_i-\lambda X_i)) .\end{aligned} \tag{7.42}$$

The operator in (7.42) acting in the space of photons with momentum $k\in\Omega$ is unity. Thus the difference between the operator in (7.42) and the analogous operator with $\Omega=\Omega(b)=\emptyset$ (i.e., $b=0$) is the unitary operator (7.42) times

$$U^{\dagger}_{\Omega(b)}(J_i(0))U_{\Omega(b)}(J_i(x_i-\lambda X_i))-I . \tag{7.43}$$

But the results of Appendix B entail that for any finite R and all

$$x_i\in\mathscr{R}_i(R,\lambda X)\equiv\{x_i:|x_i-\lambda X_i|_{\mathrm{Eucl}}\leq R\} \tag{7.44}$$

the operator in (7.43), restricted to allowed initial states, is an operator whose norm tends to zero as b tends to zero. But then

$$\lim_{b\to 0}A_{i\gamma}^{\Omega(b)'}(\psi_{j(i)},p_i,p_{i+1};k_{\alpha(\gamma,i)})=A_{i\gamma}(\psi_{j(i)},p_i,p_{i+1};k_{\alpha(\gamma,i)}) \tag{7.45}$$

exists, since the contributions from $x_i\notin\mathscr{R}_i(R,\lambda X)$ can be made arbitrarily small by taking R sufficiently large. (See the end of Appendix E.)

The amplitude $A_{i\gamma}(\psi_{j(i)},\ p_i,\ p_{i+1};\ k_{\alpha(\gamma,i)})$ is the amplitude for the process with two charged external lines. It is independent of the original process from which it came, and hence can be called $A(\psi,p_i,p_{i+1};k)$ where ψ represents the set $\psi_{j(i)}$ and k represents the set $k_{\alpha(\gamma,i)}$.

As a simple example consider the case in which there are two neutral initial particles with wave functions ψ_1 and ψ_2, and two charged final particles with physical momenta $-p_i$ and p_{i+1}. Suppose there are no external photons (i.e., no k_α) and no quantum photons [i.e., $\tilde{F}^{D}_{\mathrm{opr}}(x)$ can be replaced by $F^{D}(x)$]. Then the amplitude is

$$\begin{aligned}A_i^0(\psi_1,\psi_2,p_i,p_{i+1})&=\int d^4x_i\psi_1(x_i-\lambda X_i)\psi_2(x_i-\lambda X_i)V_ie^{-ip_i(x_i-\lambda X_i)}e^{ip_{i+1}(x_i-\lambda X_i)}\exp\left[-\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}J_{i\mu}(0)(-g^{\mu\nu})J_{i\nu}(0)\right]\\&\quad\times 2\pi\delta^{\dagger}(k^2)(e^{-i(x_i-\lambda X_i)}-1)(e^{i(x_i-\lambda X_i)\cdot k}-1)\exp\left[-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}J_{i\mu}(0)(-g^{\mu\nu})J_{i\nu}(0)\right]\\&\quad\times\frac{(e^{-i(x_i-\lambda X_i)\cdot k}+1)(e^{i(x_i-\lambda X_i)\cdot k}-1)}{(k^0+i0)^2-|\vec{k}|^2}\\&=\int d^4x_i\psi_1(x_i-\lambda X_i)\psi_2(x_i-\lambda X_i)V_ie^{-ip_i(x_i-\lambda X_i)}e^{ip_{i+1}(x_i-\lambda X_i)}\\&\quad\times\exp\left[-\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}\left[\frac{p_{i\mu}}{p_i\cdot k}-\frac{p_{i+1,\mu}}{p_{i+1}\cdot k}\right](-g^{\mu\nu})\left[\frac{p_{i\nu}}{p_i\cdot k}-\frac{p_{i+1,\nu}}{p_{i+1}\cdot k}\right]\right]\\&\quad\times 2\pi\delta^{\dagger}(k^2)(e^{-i(x_i-\lambda X_i)\cdot k}-1)(e^{+i(x_i-\lambda X_i)\cdot k}-1)\end{aligned}$$

$$\times \exp\left[-\frac{ie^2}{2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{p_{i\mu}}{p_i\cdot k}-\frac{p_{i+1,\mu}}{p_{i+1}\cdot k}\right](-g^{\mu\nu})\left[\frac{p_{i\nu}}{p_i\cdot k}-\frac{p_{i+1,\nu}}{p_{i+1}\cdot k}\right]\right]$$

$$\times\frac{1}{(k^0+i0)^2-|\vec{k}|^2}(e^{-i(x_i-\lambda X_i)\cdot k}+1)(e^{i(x_i-\lambda X_i)\cdot k}-1)\ . \qquad (7.46)$$

The factor $\exp(-ip_ix_i)$ comes from the propagator of particle i in $F^D(x)$, and the associated factor $\exp(ip_iX_i\lambda)$ comes from the factor $\exp(im_ic_i\lambda)=\exp[ip_i(X_i-X_{i-1})\lambda]$ in (7.26′) [see (7.17)]. The factor $\exp[ip_{i+1}(x_i-\lambda X_i)]$ has a similar origin.

The first integrand in an exponential in the last line of (7.46) behaves like $\delta(k^2)$ as $|k|\to 0$, and the integral is infrared convergent for any finite $x_i-\lambda X_i$.

The second integrand in an exponential has poles at $p_i\cdot k=0$ and $p_{i+1}\cdot k=0$. In the original expression, for the full triangle-diagram process before factorization, these poles were canceled by compensating zeros in the numerator. In the proofs of Appendix B a particular $i\epsilon$ resolution of the pole was introduced. One could equally well have chosen the other $i\epsilon$ resolution. But a more natural and convenient choice is the principal-value resolution. For this resolution never introduces spurious imaginary contributions.

If the principal-value resolution of these two poles is used then one may exploit the symmetry under $k\to -k$ to replace the last three factors of the final integrand in (7.46) by

$$\frac{1}{2}\left[\frac{1}{(k^0+i0)^2-|\vec{k}|^2}-\frac{1}{(k^0-i0)^2-|k|^2}\right]2i\sin(x_i-\lambda X_i)\cdot k=\tfrac{1}{2}[-2\pi i\delta^+(k^2)+2\pi i\delta^-(k^2)]2i\sin(x_i-\lambda X_i)\cdot k\ . \qquad (7.47)$$

In this form the spurious poles drop out, and the integrand goes like $\delta^{\pm}(k^2)/k$. Consequently the integral is infrared finite. In fact, insertion of (7.47) into the final integral in (7.46) allows this integral to be expressed as

$$\frac{1}{2(2\pi)^3}\int_{-K}^{K}dk^0\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\left[\frac{p_{i\mu}}{p_i(\theta,\phi)}-\frac{p_{i+1,\mu}}{p_{i+1}(\theta,\phi)}\right](-g^{\mu\nu})$$

$$\times\left[\frac{p_{i\nu}}{p_i(\theta,\phi)}-\frac{p_{i+1,\nu}}{p_{i+1}(\theta,\phi)}\right](k^0)^{-1}\sin k^0(x_i(\theta,\phi)-\lambda X_i(\theta,\phi))\ , \qquad (7.48)$$

where, for any four-vector x,

$$x(\theta,\phi)=x^0-x^3\cos\theta-x^2\sin\theta\sin\phi-x^1\sin\theta\cos\phi\ . \qquad (7.49)$$

In this form the contour in k^0 can be distorted away from the point $k^0=0$, which eliminates any possibility of infrared divergence.

The simple case treated above is very special. For one thing, the part of diagram D that corresponds to the subprocess in question consists of only one single vertex. A slightly more complicated example is obtained by taking the part of some original diagram D that corresponds to the subprocess in question to be the diagram D_1 of Fig. 4.

Consider again the case with no external photons (i.e., no k_α), and the contribution with no quantum interactions. Then $\tilde{F}^D_{\rm opr}(x)$ is reduced to $F^{D_1}(x_1,x_2,x_i,x_{i+1})$. We shall drop the subscript i on X_i and A_i, and fold in the mass-shell supported wave functions $\psi_i^{\lambda X}(p_i)$ and $\psi_{i+2}^{\lambda X}(p_{i+2})$ of the charged particles, and thus obtain

$$A^0(\psi_1,\psi_2,\psi_i,\psi_{i+2})$$
$$=\int d^4x_1d^4x_2d^4x_id^4x_{i+1}\frac{d^4p_i}{(2\pi)^4}\frac{d^4p_{i+2}}{(2\pi)^4}\psi_1(x_1-\lambda X)\psi_2(x_2-\lambda X)\psi_i(p_i)\psi_{i+2}(p_{i+2})$$
$$\times e^{-ip_i(x_i-\lambda X)}e^{ip_{i+2}(x_{i+1}-\lambda X)}F^{D_1}(x_1,x_2,x_i,x_{i+1})$$
$$\times\exp[I(p_i,p_{i+1},x_i-\lambda X)+I(p_{i+1},p_{i+2},x_{i+1}-\lambda X)+I(p_i,p_{i+1},x_i-\lambda X;p_{i+1},p_{i+2},x_{i+1}-\lambda X)]\ , \qquad (7.50)$$

where

$$I(p,p',x)=\frac{-e^2}{2}\int\frac{d^4k}{(2\pi)^4}\left[\frac{-p^2}{(p\cdot k)^2}+\frac{-(p')^2}{(p'\cdot k)^2}+\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)}\right]\{2\pi\delta(k^2)(1-\cos x\cdot k)+i2\pi[\delta^+(k^2)-\delta^-(k^2)]\sin x\cdot k\} \qquad (7.51)$$

and

$$I(p,p',x;p'',p''',x') = \frac{-e^2}{2}\text{PV}\int \frac{d^4k}{(2\pi)^4}\left[\frac{-p\cdot p''}{(p\cdot k)(p''\cdot k)}+\frac{-p'\cdot p'''}{(p'\cdot k)(p'''\cdot k)}+\frac{p\cdot p'''}{(p\cdot k)(p'''\cdot k)}+\frac{p'\cdot p''}{(p'\cdot k)(p''\cdot k)}\right]$$

$$\times\{2\pi\delta(k^2)[1+\cos(x-x')k-\cos xk-\cos x'k]$$

$$+i2\pi[\delta^+(k^2)-\delta^-(k^2)](\sin x\cdot k+\sin x'\cdot k)+ik^{-2}[-2+2\cos(x-x')\cdot k]\}\ . \tag{7.52}$$

The four-vector p_{i+1} is

$$m_{i+1}(x_{i+1}-x_i)/|x_{i+1}-x_i|_{\text{Mink}}\ ,$$

but any vector parallel to $x_{i+1}-x_i$ will do just as well.

For all x and x' in the ball of Euclidean radius R the terms in (7.52) that contain factors $\delta^+(k^2)$ and $\delta(k^2)$ are infrared finite, for reasons already given. The terms with k^{-2} are also infrared finite. In fact, the methods of Appendix B show that all contributions from $k\in\Omega(b)$ have bounds of the form $bB(R)$ where $B(R)$ is linear in R for large R.

The supports of the infinitely differentiable wave functions of the initial and final particles in $\vec{p}/p^0$ space are again taken to be disjoint. Then the contributions to the integral (7.50) from points $x\notin\mathscr{R}(R,\lambda X)$ fall off faster than any power of R^{-1}. This is shown in Appendix E. Thus the finiteness of (7.50) is assured.

The final factor in (7.50) gives the effects of the classical photons. It can be regarded as an operator that produces the modifications induced by classical photons in the wave functions of the external charged particles. Of course, the major effects of the classical photons come from the operator $U^\dagger(L(\lambda X))$ that has been incorporated into the state vectors of the final photons.

The first two terms in the final exponential in (7.50) are the classical-photon self-interaction terms for the two charged-line vertices of D_1. They are represented by the two wiggly lines of Fig. 5 that begin and end on the same vertex. The final term in this exponential is represented by the wiggly line that runs between the two charged-line vertices of Fig. 5.

It is easy to pass from (7.50) to the case in which a general diagram replaces D_1. One first writes the Feynman formula for D_1 that is analogous to (7.50), but with zero as the final exponent. Then one adds to this final exponent the terms that represent the effects of the classical photons. If the diagram that replaces D_1 has n charged-line vertices then the sum over three terms in the final exponential in (7.50) is replaced by a sum over $n(n+1)/2$ terms, one for each of the n self-interaction wiggly lines and one for each of the $n(n-1)/2$ wiggly lines that connects different vertices. If there are external photons then one must also include the two operator exponentials of (7.42) with $J_i(x_i-\lambda X_i)-J_i(0)$ replaced now by a sum of the partial currents for all n charged-particle vertices. These operators can be represented by wiggly lines coming into and going out of each of the charged-line vertices.

VIII. CONCLUDING REMARKS

Yennie, Frautchi, and Suura, at the end of a technical appendix to their paper, list a number of difficulties glossed over in their arguments, together with reasons why their approximations seem to them intuitively plausible. But they concluded that a rigorous proof of their result might by prohibitively complicated.

The difficulties in the YFS arguments cause no serious problem insofar as delicate issues can be avoided. But the applicability of quantum and spinor electrodynamics to physics requires that charged particles can continue to behave like stable particles in the presence of interactions with soft photons. Efforts to establish this property, and to derive the closely related reduction formulas, floundered, however, precisely on the delicate points not adequately treated by YFS.

The present work provides a new and fundamentally different approach to the infrared problem. It works basically with the coordinate-space representation of the sources of the electromagnetic field, and with an operator representation of the photons. Within this framework it establishes an exact result analogous to the momentum-space factorization property sought by YFs. The exact-

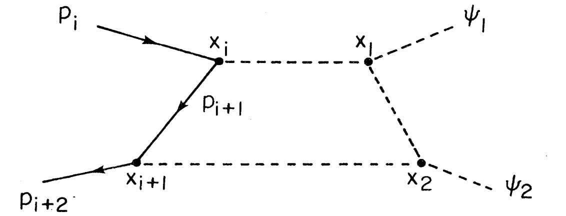

FIG. 4. Subprocess diagram D_1.

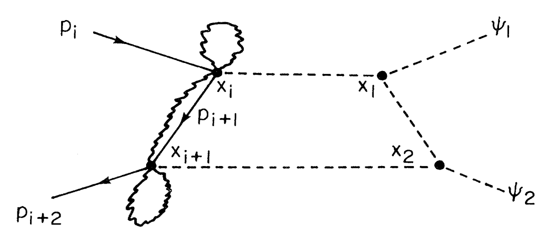

FIG. 5. The diagram D_1 with added wiggly lines representing the three classical-photon contributions to (7.50).

ness of the result allows it to be applied in the delicate situations where one sitting right on a singularity, or needs to know the precise form of the asymptotic behavior, in order to establish stability and factorization properties. Moreover, it allows gauge invariance to be fully exploited. Once approximations are introduced, in the sense that certain terms are pushed into a generalized remainder term that is not exhibited in explicit form, the full consequence of gauge invariance are no longer manifest.

The problems of completing the proof of the infrared finiteness of quantum and spinor electrodynamics, and establishing the stability and factorization properties of charged particles, though important in principle, has seemed unimportant in practice. For infrared problems seem under control in practical calculations. And physicists are generally confident that the physical effects of very soft photons are negligible, in spite of the numerous calculations that had seemed to indicate a breakdown of the stability and factorization properties. But science is a hard taskmaster: difficulties glossed over at one stage invariably crop up later. Thus the infrared problems largely ignored in quantum electrodynamics have emerged as the central problems in quantum chromodynamics. In particular, the problem of whether the stability of charged particles is upset by interactions with soft photons is the exact analog of the problem of confinement: Is the stability of colored particles upset by interactions with soft gluons? Thus the problem dealt with in detail in Sec. VII about the coordinate-space asymptotic behavior of an amplitude with a closed charged-particle loop becomes, in QCD, precisely the question of whether colored particles become asymptotically free in coordinate space.

The QCD problem of confinement is more delicate and complex than its QED couterpart. Hence the methods needed to resolve it will probably have to be at least as good as those that work in QED. And they might be expected to be a generalization of the latter.

Beyond the problems of infrared divergence and confinement there lie other related questions to which the methods of this paper may apply. These potential applications arise from the fact that the basic formula obtained here organizes the infinite series solution in a way that isolates a unitary factor that represents the classical-physics background. This type of separation may provide the technical basis needed for the full development of the idea that quantum theory must, for both physical and mathematical reasons, be arranged to be the calculation of quantum fluctuations about a classical solution. Moreover, the gathering together of infinite numbers of terms into unitary factors has the potential power of better controlling divergences, since the norm of any sum of terms that form a unitary operator is unity, in spite of any superficial indication of diverge.

ACKNOWLEDGMENTS

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APPENDIX A

Theorem. Suppose $g(p')$ is continuous, together with its first and second derivatives, and vanishes for $|\vec{p}|>R$ for some R. Let $p\equiv mv$ be any fixed mass-shell four-vector. Then

$$\lim_{\tau\to\infty}\left[\frac{2\pi i\tau}{m}\right]^{3/2} e^{im\tau}\int g(p')e^{-ip'\cdot v\tau}2m\,2\pi\delta^{+}(p^2-m^2)\times d^4p'(2\pi)^{-4}=g(p)\ . \tag{A1}$$

Proof. Transform to the variables corresponding to a frame in which $v=(1,0,0,0)$. In terms of these variables one has

$$v\cdot p'=p'^0=[m^2+(\vec{p})^2]^{1/2}=m+f[(\vec{p})^2]\ , \tag{A2}$$

where

$$f[(\vec{p})^2]=\frac{(\vec{p})^2}{2m}+\cdots>0\ . \tag{A3}$$

The introduction of the variable f in place of $(\vec{p})^2$, followed by an integration over angles, converts (A1) to

$$\frac{2}{\sqrt{\pi}}(i\tau)^{3/2}\int_0^\infty \bar{g}(f)\sqrt{f}\,e^{-if\tau}df\to\bar{g}(0)\ , \tag{A4}$$

where $\bar{g}(0)=g(0)$, and $\bar{g}(f)$ and its first and second derivatives are continuous at $f>0$. Since

$$\int_0^\infty e^{-if(\tau-i\epsilon)}\sqrt{f}\,df=\frac{\sqrt{\pi}}{2}\frac{1}{[i(\tau-i\epsilon)]^{3/2}} \tag{A5}$$

and $\bar{g}(f)$ is continuous with compact support, the required result (A4) is equivalent to

$$\lim_{\tau\to 0}\tau^{3/2}\lim_{\epsilon\to 0+}\int_0^\infty[\bar{g}(f)-\bar{g}(0)]e^{-if(\tau-i\epsilon)}\sqrt{f}\,df=0\ . \tag{A6}$$

Bounds on $(\bar{g}(f)-\bar{g}(0))$ and its first two derivatives can be obtained by writing

$$g(p)=g(r,\Omega)=g(0)+\vec{\nabla}g(0)\cdot\vec{r}+\int_0^r dr'\int_0^{r'}dr''\frac{\partial^2 g}{\partial r^2}(r'',\Omega)\ , \tag{A7}$$

where $\vec{r}=\vec{p}$ and $r=|\vec{p}|$. The integration over angles eliminates the linear term and gives

$$\frac{\bar{g}(f)}{\left[1+\dfrac{f}{2m}\right]^{1/2}}-\bar{g}(0)=\int\frac{d\Omega}{4\pi}\int_0^{r(f)}dr'\int_0^{r'}dr''\frac{\partial^2 g}{\partial r^2}(r'',\Omega)\ . \tag{A8}$$

Since the second derivative of $g(p)$ is bounded,

$$\left|\frac{\partial^2 g}{\partial r^2}\right|\le c\ , \tag{A9}$$

one has

$$\left| \frac{\bar{g}(f)}{\left[1+\dfrac{f}{2m}\right]^{1/2}} - \bar{g}(0) \right| \le \tfrac{1}{2} c r^2 \ . \tag{A10}$$

Letting F be such that

$$\bar{g}(f)=0 \text{ for } f \ge F \ ,$$

and defining $\bar{m}=F+m$, so that $\partial r^2/\partial f = 2f+2m \le 2\bar{m}$ for $f \le F$, one obtains, for $f \ge 0$,

$$|\bar{g}(f)-\bar{g}(0)| \le f c \bar{m}^2/m \ . \tag{A11}$$

Equation (A8) also yields, for $f \ge 0$,

$$|\bar{g}'(f)| \equiv \left| \frac{d}{df}\bar{g}(f) \right| \le c\bar{m}^2/m \tag{A12}$$

and, for $f>0$,

$$|\bar{g}''(f)| \le \left[\frac{1}{f} + \frac{2}{m} \right] (c\bar{m}^2/m) \ . \tag{A13}$$

An integration by parts on the integral in (A6) gives

$$\int_0^\infty [\bar{g}(f)-g(0)]\sqrt{f}\, e^{-if(\tau-i\epsilon)} df = \frac{-1}{-i(\tau-i\epsilon)} \int_0^\infty e^{-if(\tau-i\epsilon)} \frac{d}{df}\{[\bar{g}(f)-\bar{g}(0)]\sqrt{f}\} df$$
$$= \frac{1}{i(\tau-i\epsilon)} \int_o^\infty e^{-if\tau} h_\epsilon(f) df \ , \tag{A14}$$

where

$$h_\epsilon(f) = e^{-\epsilon f} \frac{d}{df}\{[\bar{g}(f)-\bar{g}(0)]\sqrt{f}\} \ . \tag{A15}$$

However,

$$\int_0^\infty e^{-if\tau} h_\epsilon(f) df = \int_0^{\pi/\tau} e^{-if\tau} h_\epsilon(f) df - \int_o^\infty e^{-if\tau} h_\epsilon(f+\pi/\tau) df$$
$$= \int_0^\infty e^{-if\tau} \tfrac{1}{2}[h_\epsilon(f)-h_\epsilon(f+\pi/\tau)]df + \tfrac{1}{2}\int_0^{\pi/\tau} e^{-if\tau} h_\epsilon(f) df \ . \tag{A16}$$

The last term in (A16) has, by virtue of (A11) and (A12), the bound $\frac{1}{2} c\bar{m}(\pi/\tau)^{3/2}$. Thus this contribution, inserted into (A14), satisfies (A6).

The first term in (A16) can be written as a sum of two terms. The first is

$$\tfrac{1}{2}\int_0^F e^{-if\tau}[h_\epsilon(f)-h_\epsilon(f+\pi/\tau)]df < \frac{1}{2}\left[\frac{\pi}{\tau}\right] \int_0^F |\max h'_\epsilon(f)|\, df \ , \tag{A17}$$

where $|\max h'_\epsilon(f)|$ is the maximum of the absolute value of $dh_\epsilon(f')/df'$ for $f' \ge f$. The bounds (A11), (A12), and (A13) ensure that the integral on the right-hand side of (A17) has a finite bound that is independent of ϵ. Thus this contribution, inserted into (A14), also satisfies (A6).

The remaining part of (A16) is

$$\tfrac{1}{2}\int_F^\infty e^{-if\tau}[h_\epsilon(f)-h_\epsilon(f+\pi/\tau)]df = \frac{-\bar{g}(0)}{4}\int_F^\infty e^{-if\tau}\left[\frac{e^{-\epsilon f}}{\sqrt{f}} - \frac{e^{-\epsilon(f+\pi/\tau)}}{(f+\pi/\tau)^{1/2}}\right] df$$
$$= \frac{-\bar{g}(0)}{4}\int_F^\infty e^{-if(\tau-i\epsilon)}\left[\frac{1}{\sqrt{f}} - \frac{1}{(f+\pi/\tau)^{1/2}}\right]$$
$$+ \frac{[-\bar{g}(0)]}{4}\int_F^\infty e^{-if(\tau-i\epsilon)}[1-e^{-\epsilon\pi/\tau}]\frac{1}{(f+\pi/\tau)^{1/2}} \ . \tag{A18}$$

The first term on the right-hand side of (A18) is bounded in magnitude by

$$\frac{|\bar{g}(0)|}{4}\left[\frac{\pi}{\tau}\right]\int_F^\infty \left[\frac{d}{df}\frac{1}{\sqrt{f}}\right] df = \frac{|g(0)|}{4}\left[\frac{\pi}{\tau}\right] F^{-1/2} \ . \tag{A19}$$

Thus this contribution, inserted into (A14) also satisfies (A6). The second term on the right-hand side of (A18) can be written

$$\frac{-g(0)}{4}[1-e^{-\epsilon\pi/\tau}]e^{+i(\pi/\tau)(\tau-i\epsilon)}\int_{F+\pi/\tau}^\infty e^{-if(\tau-i\epsilon)}\frac{1}{\sqrt{f}}df = \frac{g(0)}{4}[e^{\epsilon\pi/\tau}-1]$$
$$\times \left[\int_0^\infty e^{-if(\tau-i\epsilon)}\frac{1}{\sqrt{f}}df - \int_0^{F+\pi/\tau} e^{-if(\tau-i\epsilon)}\frac{df}{\sqrt{f}}\right]$$
$$= \frac{g(0)}{4}[e^{\epsilon\pi/\tau}-1]\left[\left[\frac{\pi}{-i(\tau-i\epsilon)}\right]^{1/2} - \int_0^{F+\pi/\tau} e^{-if(\tau-i\epsilon)}\frac{df}{\sqrt{f}}\right] \ .$$

This term vanishes when we take the limit $\epsilon\to 0$ in (A6). Thus all the contributions satisfy (A6).

APPENDIX B

The unitary operator $U(L(x))$ has the form

$$U(L(x))=\exp(\langle a^*\cdot J\rangle)\exp(-\langle J^*\cdot a\rangle)\exp(-\tfrac{1}{2}\langle J^*\cdot J\rangle)\exp\left[-\frac{i}{2}\langle J^*\cdot J\rangle_{\rm PV}\right], \tag{B1a}$$

$$=\exp(\langle \bar{a}\cdot J\rangle)\exp(\langle \bar{J}\cdot a\rangle)\exp(\tfrac{1}{2}\langle \bar{J}\cdot J\rangle)\exp\left[\frac{i}{2}\langle \bar{J}\cdot J\rangle_{\rm PV}\right], \tag{B1b}$$

where $J=J(L(x))$, and the bracket products are defined in (2.18), (2.20), (2.21), and (5.8).

Let $J(L(\lambda(X))$ be abbreviated by J_1. Then

$$\begin{aligned}U(L(x))U^{-1}(L(\lambda X))&=U(L(x))U^{\dagger}(L(\lambda X))\\&=\exp(\langle a^*\cdot J\rangle)\exp(-\langle J^*\cdot a\rangle)\exp(-\tfrac{1}{2}\langle J^*\cdot J\rangle)\exp\left[-\frac{i}{2}\langle J^*\cdot J\rangle_{\rm PV}\right]\\&\quad\times\exp(-\langle a^*\cdot J_1\rangle)\exp(\langle J_1^*\cdot a\rangle)\exp(-\tfrac{1}{2}\langle J_1^*\cdot J_1\rangle)\exp\left[\frac{i}{2}\langle J_1^*\cdot J_1\rangle_{\rm PV}\right].\end{aligned} \tag{B2}$$

The commutation relation

$$[\langle J^*\cdot a\rangle,\langle a^*\cdot J_1\rangle]=\langle J^*\cdot J_1\rangle \tag{B3}$$

gives

$$[\exp(-\langle J^*\cdot a\rangle),-\langle a^*\cdot J_1\rangle]=\langle J^*\cdot J_1\rangle\exp(-\langle J^*\cdot a\rangle), \tag{B4}$$

which gives

$$\exp(-\langle J^*\cdot a\rangle\exp(-\langle a^*\cdot J_1\rangle)=\exp(-\langle a^*\cdot J_1\rangle)\exp(-\langle J^*\cdot a\rangle)\exp(\langle J^*\cdot J_1\rangle, \tag{B5}$$

which gives

$$\begin{aligned}U(L(x))U^{-1}(L(\lambda X))&=\exp[\langle a^*\cdot(J-J_1)\rangle]\exp[-\langle(J-J_1)^*\cdot a\rangle]\exp[-\tfrac{1}{2}\langle(J-J_1)^*\cdot(J-J_1)\rangle]\\&\quad\times\exp[\tfrac{1}{2}\langle J^*\cdot J_1\rangle-\tfrac{1}{2}\langle J_1^*\cdot J\rangle-\tfrac{1}{2}i\langle J^*\cdot J\rangle_{\rm PV}+\tfrac{1}{2}i\langle J_1^*\cdot J_1\rangle_{\rm PV}]\end{aligned} \tag{B6}$$

$$=U'(L(x)-L(\lambda X))\exp[i\Phi(J,J_1)], \tag{B7}$$

where $U'(L)$ is the function defined in (B1) without the final (i.e., Coulomb) exponential factor, and $\Phi(J,J_1)$ is $-i$ times the argument of the final exponential in (B6). The phase $\Phi(J,J_1)$ can be expressed in the form

$$\begin{aligned}\Phi(J,J_1)&=-\tfrac{1}{2}\langle(J-J_1)^*\cdot(J+J_1)/2\rangle_{\rm PV}-\frac{i}{2}\langle(J-J_1)^*\cdot(J+J_1)/2\rangle\\&\quad-\tfrac{1}{2}\langle(J+J_1)^*\tfrac{1}{2}\cdot(J-J_1)\rangle_{\rm PV}+\frac{i}{2}\langle(J+J_1)^*\tfrac{1}{2}\cdot(J-J_1)\rangle\\&=-\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}(J_\mu(k)-J_{1\mu}(k))^*(-g^{\mu\nu})(J_\nu(k)+J_{1\nu}(k))/2\left[{\rm PV}\frac{1}{k^2}+i2\pi\delta^+(k^2)\right]\\&\quad-\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}(J_\mu(k)+J_{1\mu}(k))^*\tfrac{1}{2}(-g^{\mu\nu})(J_\nu(k)-J_{1\nu}(k))\left[{\rm PV}\frac{1}{k^2}-i2\pi\delta^+(k^2)\right]\\&=\int\frac{d^4k}{(2\pi)^4}(\bar{J}_\mu(k)-\bar{J}_{1\mu}(k))(-g^{\mu\nu})(J_\nu(k)+J_{1\nu}(k))/2\left[{\rm PV}\frac{1}{k^2}+i\pi(\theta(k^0)-\theta(-k^0))\delta(k^2)\right]\\&=\int\frac{d^4k}{(2\pi)^4}\frac{(\bar{J}_\mu(k)-\bar{J}_{1\mu}(k))(-g^{\mu\nu})(J_\nu(k)+J_{1\nu}(k))/2}{(k^0-i0)^2-|\vec{k}|^2}\\&=\int\frac{d^4k}{(2\pi)^4}\frac{\tfrac{1}{2}(\bar{J}_\mu(k)+\bar{J}_{1\mu}(k))(-g^{\mu\nu})(J_\nu(k)-J_{1\nu}(k))}{(k^0+i0)^2-|\vec{k}|^2}\\&\equiv\langle\tfrac{1}{2}(\bar{J}+\bar{J}_1)\cdot(J-J_1)\rangle_r,\end{aligned} \tag{B8}$$

where the subscript r indicates the retarded propagator. Thus

$$U(L(x))U^{-1}L(\lambda X)=\exp[\langle \bar{a}\cdot(J-J_1)\rangle]\exp[\langle \bar{J}-\bar{J}_1)\cdot a\rangle]\exp[\tfrac{1}{2}\langle \bar{J}-\bar{J}_1)\cdot(J-J_1)\rangle+\frac{i}{2}\langle \bar{J}+\bar{J}_1)\cdot(J-J_1)\rangle_r]\ , \tag{B9}$$

where $J\equiv J(L(x))$ and $J_1\equiv J(L(\lambda X))$.

Our interest here is in the restriction $U_\Omega(L(x))U_\Omega{}^{-1}(L(\lambda X))$ of $U(L(x))U^{-1}(L(\lambda X))$ to the soft-photon region Ω. This restriction is made by restricting the domain of integration to points k in Ω. The integrals occurring in (B9) when restricted to any bounded region Ω are all well defined.

The variable x will initially be confined to the region

$$\mathscr{R}(R,\lambda X)\equiv\{x\in R^{4n};\ |x_i-\lambda X_i|_{\text{Eucl}}\langle R\}\ , \tag{B10}$$

where $R>0$ is fixed. The time components of the timelike differences X_i-X_{i-1} are all taken to be greater than unity. Then for some $\Lambda>1$ one has, for all x in $\mathscr{R}(R,\lambda X)$ and all $\lambda\geq\Lambda-1$,

$$(x_i-x_{i-1})^2\rangle 1 \tag{B11a}$$

and

$$\text{sgn}(x_i{}^0-x_{i-1}{}^0)=\text{sgn}(X_i^0-X_{i-1}^0)\ . \tag{B11b}$$

The function $J_\nu(k)$ appearing in the integrand of (B8) is

$$\begin{aligned} J_\nu(k)&\equiv J_\nu(L(x),k)\\ &=e\sum_{i=1}^{n}(e^{ikx_i}-e^{ikx_{i-1}})\frac{(x_i-x_{i-1})_\nu}{(x_i-x_{i-1})\cdot k}\\ &=e\sum_{i=1}^{n}e^{ikx_i}(1-e^{ik(x_{i-1}-x_i)})\frac{(x_i-x_{i-1})_\nu}{(x_i-x_{i-1})\cdot k}\ . \end{aligned} \tag{B12}$$

The superficial pole at $(x_i-x_{i-1})\cdot k=0$ is canceled by the like factor in the numerator. Thus one can shift the contour infinitesimally away from the zero of $(x_i-x_{i-1})\cdot k$ in any convenient manner. Here the contour is fixed by replacing $(x_i-x_{i-1})\cdot k$ by

$$(x_i-x_{i-1})\cdot k+i0\,\text{sgn}(X_i{}^0-X_{i-1}{}^0)\ . \tag{B13}$$

Thus the k^0 contour is shifted into the upper-half plane. The denominator zero of $\bar{J}_{1\mu}(k)$ is treated in the same way, as are the zeros of $\bar{J}_\mu(k)+\bar{J}_{1\mu}(k)$. Thus the k^0 contour is distorted always into the upper-half plane.

The domain Ω will be taken to be of the form $\{|k^0|\leq 2b,\ |\vec{k}|\leq b\}$, and the notation

$$\Delta_i\equiv x_i-\lambda X_i \tag{B14}$$

is introduced.

Consider first the contribution to $\Phi(J,J_1)$ coming from the part of $\bar{J}_{1\mu}(k)$ corresponding to the line from 1 to 2 in Fig. 1, and from the part of $J_{1\mu}(k)$ corresponding to the line from 2 to 3. This contribution is minus one times

$$\Phi_{(2,1)(3,2)}(J_1)=\frac{e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}(e^{-ik\lambda X_2}-e^{-ik\lambda X_1})(e^{ik\lambda X_3}-e^{ik\lambda X_2})$$
$$\times\frac{(X_2-X_1)_\mu(-g^{\mu\nu})(X_3-X_2)_\nu}{((X_2-X_1)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_2)\cdot k+i0)}\ . \tag{B15}$$

By virtue of the time ordering $X_3{}^0>X_2{}^0>X_1{}^0$ in Fig. 1 one may push the k^0 contour a finite distance into the upper-half plane without encountering any exponentials that increase as $\lambda\to\infty$. One may take it to be a semicircle of radius $2b$. The integrand and integral are then uniformly bounded over the domain $\lambda\geq 0$.

Consider next the contribution that arises from replacing $J_{1\mu}(k)$ in the above expression by $J_\mu(k)$:

$$\Phi'_{(2,1)(3,2)}(J_1,J)=\frac{e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}(e^{-ik\lambda X_2}-e^{-ik\lambda X_1})(e^{ik\lambda X_3+ik\Delta_3}-e^{ik\lambda X_2+ik\Delta_2})$$
$$\times\frac{(X_2-X_1)_\mu(-g^{\mu\nu})(X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})_\nu}{((X_2-X_1)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})\cdot k+i0)}\ . \tag{B16}$$

For $\lambda\geq\Lambda$ one may again distort the k^0 contour into a semicircle in the upper-half plane and obtain an integrand and integral that are uniformly bounded over $\lambda\geq\Lambda$.

Consider now the contribution to the integral in (B16) that arises from the terms $(\Delta_3\lambda^{-1})_\nu$ and $(\Delta_2\lambda^{-1})_\nu$. Each of these contributions has, by virtue of the bound

$$|(\lambda(X_2-X_1)\cdot k)^{-1}(e^{-ik\lambda X_2}-e^{-ik\lambda X_1})|\leq 1\ , \tag{B17}$$

a bound of the form bB, where B is a number that independent of b and λ, but can depend on R. For $\lambda\geq\Lambda$ one may, for points on the semicircle $|k|=2b$, write

$$((X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})\cdot k)^{-1}=((X_3-X_2)\cdot k)^{-1}+\frac{1}{\lambda}f(k,\lambda)$$

with bounded $f(k,\lambda)$. For the second term one may again use (B17) to obtain a bound on the contribution to (B16) of the form bB. Thus one has

$$\Phi'_{(2,1)(3,2)}(J_1,J)-\Phi_{(2,1)(3,2)}(J_1)=O(b)+\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(e^{-k\lambda X_2}-e^{-k\lambda X_1})(e^{ik\lambda X_3}(e^{ik\Delta_3}-1)-e^{ik\lambda X_2}(e^{ik\Delta_2}-1))$$
$$\times\frac{(X_2-X_1)_\mu(-g^{\mu\nu})(X_3-X_2)_\nu}{((X_2-X_1)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_2)\cdot k+i0)}, \tag{B18}$$

where the magnitude of the term $O(b)$ is bounded for all $b>0$ and all $\lambda\geq\Lambda$ by an expression of the form bB. But then the bound

$$|e^{ik\Delta}-1|\leq|k\Delta| \tag{B19}$$

gives

$$|\Phi'-\Phi|\leq bB \tag{B20}$$

for all $b>0$ and all $\lambda\geq\Lambda$. Here B is some finite number that is independent of b and λ, but can depend on R. In what follows B will be a generic number with these properties: it need not always be the same number.

Consider next the contribution to $\Phi(J,J_1)$ in which the roles of the lines from 1 to 2 and 2 to 3 are interchanged:

$$\Phi_{(3,2)(2,1)}(J_1)=\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(e^{-ik\lambda X_3}-e^{-ik\lambda X_2})(e^{ik\lambda X_2}-e^{ik\lambda X_1})$$
$$\times\frac{(X_3-X_2)_\mu(-g^{\mu\nu})(X_2-X_1)_\nu}{((X_3-X_2)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_2-X_1)\cdot k+i0)} \tag{B21a}$$

and

$$\Phi'_{(3,2)(2,1)}(J_1,J)=\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(e^{ik\lambda X_3}-e^{-ik\lambda X_2})(e^{ik\lambda X_2+ik\Delta_2}-e^{ik\lambda X_1+ik\Delta_1})$$
$$\times\frac{(X_3-X_2)_\mu(-g^{\mu\nu})(X_2-X_1+\Delta_2\lambda^{-1}-\Delta_1\lambda^{-1})_\nu}{((X_3-X_2)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)(X_2-X_1+\Delta_2\lambda^{-1}-\Delta_1\lambda^{-1})\cdot k+i0)}. \tag{B21b}$$

Consider the difference $\Phi'-\Phi$ of the integrals defined in (B21b) and (B21a). For $\lambda\geq\Lambda$ one may complete the k^0 contour by adding in the lower-half plane a semicircle at $|k|=2b$. The arguments that led to (B20) show that the contribution from this semicircle also has a bound of the form (B20).

The completed contour can now be collapsed onto the poles, which are located at $k^0=\pm|\vec{k}|$. This leaves a d^3k integration in which the three remaining denominators all contain factors of $|\vec{k}|$. With the factor $|\vec{k}|^3$ separated out the denominator is left in a form that remains finite in the angular integration, due to the timelike character of the vectors (X_i-X_{i-1}) and $(X_i-X_{i-1}+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})$. Thus the quantities $\Delta_2\lambda^{-1}$ and $\Delta_3\lambda^{-1}$ in (B12b) again give corrections of order λ^{-1}, for $\lambda\geq\Lambda$, and by virtue of (B17), give a contribution to the integral that enjoys a bound bB. The difference of the remaining integral in (B21b) with the function Φ defined in (B21a) again enjoys a bound bB, due to (B19). Thus the difference $\Phi'-\Phi$ of the functions defined in (B21) enjoys a bound of the form (B20).

Consider next the contribution

$$\Phi_{(3,2)(3,1)}(J_1)=\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(e^{-ik\lambda X_3}-e^{-ik\lambda X_2})(e^{ik\lambda X_3}-e^{ik\lambda X_1})$$
$$\times\frac{(X_3-X_2)_\mu(-g^{\mu\nu})(X_3-X_1)_\nu}{((X_3-X_2)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_1)\cdot k+i0)}. \tag{B22a}$$

It will be taken together with

$$\Phi'_{(3,2)(3,1)}(J)=\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(e^{-ik\lambda X_3-ik\Delta_3}-e^{-ik\lambda X_2-ik\Delta_2})(e^{ik\lambda X_3+ik\Delta_3}-e^{ik\lambda X_1+ik\Delta_1})$$
$$\times\frac{(X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})_\mu(-g^{\mu\nu})(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})_\nu}{((X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})\cdot k+i0)} \tag{B22b}$$

Consider now the difference $\Phi'-\Phi$ of these two functions. Due to the inequalities $X_3^0>X_2^0>X_1^0$ one may, for $\lambda\geq\Lambda$ and for the terms containing factors $\exp(ik\lambda X_3)$ or $\exp(ik\lambda X_3+ik\Delta_3)$, distort the k^0 contour into the upper-half plane and obtain, as before, for these contributions to $\Phi'-\Phi$ a bound bB. For the remaining terms, which contain the factor $\exp(ik\lambda X_1)$ or $\exp(ik\lambda X_1+ik\Delta_1)$, one can complete the k^0 contour by a semicircle in the lower-half plane: the added contribution to $\Phi'-\Phi$ has, as before, a bound bB. The completed contour can now be contracted to the poles. The poles at $k^0=\pm|\vec{k}|$ again give terms with a bound bB.

The contribution to the integral in (B22a) from the pole at $(X_3-X_1)\cdot k=0$ is

$$\Phi^{\text{pole}}_{(3,2)(3,1)}(J_1)=\frac{e^2}{2}(-i)\int_\Omega \frac{d^3k}{(2\pi)^3}(e^{ik\lambda(X_3-X_2)}-1)\frac{(X_3-X_2)_\mu(-g^{\mu\nu})(X_3-X_1)_\nu}{((X_3-X_2)\cdot k)((k^0)^2-|\vec{k}|^2)(X_3{}^0-X_1{}^0)}\,, \tag{B23a}$$

where

$$k^0=\frac{(\vec{X}_3-\vec{X}_1)\cdot k}{X_3^0-X_1^0}\,. \tag{B23a$'$}$$

The companion pole contribution is

$$\Phi'^{\text{pole}}_{(3,2)(3,1)}(J)=\frac{e^2}{2}(-i)\int_\Omega \frac{d^3k}{(2\pi)^3}(e^{ik\lambda(X_3-X_2)+ik(\Delta_3-\Delta_2)}-1)$$

$$\times\frac{(X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})_\mu(-g^{\mu\nu})(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})}{(X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})\cdot k((k^0)^2-|\vec{k}|^2)(X_3^0-X_1^0+\Delta_3^0\lambda^{-1}-\Delta_1^0\lambda^{-1})}\,, \tag{B23b}$$

where

$$k^0=\frac{(\vec{X}_3-\vec{X}_1+\vec{\Delta}_3\lambda^{-1}-\Delta_1\lambda^{-1})\cdot\vec{k}}{(X_3^0-X_1^0+\Delta_3^0\lambda^{-1}-\Delta_1^0\lambda^{-1})}\,. \tag{B23b$'$}$$

The terms $(\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})_\nu$, $(\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})_\mu$, and $(\Delta_3^0\lambda^{-1}-\Delta_1^0\lambda^{-1})$ give contributions to (B23b) having a bound bB, by virtue of (B17) with X_1 replaced by X_3. The factor $((k^0)^2-|\vec{k}|^2)^{-1}$ evaluated as specified in (B23b$'$) is nonzero in the domain of integration and can be expressed as its value at $\lambda=\infty$ plus a correction term of the form $f(k/\lambda)/\lambda$, where f is bounded in the domain of integration for all $\lambda\geq\Lambda$. This term f/λ gives a contribution to the integral in (B23) that has a bound bB, by virtue of (B17) with X_1 replaced by X_3.

Insertion of the value of k^0 specified in (B23b$'$) gives

$$k\cdot(X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})=\vec{k}\cdot V(\lambda^{-1})=\vec{k}\cdot\vec{V}_0+\vec{k}\cdot\vec{W}\lambda^{-1}\,, \tag{B24a}$$

where

$$\vec{k}\cdot\vec{V}_0=k\cdot(X_3-X_2)\,|_{k\cdot(X_3-X_1)=0}=\vec{k}\cdot\left[-\frac{\vec{X}_3-\vec{X}_2}{X_3^0-X_2^0}+\frac{\vec{X}_3-\vec{X}_1}{X_3^0-X_1^0}\right](X_3^0-X_2^0) \tag{B24b}$$

and

$$\vec{W}=-(\vec{\Delta}_3-\vec{\Delta}_2)+(\vec{\Delta}_3-\vec{\Delta}_1)\left[\frac{X_3^0-X_2^0+\Delta_3^0\lambda^{-1}-\Delta_2^0\lambda^{-1}}{X_3^0-X_1^0+\Delta_3^0\lambda^{-1}-\Delta_1^0\lambda^{-1}}\right]$$

$$+(\vec{X}_3-\vec{X}_1)\left[\frac{\Delta_3^0-\Delta_2^0}{X_3^0-X_2^0}-\frac{\Delta_3^0-\Delta_1^0}{X_3^0-X_1^0}\right]\frac{X_3^0-X_2^0}{(X_3^0-X_1^0+\Delta_3^0\lambda^{-1}-\Delta_1^0\lambda^{-1})}\,. \tag{B24c}$$

Thus the difference of the pole terms shown in (B23a) and (B23b) can be expressed as

$$\Phi'^{\text{pole}}-\Phi^{\text{pole}}=0(b)+(X_3-X_2)_\mu(-g^{\mu\nu})(X_3-X_1)_\nu(X_3^0-X_1^0)$$

$$\times\frac{e^2}{2}(-i)\int_{|\vec{k}|<b}\frac{d^3k}{(2\pi)^3}\left.\frac{1}{(k^0)^2-|\vec{k}|^2}\right|_{k\cdot(X_3-X_1)=0}(-1)\left[\frac{e^{i\lambda\vec{k}\cdot\vec{V}}-1}{\vec{k}\cdot\vec{V}}-\frac{e^{i\lambda\vec{k}\cdot\vec{V}_0}-1}{\vec{k}\cdot\vec{V}_0}\right]. \tag{B25}$$

Let $v=|\vec{V}(\lambda^{-1})|=v(\lambda^{-1})$ and $v_0=|\vec{V}_0|=v(0)$. Let $\cos\theta$ and $\cos\theta_0$ be defined by $\vec{k}\cdot\vec{V}=kv\cos\theta$ and $\vec{k}\cdot\vec{V}_0=kv_0\cos\theta_0$, respectively. Then one may define

$$f(v\cos\theta,\lambda^{-1})=v^{-1}\int_0^{2\pi}d\phi\left.\frac{|\vec{k}|^2}{(k^0)^2-|\vec{k}|^2}\right|_{\substack{k\cdot(X_3-X_1)=0\\ \cos\theta\ \text{fixed}}} \tag{B26a}$$

and

$$f_0(v_0\cos\theta_0)=v_0{}^{-1}\int_0^{2\pi}d\phi_0\,\frac{|\vec{k}|^2}{(k^0)^2-|\vec{k}|^2}\Bigg|_{\substack{k\cdot(X_3-X_1)=0\\ \cos\theta_0\text{ fixed}}}\,, \tag{B26b}$$

where (θ,ϕ) and (θ_0,ϕ_0) are two sets of angular coordinates. The function $f_0(v_0\cos\theta)$ is the limit of $f(v(\lambda^{-1})\cos\theta,\lambda^{-1})$ as $\lambda^{-1}\to 0$, and

$$f(v(\lambda^{-1})\cos\theta,\lambda^{-1})=f_0(v_0\cos\theta)+\lambda^{-1}f_1(v_0\cos\theta,\lambda^{-1})\,, \tag{B26c}$$

where $f_1(\cos\theta,\lambda^{-1})$ is bounded for $\lambda\ge\Lambda$ and $1\ge\cos\theta\ge-1$. Because of symmetry only the real parts of $\exp(i\lambda\vec{k}\cdot\vec{V})$ and $\exp(i\lambda\vec{k}\cdot\vec{V}_0)$ contribute to the integral in (B25). Thus, using (B26), one may write $(-i)$ times this integral as

$$\begin{aligned}
&\frac{(-i)(-1)}{(2\pi)^3}\int_0^b dk\int_{-1}^{1}d\cos\theta\left[vf(v\cos\theta,\lambda^{-1})\frac{(e^{i\lambda kv\cos\theta}-1)}{kv\cos\theta}-v_0f_0(v_0\cos\theta)\left[\frac{e^{i\lambda kv_0\cos\theta}-1}{kv_0\cos\theta}\right]\right]\\
&=\frac{(-1)}{(2\pi)^3}\int_0^b\frac{dk}{k}\int_{-v\lambda k}^{v\lambda k}dx\,f(x/\lambda k,\lambda^{-1})\frac{\sin x}{x}+\frac{(+1)}{(2\pi)^3}\int_0^b\frac{dk}{k}\int_{-v_0}^{v_0}dx\,f_0(x/\lambda k)\frac{\sin x}{x}\\
&=\frac{(-1)}{(2\pi)^3}\int_0^b\frac{dk}{k}\frac{1}{\lambda}\int_{-v_0\lambda k}^{v_0\lambda k}dx\,f_1(x/\lambda k,\lambda^{-1})\frac{\sin x}{x}-\frac{1}{2\pi^3}\int_0^b\frac{dk}{k}\left[\int_{v_0\lambda k}^{v\lambda k}dx+\int_{-v\lambda k}^{-v_0\lambda k}dx\right]f(x/\lambda k,\lambda^{-1})\frac{\sin x}{x}\,.
\end{aligned} \tag{B27}$$

By virtue of the boundedness of $f(x/\lambda k,\lambda^{-1})$ and $f_1(x/\lambda k,\lambda^{-1})$ both integrals in the last line of (B27) enjoy bounds of the form bB. Hence the difference $\Phi'^{\text{pole}}-\Phi^{\text{pole}}$ of the pole contributions defined in (B23) enjoy a bound of this form.

Consider next the contributions

$$\begin{aligned}\Phi_{(3,2)(3,1)}(J_1,J)=\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}&(e^{-ik\lambda X_3}-e^{-ik\lambda X_2})(e^{+ik\lambda X_3+ik\Delta_3}-e^{ik\lambda X_1+ik\Delta_1})\\
&\times\frac{(X_3-X_1)_\mu(-g^{\mu\nu})(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})_\nu}{((X_3-X_2)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})\cdot k+i0)}\end{aligned} \tag{B28a}$$

and

$$\begin{aligned}\Phi''_{(3,1)(3,2)}(J,J_1)=\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}&(e^{-ik\lambda X_3-ik\Delta_3}-e^{-ik\lambda X_1-ik\Delta_1})(e^{ik\lambda X_3}-e^{ik\lambda X_2})\\
&\times\frac{(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})_\mu(-g^{\mu\nu})(X_3-X_2)_\nu}{((X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-}1)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_2)\cdot k+i0)}\,.\end{aligned} \tag{B28b}$$

In Φ one pushes the k^0 contour into the upper-half plane for the terms with $\exp(i\lambda kX_3+ik\Delta_3)$, and completes the contour in the lower-half plane for terms with $\exp(i\lambda kX_1+ik\Delta_1)$. In Φ'' one pushes the k^0 contour into the upper-half plane for the terms with $\exp(-i\lambda kX_1-ik\Delta_1)$ and completes the contour in the lower-half plane for terms with $\exp(-i\lambda kX_3-ik\Delta_3)$. The importance of this grouping $\Phi''-\Phi$ is that the contributions from the poles at $(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})\cdot k=0$ cancel exactly, by virtue of the antisymmetry of this pole contribution to $\Phi''-\Phi$.

For the remaining partial cancellations that give the bounds of the form bB one groups Φ of (B28a) with

$$\begin{aligned}\Phi'_{(3,2)(3,1)}(J,J_1)=\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}&(e^{-ik\lambda X_3-ik\Delta_3}-e^{-ik\lambda X_2-ik\Delta_2})(e^{ik\lambda X_3}-e^{ik\lambda X_1})\\
&\times\frac{(X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2{}^{-1})_\mu(-g^{\mu\nu})(X_3-X_1)_\nu}{((X_3-X_2+\Delta_3\lambda^{-1}-\Delta_2\lambda^{-1})\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)((X_3-X_1)\cdot k+i0)}\,.\end{aligned} \tag{B28c}$$

The proof of the bound $|\Phi'-\Phi|\le bB$ goes as before, except that one need not consider contributions from the poles at $(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})\cdot k=0$ and $(X_3-X_1)\cdot k=0$, due to the cancellation mentioned above, and the analogous cancellation between the poles of $\Phi'_{(3,2)(3,1)}(J,J_1)$ and $\Phi'''_{(3,1)(3,2)}(J_1,J)$ at $(X_3-X_1)\cdot k=0$.

Consider next the contributions to $\Phi(J,J_1)$ coming from the (3,1) contribution to $\bar{J}_{1\nu}(k)$ and the (3,1) contribution to $J_{1\mu}(k)$:

$$\Phi_{(3,1)(3,1)}(J_1)=\frac{e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}(e^{-ik\lambda_3}-e^{-ik\lambda X_1})(e^{ik\lambda X_3}-e^{ik\lambda X_1})\frac{(X_3-X_1)_\mu(-g^{\mu\nu})(X_3-X_1)_\nu}{((X_3-X_1)\cdot k+i0)^2((k^0+i0)^2-|\vec{k}|^2)}\,. \tag{B29}$$

In the contributions with a factor $\exp(ik\lambda X_3)$ one can move the k^0 contour into the upper-half plane without encountering any exponentials that become large as $\lambda\to\infty$. Thus one finds a uniform bound as $\lambda\to\infty$. The remaining terms are

$$\Phi^{\mathrm{rem}}_{(3,1)(3,1)}(J_1)=\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(1-e^{-ik\lambda(X_3-X_1)})\frac{(X_3-X_1)_\mu(-g^{\mu\nu})(X_3-X_1)_\nu}{((X_3-X_1)\cdot k+i0)^2((k^0+i0)^2-|\vec{k}|^2)} . \tag{B30}$$

The $(X_3-X_1)\cdot k$ contour in (B30) can be completed by a path in the lower-half plane, and then contracted to the poles. The poles at $k^0=\pm|\vec{k}|$ give contributions that enjoy a bound of the form $C+D\ln(b\lambda)\theta(b\lambda-1)$. The contribution from the double pole arises from the derivative of the remaining factors, evaluated at the pole. This derivative acting on the factor $k^{-2}\chi_\Omega(k)$ gives no contribution, due to the zero in the numerator, but acting on the exponential it gives the contribution

$$\Phi^{\mathrm{pole}}_{(3,1)(3,1)}=\frac{e^2}{2}\frac{\lambda(X_3-X_1)_\mu(-g^{\mu\nu})(X_3-X_1)_\nu}{(X_3{}^0-X_1{}^0)}\int_\Omega \frac{d^3k}{(2\pi)^3}\frac{1}{(k^0)^2-|\vec{k}|^2}\bigg|_{k\cdot(X_3-X_1)=0} . \tag{B31}$$

This contribution to Φ increases linearly with the distance $(\lambda X_3-\lambda X_1)$. It gives a contribution to $\exp i\Phi$ that is the same as that of a mass term. The magnitude of the effective mass shift induced by this term equals the classical-photon contribution to the usual lowest-order Dirac-particle self-energy diagram, apart from the factor of $-\frac{1}{2}$ stemming from the occurrence of this factor in $-\frac{1}{2}J_{1\mu}$.

The Dirac-particle self-energy counterterm has not yet been taken into account. It cancels precisely the above self-energy contribution to Φ: one may omit the self-energy contribution to the operators $U(L(x))$, and consider the mass m to be the physical mass of the particle.

Consider next the contribution to $\Phi_{(3,1)(3,1)}$ coming from the (3,1) part of $\bar{J}_{1\mu}(k)$ and the (3,1) part of $J_\mu(k)$:

$$\Phi'_{(3,1)(3,1)}(J_1,J)=\frac{e^2}{2}\int_\Omega \frac{d^4k}{(2\pi)^4}(e^{-ik\lambda X_3}-e^{-ik\lambda X_1})(e^{ik\lambda X_3+ik\Delta_3}-e^{ik\lambda X_1+ik\Delta_1})$$
$$\times\frac{(X_3-X_1)_\mu(-g^{\mu\nu})(X_3-X_1+\Delta_3/\lambda-\Delta_1/\lambda)_\nu}{((X_3-X_1)\cdot k+i0)((k^0+i0)^2-|\vec{k}|^2)}\frac{1}{(X_3-X_1+\Delta_3/\lambda-\Delta_1/\lambda)\cdot k+i0} . \tag{B32}$$

In the two terms containing $\exp(ik\lambda X_3+ik\Delta_3)$ one may distort the k^0 contour into the upper-half plane. They combine with the like contributions to (B29) to give a difference $\Phi'-\Phi$ whose magnitude enjoys a bound bB. In the remaining two terms one completes the k^0 contour in lower-half plane. This contributes to $\Phi'-\Phi$ a term with bound bB. Then contracting the completed contour to the poles one obtains from the poles at $k^0=\pm|\vec{k}|^2$ contributions to Φ' that combine with those of Φ to give contributions to $\Phi'-\Phi$ with a bound bB. The other pole gives a contribution to Φ' of the form

$$\Phi'^{\mathrm{pole}}_{(3,1)(3,1)}(J_1,J)=\frac{e^2}{2}(-i)\int_\Omega \frac{d^3k}{(2\pi)^3}(e^{ik\Delta_3}-e^{ik\Delta_1})\frac{(X_3-X_1)_\mu(-g^{\mu\nu})(X_3-X_1+\Delta_3\lambda^{-1}-\Delta_1\lambda^{-1})_\nu}{(X_3{}^0-X_1{}^0)[(k^0)^2-|\vec{k}|^2]}\frac{1}{(\Delta_3/\lambda-\Delta_1/\lambda)\cdot k} , \tag{B33}$$

$$=\frac{e^2}{2}(-i)\int_\Omega \frac{d^3k}{(2\pi)^3}\left[\frac{e^{ik\Delta_3}-e^{ik\Delta_1}}{(\Delta_3-\Delta_1)\cdot k}\right]\frac{(X_3-X_1)_\mu(-g^{\mu\nu})[\lambda(X_3-X_1)+\Delta_3-\Delta_1]_\nu}{(X_3{}^0-X_1{}^0)[(k^0)^2-|\vec{k}|^2]} , \tag{B34}$$

where k^0 is evaluated by using

$$(X_3-X_1)\cdot k=-\left[\frac{\Delta_3}{\lambda}-\frac{\Delta_1}{\lambda}\right]\cdot k . \tag{B35}$$

This contribution comes from $\bar{J}_{1\mu}(k)(-g^{\mu\nu})J_\nu(k)$. The similar contribution from $\bar{J}_\mu(k)(-g^{\mu\nu})J_{1\nu}(k)$ is obtained by replacing k by $-k$. These two integrals are equal, because of the symmetry of the integral under the replacement of the variable k by $-k$. Thus their difference vanishes. Hence the only contributions linear in λ come from the terms $\bar{J}_{1\mu}(k)(-g^{\mu\nu})J_{1\nu}(k)$ and $\bar{J}_\mu(k)(-g^{\mu\nu})J_\nu(k)$. The contributions from these two forms that increase with λ cancel, even without considering the self-mass counterterms. And the remaining terms have a bound of the form bB. Thus the sum of the (3,1)(3,1) contributions enjoys a bound of the form bB.

All remaining contributions succumb to the methods shown above, and one obtains the bound

$$|\Phi_\Omega(J,J_1)|\le bB , \tag{B36}$$

where B is some number that is independent of b and λ.

According to (B7) one has $U(L(x))U^{-1}(L(\lambda X))=U'(L(x)-L(\lambda X))\exp(i\Phi)$. Transposing the two operators on the left-hand side gives

$$U_\Omega{}^{-1}(L(\lambda X))U(L(x))=U'(L(x)-L(\lambda X))\exp(-i\Phi) . \tag{B37}$$

Thus

$$U_\Omega{}^{-1}(L(\lambda X))U_\Omega(L(x))=U\exp[-i\Phi_\Omega(J,J_1)] , \tag{B38}$$

where

$$U=\exp(\langle a^*\cdot J\rangle)\exp(-\langle J^*\cdot a\rangle)\exp(-\tfrac{1}{2}\langle J^*\cdot J\rangle) \tag{B39}$$

and J represents the vector function with components

$$J_\mu(L(x)-L(\lambda X),k)=e\sum_{i=1}^{3}\left[\frac{(e^{ik\lambda X_i+ik\Delta_i}-e^{ik\lambda X_{i-1}+ik\Delta_{i-1}})((X_i-X_{i-1})+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})_\mu}{(X_i-X_{i-1}+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})\cdot k} - \frac{(e^{ik\lambda X_i}-e^{ik\lambda X_{i-1}})(X_i-X_{i-1})_\mu}{(X_i-X_{i-1})\cdot k}\right]_\Omega . \quad \text{(B40)}$$

In calculating U this function J is evaluated at $k^2=0$. Owing to the spacelike character of (X_i-X_{i-1}) and $(X_i-X_{i-1}+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})$ each of the denominators in (B38), evaluated at $k^2=0$, is $|\vec{k}|$ times a function of angles that is nonvanishing over the physical domain of integration. Thus for $\lambda\geq\Lambda$ and physical k satisfying $k^2=0$ one may write

$$\frac{(X_i-X_{i-1}+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})_\mu}{(X_i-X_{i-1}+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})\cdot k}=\frac{(X_i-X_{i-1})_\mu}{(X_i-X_{i-1})\cdot k}+\frac{1}{(X_i-X_{i-1}+\Delta_i\lambda^{-1}-\Delta_{i-1}\lambda^{-1})\cdot k}\frac{f_\mu(\lambda,\theta,\psi)}{\lambda} , \quad \text{(B41)}$$

where $f_\mu(\lambda,\theta,\phi)$ is bounded for $\lambda\geq\Lambda$ and (θ,ϕ) in the physical range. This expression (B41) may be inserted into (B40). The second term of (B41) then gives a contribution to $J_\mu(k)$ that is bounded for $\lambda\geq\Lambda$ and (θ,ϕ) in the physical range. The first term in (B41) gives a contribution to (B40) that combines with the second term of (B40) to give a contribution to $J_\mu(k)$ that also is bounded for $\lambda\geq\Lambda$ and (θ,ψ) in the physical region.

Because $J(k)$ is bounded

$$N\equiv(\langle J^*\cdot J\rangle)^{1/2} \quad \text{(B42)}$$

is of order b.

One may introduce a set of orthonormal basis functions $f_i(k)$ over the portion Ω of k space such that the first of these functions is $f_1(k)=J(k)/N$. Then the operator U of (B39) has the form

$$U(N)=\exp(\langle a^*\cdot f_1\rangle)\exp(-\langle f_1^*\cdot a\rangle N)\exp(-\tfrac{1}{2}N^2) , \quad \text{(B43)}$$

where N is order b.

In the formula for transition probabilities the contribution from $A_{\rm rem}(\lambda x)$ has, according to (7.2), (7.3), and (7.4), a factor

$$F(N)=(U(N)e^{-i\Phi(J,J_1)}-I_\Omega)\tilde{F}^{D'}_{{\rm opr}\Omega}\rho_{{\rm in}\Omega} . \quad \text{(B44)}$$

To calculate the dependence of F upon b one may introduce the coherent states[5,10]

$$|z\rangle=(e^{\langle a^*\cdot f_1\rangle z}e^{-\langle f_1^*\cdot a\rangle z^*}e^{-zz^*/2})|0\rangle . \quad \text{(B45)}$$

Then

$$U(N)|z\rangle=|z+N\rangle e^{-N(z-z^*)/2} . \quad \text{(B46)}$$

Thus for small N and Φ one has

$$(U(N)e^{-i\Phi}-1)|z\rangle\simeq|z+N\rangle-|z\rangle-i\Phi|z\rangle - \tfrac{1}{2}N(z-z^*)|z\rangle . \quad \text{(B47)}$$

The vector $|z+N\rangle-|z\rangle$ is small for small N and $N|z\rangle$[11]:

$$|(z+N\rangle-|z\rangle|\leq\sqrt{2}(|z|+|z+N|)^{1/2}N^{1/2} . \quad \text{(B48)}$$

The normalization factor N is of order b. But what is z?

Consider first the contribution to (B44) coming from the part $\tilde{F}^{D0}_{{\rm opr}\Omega}$ of $\tilde{F}^{D}_{{\rm opr}\Omega}$ that corresponds to the original diagram D. This factor $\tilde{F}^{D0}_{{\rm opr}\Omega}$ gives no contribution to the photon space operator. Thus the amplitude of state $|z\rangle$ is given by the decomposition[12]

$$\rho_{{\rm in}\Omega}=\int\frac{d^2z}{\pi}|z\rangle\langle z|\rho_{{\rm in}\Omega} . \quad \text{(B49)}$$

Now the expectation value of the number of photons in the state $|z\rangle$ is $|z|^2$.[13] And the expectation value of the energy in this state is

$$\bar{E}=z^2E_1 , \quad \text{(B50)}$$

where E_1 is the expectation value of the energy in the state $\langle a^*\cdot f_1\rangle|0\rangle$. Since the wave function $f_1(k)$ in this state is $\sim\theta(b-k)/b$ the energy E_1 is

$$E_1\sim\int_0^b\frac{dk^3}{k}k\cdot(1/n)^2 \sim b . \quad \text{(B51)}$$

By the principle of equipartition of energy the energy residing in each low-energy mode of the photon field should be approximately the same. Thus one should expect the $\bar{E}$ in (B4a) to be roughly independent of the mode. But then the expected dependence of z on b is given by

$$|z|\sim b^{-1/2} . \quad \text{(B52)}$$

But if $\langle z|\tilde{F}^{D0}_{{\rm opr}\Omega}\rho_{{\rm in}\Omega}$ is concentrated near values of z satisfying (B52) then (B47), (B48), and (B36) show that

$$|F(N)|\to 0 \quad \text{(B53)}$$

as $b\to 0$. In fact, one could tolerate a growth as large as $|z|\sim b^{-1+\epsilon}$ $(\epsilon>0)$ and still obtain the result (B53).

The results in paper II will show that the very shoft photons emitted and absorbed by the operator part of $F^D_{\rm opr}(x)$ produce only very mild effects that do not upset this result (B53).

The bounds obtained above refer to the contributions from the points x in

$$\mathscr{R}(R,\lambda X)=\{x;\,|x_i-\lambda X_i|_{\rm Eucl}\leq R\} . \quad \text{(B54)}$$

To obtain a bound on the contributions to $A_{\text{rem}}(\lambda X)$ from points outside $\mathscr{R}(R,\lambda X)$ consider first the points x outside the set $\mathscr{R}(\lambda^{\eta},\lambda X)$, where $\eta=0.01$. And consider initially the part $A_{\text{rem}}{}^{0}(\lambda X)$ of $A_{\text{rem}}(\lambda X)$ that comes from the $F^{D}(x)$ part of $F^{D}_{\text{opr}}(x)$.

Equation (7.3) shows that the operator part of the integrand in $A_{\text{rem}}(\lambda X)$ has norm ≤ 2. And the function $F^{D}(x)$ is bounded. (Ultraviolet cutoffs are assumed.) The product of the wave functions falls off faster than any power of $|x-\lambda X|$. Thus for any $\epsilon>0$, however small, and any $C>0$, however small, one can find a $\Lambda(\epsilon,C)\equiv\Lambda_1$ such that for all $\lambda>\Lambda_1$ the sum of contributions to $A^{0}_{\text{rem}}(\lambda X)$ from points x outside $\mathscr{R}(\lambda^{\eta},\lambda X)$ is an operator with norm less than $(\epsilon/4)C\lambda^{-9/2}$:

$$|A^{0}_{\text{rem}}(\lambda X)^{\mathscr{R}(\lambda^{\eta},\lambda)}|<\frac{\epsilon}{4}C\lambda^{-9/2}(\lambda>\Lambda_1)\ . \tag{B55}$$

Consider next the contributions to $A^{0}_{\text{rem}}(\lambda X)$ from points x inside $\mathscr{R}(\lambda^{\eta},\lambda X)$ and outside $\mathscr{R}(R,\lambda X)$. The operator part of the integrand still has norm ≤ 2. The function $|F^{D}(x)|$ has, for all points $x\in\mathscr{R}(\lambda^{\eta},\lambda X)$ for $\lambda>\Lambda_2\gg 1$, a bound of the form

$$|F^{D}(x)|\leq C'\lambda^{-9/2}\ \ [x\in\mathscr{R}(\lambda^{\eta},\lambda X)\lambda>\Lambda_2]\ . \tag{B56}$$

Inserting the bound $2C'\lambda^{-9/2}$ on the norm of the parts of the integrand other than the wave functions one may obtain a weaker bound by extending the region of integration of the magnitude of the product of the wave functions to all points x outside $\mathscr{R}(R,\lambda X)$. The faster than any power falloff of the absolute value of the products of the wave functions ensures the convergence of this new bounding integral. This procedure gives a bound that depends on λ only via the factor $\lambda^{-9/2}$, and that falls off faster than any power of $\mathscr{R}$, due to the falloff of the absolute value of the products of the wave functions. Thus for some sufficiently larger R the contribution to $A^{0}_{\text{rem}}(\lambda X)$ from points x inside $\mathscr{R}(\lambda^{\eta},\lambda X)$ and outside $\mathscr{R}(R,\lambda X)$ has a bound of the form $(\epsilon/4)C\lambda^{-9/2}$:

$$|A^{0}_{\text{rem}}(\lambda X)^{\mathscr{R}(R,\lambda X)}_{\mathscr{R}(\lambda^{\eta},\lambda X)}|<\frac{\epsilon}{4}C\lambda^{-9/2}\ . \tag{B57}$$

For the remaining points x in $\mathscr{R}(R,\lambda X)$ one uses the main result of this appendix: for some fixed Λ and for any R, however large, the norm

$$|U_{\Omega(b)}{}^{-1}(L(\lambda X))U_{\Omega(b)}(L(x))-1| \tag{B58}$$

tends to zero with b uniformly over the set

$$\{(\lambda,x);\lambda>\Lambda,x\in\mathscr{R}(R,\lambda X)\}\ .$$

This constant Λ can be made larger than Λ_1 and Λ_2. Then combining this bound on (B58) with (B56) one concludes that for some sufficiently small $b\equiv b(\epsilon,c,R)>0$ the contribution to $A^{0}_{\text{rem}}(\lambda X)$ for points $x\in\mathscr{R}(R,\lambda X)(\lambda>\Lambda)$ satisfies

$$|A^{0}_{\text{rem}}(\lambda X)_{\mathscr{R}(R,\lambda X)}|<\frac{\epsilon}{2}C\lambda^{9/2}\ \ (\lambda>\Lambda)\ . \tag{B59}$$

Then the sum of (B59), (B57), and (B55) gives

$$|A^{0}_{\text{rem}}(\lambda X)|<\epsilon C\lambda^{-9/2}\ \ (\lambda>\Lambda)\ . \tag{B60}$$

The constant $\epsilon<0$ is taken to be the number occurring in (7.13), and the constant C is constructed from the $F^{D}(x)$ parts of the three functions defined in (7.45). [See also (7.26).]

The above discussion dealt with the part $A^{0}_{\text{rem}}(\lambda X)$ of $A_{\text{rem}}(\lambda X)$. However, the good infrared properties of $F^{D}_{\text{opr}}(x)$ ensure that the arguments carry over to the full operator $A_{\text{rem}}(\lambda X)$. In particular, the crucial property (B56) holds also for $F^{D}_{\text{opr}}(x)$, and the soft photons emitted and absorbed by F^{D}_{opr} do not upset the required operator properties. A detailed justification of the extension to $F^{D}_{\text{opr}}(x)$ depends on the detailed results to be described in paper II.

APPENDIX C

The self-energy and wave-function renormalization effects of classical photons on charged-particle propagators are calculated in this appendix.

The starting point is the one-particle propagator with a single classical-photon correction:

$$S_F{}^{1}(z)=\frac{e^2}{2}\int\frac{d^4p}{(2\pi)^4}e^{-ipz}\int_{\Omega}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i0}\frac{1}{(\hat{z}\cdot k)^2}\left[\frac{1}{\not p-m}\not k\frac{1}{\not p+\not k-m}\not k\frac{1}{\not p-m}+\frac{1}{\not p-m}\not k\frac{1}{\not p-\not k-m}\not k\frac{1}{\not p-m}\right]. \tag{C1}$$

The two terms arise from the cases in which the photon enters the charged line before or after the point at which it leaves this line, respectively. The two terms are equal if the integration region Ω and the factor $(\hat{z}\cdot k)^2$ are invariant under the transformation $k\to -k$.

A double application of the Ward identity (2.8) gives

$$S_F{}^{1}(z)=\frac{e^2}{2}\int\frac{d^4p}{(2\pi)^4}e^{-ipz}\int_{\Omega}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i0}\frac{1}{(\hat{z}\cdot k)^2}\times\left[\frac{1}{\not p-m}\not k\frac{1}{\not p-m}-\frac{1}{\not p-m}+\frac{1}{\not p+\not k-m}+\frac{1}{\not p-m}(-\not k)\frac{1}{\not p-m}-\frac{1}{\not p-m}+\frac{1}{\not p-\not k-m}\right]. \tag{C2}$$

If $(\hat{z}\cdot k)^{-2}=z^2(z\cdot k)^{-2}$ is resolved by the principal-value rule, or has the form $(\hat{z}\cdot k+i0)(\hat{z}\cdot k-i0)$, and is therefore symmetric under $k\to -k$, and if the region Ω is symmetric, then the two terms with double pole $(\not p-m)^{-2}$ are individually zero by symmetry. In any case they cancel and leave

$$S_F^1(z)=\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ipz}}{\not p-m}\left[\frac{e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i0}\frac{z_\mu(-g^{\mu\nu})z_\nu}{(z\cdot k)(z\cdot k)}(-2+e^{-ikz}+e^{+ikz})\right]$$

$$=S_F(z)i\Delta(z)\ , \tag{C3}$$

where

$$i\Delta(z)=\frac{-e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i0}\frac{z_\mu(-g^{\mu\nu})z_\nu}{(z\cdot k)^2}(e^{ikx_2}-e^{ikx_1})(e^{-ikx_2}-e^{-ikx_1})$$

$$=\frac{-e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\frac{i(-g^{\mu\nu})}{k^2+i0}\int_{x_1}^{x_2}dx_\mu e^{ikx}\int_{x_1}^{x_2}dx'_\nu e^{-ikx'}\ . \tag{C4}$$

Inclusion of the contributions from all classical photons gives

$$S'_F=S_F(z)e^{i\Delta(z)}\ , \tag{C5}$$

which is closely connected to (2.14) and (2.17).

The function $\Delta(z)$ is

$$\Delta(z)=\frac{-e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i0}\left[\frac{1}{\hat z\cdot k}\right]^2(e^{-ikz}+e^{+ikz}-2)$$

$$=-z\Delta m+a+ib+r(z)+is(z)\ , \tag{C6}$$

where, for $\hat z^2>0$ and $\hat z^0>0$, and with $\omega=+(\vec k\cdot\vec k)^{1/2}$,

$$\Delta m=\frac{-e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}2\pi\delta(\hat z\cdot k)\ , \tag{C7}$$

$$a=\frac{e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^0+i0)^2-\omega^2}\frac{1}{(\hat z\cdot k+i0)^2}+\frac{1}{(k^2-i0)^2-\omega^2}\frac{1}{(\hat z\cdot k-i0)^2}\right]\ , \tag{C8}$$

$$b=+\frac{e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\left[\frac{2\pi\delta(\omega+k^0)-2\pi\delta(\omega-k^0)}{2\omega(\hat z\cdot k)^2}\right]\ , \tag{C9}$$

$$r(z)=\frac{-e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\left[\frac{e^{ikz}}{(k^0+i0)^2-\omega^2}\frac{1}{(\hat z\cdot k+i0)^2}+\frac{e^{-ikz}}{(k^0-i0)^2-\omega^2}\frac{1}{(\hat z\cdot k-i0)^2}\right]\ , \tag{C10}$$

and

$$s(z)=\frac{-e^2}{2}\int_\Omega\frac{d^4k}{(2\pi)^4}\left[\frac{2\pi\delta(\omega+k^0)e^{ikz}-2\pi\delta(\omega-k^0)e^{-ikz}}{2\omega(\hat z\cdot k)^2}\right]\ . \tag{C11}$$

The quantity Δm is a mass shift, and a is a wave-function renormalization. The quantities b and s are zero if Ω and $(\hat z\cdot k)^2$ are symmetric under $k\to -k$. The function $r(z)$ tends to zero as z tends to infinity.

The self-energy contribution (C7) is the classical-photon part of the full self-energy. As such it is canceled by the classical-photon part of the self-energy counterterm.

In the context of the calculation of (7.20) the above calculations take into account all contributions in which there is a double pole $(\hat z\cdot k)^{-2}$. Taking together all four contributions of this kind yields the numerator factor $(-2+e^{ikz}+e^{-ikz})$, which vanishes for $\hat z\cdot k=0$. The vanishing of the numerator at $\hat z\cdot k=0$ is important: it means that the derivative associated with the double pole $(\hat z\cdot k)^{-2}$ acts only on the exponentials in the factor $(-2+e^{ik\cdot z}+e^{-ik\cdot z})$.

To take advantage of this numerator zero one should, in the calculation of (7.20), initially combine all double-pole contributions in the way done here, and then afterward associate the z-independent contribution $a/2$ with the vertex on each end of the line under consideration.

At a later stage of the calculations [cf. (7.38)] the coherent states generated by $U(L(\lambda X))$ are introduced, and the operator $U(L(x))$ is replaced by $U^{-1}(L(\lambda X)U(L(x))$. The various contributions to $U(L(x))$ from the terms $J_i^*J_j$ with $i\neq j$ are either mass-renormalization terms, which are canceled by counterterms, or do not contribute in the large (x_i-x_j) limit, or have the form e^a, with a independent of x. These latter terms drop out of $U^{-1}(L(\lambda X))U(L(x))$. Thus only the $J_i^*J_i$ terms survive. For each of these individual terms $J_i^*J_i$ one can perform the transformation shown in (7.42), in order to obtain the results given by (7.47) and (7.52). Note that no double poles appear in these final formulas.

APPENDIX D

The purpose of this appendix is to show that the contributions to the probability $P_{\rm dom}(\lambda X)$ from the $J_i^* J_j$ $(i \neq j)$ contributions to the phase $\Phi^{\Omega}(L(x))$ defined in (7.20a) fall off faster than λ^{-9}.

The full current $J_\mu(L(x),k)$ defined in (7.21) is a sum of three terms, one for each line of $L(x)$. Thus J^*J decomposes into nine terms. The diagonal terms, which correspond to the contribution from the same line in both J and J^*, were dealt with in Appendix C.

Let J_{ij} be the contribution to J corresponding to the line segment of $L(x)$ that runs between vertex i and j:

$$J_{ij\mu}(L(x),k) = -ie\frac{(x_i - x_j)_\mu}{(x_i - x_j)\cdot k}(e^{ikx_i} - e^{ikx_j}) \ . \qquad \text{(D1)}$$

Consider first the points x in $\mathscr{R}(\lambda^\eta, \lambda X)$, for $\lambda > \Lambda \gg 1$, and $0 < \eta \ll 1$. Then $x_3^0 > x_2^0 > x_1^0$, and the k^0 contour may therefore be distorted into the lower-half plane for the term $J_{32}^* J_{21}$ and into the upper-half plane for the terms $J_{21}^* J_{32}$. Since there are no actual poles at the points $(x_i - x_j)\cdot k = 0$ this distortion is allowed, provided one adds appropriate contributions $\delta^{\pm}(k^2)$ corresponding to the poles of k^2 that have to be crossed. These $\delta^{\pm}(k^2)$ contributions are similar to the ones already discussed in connection with (7.20b), and give faster than λ^{-9} falloff.

With the contours distorted in this way there is exponential falloff as $\lambda \to \infty$ for the $J_i^* J_j$ $(i \neq j)$ parts, except for the contributions from the ends of the k_0 contours. But the end-point contributions fall off linearly with λ^{-1}, as one sees from the fact that

$$(-i\lambda) \int_0^{i\epsilon} e^{ik\lambda} dk = (1 - e^{-\epsilon\lambda}) \qquad \text{(D2)}$$

tends to unity as λ tends to infinity with ϵ fixed.

Having established the linear falloff of this integral the rest of the argument proceeds as in the text: The bound $C\lambda^{-9+8\eta}$ on the remaining factors in $P_{\rm dom}(\lambda X)$ of (7.18) arises from the $C'\lambda^{-9/2}$ bound on $|F_{\rm opr}^D(x)|$ for x in $\mathscr{R}(\lambda^\eta, \lambda X)$, and from the bound $C''\lambda^{4\eta}$ on the volume of $\mathscr{R}(\lambda^\eta, \lambda X)$. Thus for $\eta < \frac{1}{8}$ the λ^{-1} falloff overcomes the $\lambda^{8\eta}$ increase, and one is left with a better than λ^{-9} falloff.

For the term $J_{32}^* J_{31}$ one may distort the k contour into the region

$$\{k; {\rm Im}k\cdot(x_3 - x_1) < 0, {\rm Im}k\cdot(x_2 - x_1) < 0, {\rm Im}k\cdot(x_3 - x_2) > 0\} \ . \qquad \text{(D3)}$$

This distortion into the imaginary k space has a spacelike direction, but yields the same λ^{-1} falloff that was obtained above for the pure timelike distortion. The rest of the argument then follows as before.

For the term $J_{31}^* J_{32}$ one distorts into the image of (D3) under inversion $k \to -k$. The other terms are dealt with similarly. In this way every $J_i^* J_j (i \neq j)$ part of J^*J gives a contribution to (7.20a) that falls off at least linearly in λ^{-1}, and hence a contribution to $P_{\rm dom}(\lambda X)$ that falls off faster than λ^{-9}.

APPENDIX E

Consider first the Feynman coordinate-space function $F(x)$ corresponding to the diagram D_1 of Fig. 4. Introduce the following relabeling: let $i = (1,2,3,4)$ label cyclically the internal lines of D_1, and also the vertices of D_1. The function $F(x)$ is then essentially a product of the four Feynman propagators $D_i(x_i - x_{i-1})$, one for each of the four internal lines of D_1.

Each propagator $D_i(z_i)$ is expressed as in integral over a momentum-energy four-vector p_i. A partition of unity is introduced into each P_i space. For each pair (i,j) the corresponding partition function $\chi_{ij}(p_i)$ is an infinity differentiable function of tiny compact support centered at $p_i = p_{ij}$. Consequently, each partial propagator

$$D_{ij}(z_i) = \int d^4 p_i \frac{e^{-ip_i z_i}}{p_i{}^2 - m_i{}^2 + i0} \chi_{ij}(p_i) \qquad \text{(E1)}$$

will, by virtue of the result proved in Sec. IV 3a in the first citation of Ref. 8, fall off faster than any inverse power of the Euclidean norm of the four-vector z_i for all directions outside the set of "casual" directions C_{ij}. This casual set C_{ij} is the set of (signed) directions of the set of covariant four-vectors p_i that lie in the intersection of the mass-shell surface $p_i{}^2 = m_i{}^2$ with the support of $\chi_{ij}(p_i)$. All directions in the casual set C_{ij} will lie close to the direction of P_{ij}. The rate of falloff of $D_{ij}(z_i)$ is uniform over any closed set of directions of the four-vector z_i that does not intersect C_{ij}. Each casual set C_{ij} can also be considered to be a closed spacetime cone minus its apex at the origin.

The function $F[\psi]$ is obtained by folding $F(x)$ into the four coordinate-space wave functions $\psi_i(x_i)$ corresponding to the four external lines of D_1. Each $\psi_i(x_i)$ is the Fourier transform of a function $\widetilde{\psi}_i(p_i) = \widetilde{\psi}_i'(p_i)\delta^+(p_i{}^2 - m_i{}^2)$ or $\widetilde{\psi}_i'(p_i)\delta^-(p_i{}^2 - m_i{}^2)$, where $\psi_i'(p_i)$ is an infinitely differentiable function of (say tiny) compact support around $p_i = P_i(P_i{}^2 = m_i{}^2)$. These four supports define four four-dimensional closed casual bi-cones C_i $(i = 1,2,3,4)$, which are taken to be disjoint, except at the origin. [The supports of the $\widetilde{\psi}_i'(p_i)$ can be made tiny by other partitions of unity.]

The separation of each propagator D_i into its parts D_{ij} induces a separation of $F(x)$ into a finite sum of terms $F_\alpha(x)$. Let $\{i, j(\alpha,i); i \in (1,2,3,4)\}$ specify the four functions $D_{ij}(\alpha,i)$ corresponding to α. Then a transformation to momentum-space shows that the function $F_\alpha[\psi]$ vanishes unless there is, for that α, a set $\{p_{i\alpha}, p_{i,j(\alpha,i)}; i = 1,2,3,4\}$ such that, for all $i \in (1,2,3,4)$,

$$p_{i\alpha} \in {\rm supp}\widetilde{\psi}_i \ , \qquad \text{(E2a)}$$

$$p_{i,j(\alpha,i)} \in {\rm supp}\chi_{ij(\alpha,i)} \ , \qquad \text{(E2b)}$$

and

$$p_{i\alpha} = p_{i,j(\alpha,i)} - p_{i+1,j(\alpha,i+1)} \ . \qquad \text{(E2c)}$$

Equation (E2c) expresses momentum-energy conservation at vertex i. The conditions (E2) entail that $F_\alpha[\psi]$ vanishes if momentum-energy conservation $P_i = P_{i,j(\alpha,i)} - P_{i+1,j(\alpha,i+1)}$ fails by more than the tiny amounts corresponding to the tiny supports of the functions χ_{ij} and $\widetilde{\psi}_i$.

Let the nonvanishing functions $F_\alpha[\psi]$ be those with α in the set A. The integrals $F_\alpha[\psi],\alpha\in A$, can be reconverted back into coordinate space, and one can then examine the contributions to the x_i-space integrals from regions in which one or more of the four points x_i tends to infinity.

For any $F_\alpha[\psi],\alpha\in A$, one has approximate energy-momentum conservation at each vertex. This approximate energy-momentum conservation together with the stability conditions on the masses of the stable particles, and the three-particle character of the vertices of D_1, entail that for any $\alpha\in A$ and any $i\in(1,2,3,4)$ either

$$\text{supp}\chi_{i,j(\alpha,i)}\cap\{p_i;p_i{}^2=m_i{}^2\}=\varnothing \tag{E3a}$$

or

$$\text{supp}\chi_{i+1,j(\alpha,i+1)}\cap\{p_{i+1};p_{i+1}{}^2=m_{i+1}{}^2\}=\varnothing \tag{E3b}$$

provided the supports of the functions $\chi_{ij}(p_i)$ and $\psi_i(p_i),i\in(1,2,3,4)$, have been taken sufficient small. Consequently, for each $i\in(1,2,3,4)$ and any $\alpha\in A$, at least one of the two partial propagators $D_{i,j(\alpha,i)}(z)$ or $D_{i+1,j(\alpha,i+1)}(z)$ will fall off faster than any power of $|z|_{\text{Eucl}}^{-1}$ uniformly over all directions.

This uniform fast fall-off of at least one of any two neighboring pair of partial propagators, $D_{i,j(\alpha,i)}(z_i)$ or $D_{i+1,j(\alpha,i+1)}(z_{i+1}),\alpha\in A$, coupled with the uniform faster than any power of $|x_i|^{-1}$ falloff of each coordinate space function $\psi_i(x_i)$ on compact sets lying outside any closed bi-cone C_i' centered at the origin that contains in its interior the set of casual directions C_i [cf. Ref. 7, Eq. (2.17)] entails the rapid (i.e., faster than any power of R^{-1}) falloff of the contribution to the x-spin integral for $F_\alpha[\psi]$ from points $x=(x_1,x_2,x_3,x_4)$ lying outside the set

$$\mathscr{R}'(R,\lambda X)\equiv\{x;|x_i-\lambda X|\le R,\ \text{all } i\in(1,2,3,4)\}\ . \tag{E4}$$

To prove this asserted falloff property one may separate the $x=(x_1,x_2,x_3,x_4)$-space integration region into four parts P_i, where the condition $|X_i|_{\text{Eucl}}\le|X_j|_{\text{Eucl}}$ (all j) holds for all x in P_i. Then the 16 variables $(x_1^0,\ldots,x_4^3)$ of x can be transformed to one radial variable R, which is $|x_i|_{\text{Eucl}}$ in P_i, and 15 "angle" variables u. The variable R ranges from zero to infinity, whereas for any fixed R the range of u is bounded.

The variables u can be specified by a set of four four-vectors $u_i,i\in(1,2,3,4)$. One of these four four-vectors u_i lies on the unit sphere, and the other three lie on or inside this sphere.

This unit sphere is centered at the origin. Four bi-cones C_i' centered at the origin can then be drawn. There is one bi-cone C_i' for each external particle i. These bi-cones are taken to be disjoint, except at the origin, and the vectors p_i in the support of $\tilde{\psi}_i(p_i)$ are contained in the interior of C_i'.

Let the set C_1'' consist of C_i' and the ball of radius 10^{-2} centered at the origin. If the point u_i corresponding to external particle i does not lie in C_i'' then the integral will have a factor that falls off faster than any power of R^{-1} due to the fast falloff of the wave functions $\psi_i(Ru_i)$ (cf. Ref. 7). But if each point u_i lies in the corresponding set C_i'', and one of these points u_i lies on the unit sphere, then both

$$x_i-x_{i-1}=R(u_i-u_{i-1})$$

and

$$x_{i+1}-x_i=R(u_{i+1}-u_i)$$

must increase linearly with R. Thus either $S_{ij}(x_i-x_{i-1})$ or $S_{i+1,j}(x_{i+1}-x_i0$ will fall off faster than any power of R^{-1}. The remaining factors in the integrand are bounded. Hence the total contribution to $F[\psi]$ from the coordinate-space region lying outside a sphere of radius R must also fall off faster than any power of R.

The integral of actual interest is given in (7.50). The integrand has in addition to the Feynman function $F^{D_1}(x)$ and the four external-particle wave functions $\psi_i(x_i)$, also several exponential factors. Some of these exponentials appear with imaginary exponents. These factors are bounded and do not affect the result. However, there is also an exponential with a real exponent. This real exponent consists of a sum of terms of the form

$$\int^K \frac{d^4k}{(2\pi)^4}2\pi\delta(k^2)\frac{1}{p\cdot k}\frac{1}{p'\cdot k}(1-\cos y\cdot k)\ , \tag{E5}$$

where y can be $x_i-\lambda X$ or x_i-x_i', and can become large.

It is sufficient to show that this integral (E5) can increase no faster than $c\ln|y|$ as $|y|\to\infty$. For in this case the exponential itself increases at most linearly in $|y|$. But any such linear increase is damped out by the just established faster than any power of $|y|^{-1}$ decrease of the remaining factors [note that $|x_i'-x_i|\ge a$ implies $|x_i'-\lambda X|\ge a/2$ or $|x_i-\lambda X|\ge a/2$. Hence the faster than any inverse power of R falloff of the contributions for x or x' outside $\mathscr{R}'(R,\lambda X)$ entails a faster than any inverse power falloff also in $|x_i'-x_i|$].

To obtain this logarithmic bound write

$$y\equiv\lambda\hat{y}\ , \tag{E6}$$

where $\hat{y}$ has Euclidean norm unity, and write, for $k^2=0$,

$$y\cdot k\equiv\lambda|\vec{k}|\beta\ , \tag{E7}$$

where β is a function of the angle θ between the three vectors $\vec{y}$ and $\vec{k}$. Then the integral (E5) can be written (with k now $|\vec{k}|$) as

$$\int_0^K\frac{k^2dk}{2k^3}2\pi\int_{-1}^{1}d(\cos\theta)f(\cos\theta)(1-\cos\lambda x\beta)\ , \tag{E8}$$

where $|f(\cos\theta)|$ is bounded.

To prove an asymptotic logarithmic bound $c\ln\lambda$ on the magnitude of (E8) for large λ it is sufficient to exhibit a bound c'/λ ($c'<c$) on the magnitude of the λ derivative

$$\begin{aligned}\int_0^K\frac{k^2dk}{2k^3}2\pi\int_{-1}^{1}&d\cos\theta f(\cos\theta)k\beta\sin\lambda k\beta\\ &=\pi\int_{-1}^{1}d\cos\theta f(\cos\theta)\beta\int_0^K dk\,\sin\lambda k\beta\\ &=\frac{\pi}{\lambda}\int_{-1}^{1}d\cos\theta f(\cos\theta)(1-\cos\lambda K\beta)\ .\end{aligned} \tag{E9}$$

The magnitude of (E9) has the bound $4\pi|f|_{\max}/\lambda$, and hence the convergence of (7.50) is assured.

The convergence of the x integration in (7.46) is assured by essentially the same argument.

The fact that the partial propagators $D_{ij}(z_i)$ enjoy rapid falloff in $|z_i|_{\rm Eucl}$ for directions of z_i lying outside the casual set C_{ij} was not used in the above discussion. However, this falloff property is needed to cover the general case in which D_1 is replaced by some other diagram D'_1. These rapid falloff conditions, together with the approximate momentum-energy conservation equations mentioned below (E2), ensure a rapid falloff in R of the contributions to the analogs of (7.50) from points x outside $\mathscr{R}(R,\lambda X)$ unless the momentum-energies of the external lines of D'_1 lie close to a singularity surface of D'_1. And even in this case there is a rapid falloff of the contributions not lying near the regions in x space such that the spacetime diagram $D'_1(x)$ corresponds to a classically allowed physical process with the specified external momentum-energies.

This property is needed in the extension of the arguments given in this paper to the general case. It entails, generally, that the contributions to the transition amplitudes from regions of x space that are far away from the regions that correspond to the classically allowed processes fall off rapidly as the distances from the classically allowed configurations increase.

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