Operator-product expansion and the asymptotic behavior of spontaneously broken scalar field theories

C. Taylor and B. McClain
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 2 March 1983)

We reexamine a recent study of the operator-product expansion in spontaneously broken scalar field theories. First, the asymptotic behavior of the propagator in a spontaneously broken $\lambda\phi^4$ theory is calculated to lowest nontrivial order. The use of the operator-product expansion in the "naive" vacuum, with operators developing nonvanishing vacuum expectation values, is found to correctly reproduce the usual perturbative analysis of the shifted theory when carried out to the same order. The renormalization-group improvement of this result is studied. We find that γ_{ϕ^2}, the renormalization-group coefficient of the operator ϕ^2, is nonzero at first order in λ. This contradicts the result of the study of Gupta and Quinn. The generalization of this analysis to all Green's functions at all orders in perturbation theory is outlined. We argue that the renormalization-group improvement of the perturbation theory should yield the same answer for the two methods of calculating the asymptotic limit. Finally, we discuss the implications of this study for gauge theories.

I. INTRODUCTION

The operator-product expansion (OPE) is a powerful tool for calculating observables in QCD.[1] An important assumption in these calculations is that any nonperturbative effects present can be absorbed into operator matrix elements. It is, of course, assumed that the large-momentum behavior of the coefficients can be found from renormalization-group-improved perturbation theory.

In a recent paper,[2,3] Gupta and Quinn have questioned the validity of these assumptions. To test them in a specific example, they studied $\lambda\phi^4$ theory with spontaneous symmetry breaking due to a negative mass term. The effect of the symmetry breaking on the vacuum of the $\lambda\phi^4$ theory is used as a model of nonperturbative effects which presumably are responsible for a nontrivial vacuum structure in QCD. This study should be understood as an attempt to examine a known theory from behind the same veil of ignorance shielding us from a complete understanding of QCD. In particular, their study assumes no knowledge of the vacuum structure of the theory such as vacuum expectation values of operator products. The key question, then, is whether or not the OPE can be used to calculate the asymptotic behavior of Green's functions in the absence of such knowledge. It is the purpose of this paper to reconsider the example of Gupta and Quinn in an attempt to distinguish between models of potentially erroneous calculations in gauge theories, and the legitimate analysis of spontaneously broken scalar field theories.

Gupta and Quinn suggest calculating the large-q^2 behavior of the propagator via two different ways.

The first method defines subtractions of the theory so that composite operators have zero expectation value in a "naive" vacuum, defined as an expansion around $\langle\phi\rangle=0$. The coefficients of the operator-product expansion are calculated in perturbation theory in this "naive" vacuum. Of course, because of the symmetry breaking, this is not the true vacuum. This is handled by simply allowing operator expectation values in the physical vacuum to be nonvanishing.

The second method is to take the symmetry breaking into account from the beginning by shifting variables $\phi=\rho+v$ where v is the expectation value of ϕ and $\langle\rho\rangle=0$. Thus the expansion is done about the physical vacuum. The Green's functions are calculated using renormalization-group-improved perturbation theory.

Clearly these two methods should give the same large-q^2 evolution of the Green's functions if our initial assumptions are correct. Indeed, the perturbative equivalence of the shifted and unshifted theories is well known.[4] In Sec. IV, we shall show this in the context of the operator-product expansion.

This sort of argument always uses some knowledge of the effects of symmetry breaking. We use the fact that the two expansions are related by a shift of the field operator $\phi=\langle\phi\rangle+\rho$. The question to be addressed, then, is whether, if we choose to be "naive" and to ignore such information about the vacuum structure (which we certainly do not have in QCD), can the operator-product expansion correctly predict the asymptotic behavior of the shifted theory?

Gupta and Quinn suggest that the answer is "no." They argue that calculating the large-q^2 behavior of the first higher-twist term of the propagator via a "naive" use of the operator-product expansion with renormalization-group-improved coefficients does not yield the same asymptotic behavior as the calculation in the shifted theory. Specifically, they assume that asymptotically as the momentum $q^2\to\infty$, the propagator $\Delta(q)$ can be written as

$$\Delta(q) \sim \frac{1}{q^2} + \frac{m^2}{q^4}\left[\frac{q^2}{m^2}\right]^{\gamma_\theta/2}, \tag{1.1}$$

where γ_θ is an "anomalous dimension." They argue that, to first order in λ, γ_θ is nonzero in the shifted theory, but vanishes when calculated using the "naive" OPE. Such a result would suggest that difficulties with the operator-product expansion would arise in QCD where nonperturbative information is not available.

In Sec. II we calculate the propagator to the one-loop level using both methods suggested by Gupta and Quinn. Using our knowledge of $\langle\phi^2\rangle$ in the shifted theory, we find that both methods give the same result.

Section III discusses the renormalization-group improvement of these calculations. We find (1) that γ_θ, the "anomalous dimension" associated with the operator ϕ^2, is nonzero to first order in λ and (2) the renormalization group shows that the sum of leading logs in the propagator does *not* exponentiate as assumed in (1.1), but is related to the scaling of the four-point function at exceptional momentum. Thus we find

$$\Delta(q) \sim \frac{1}{q^2} + \frac{\langle\phi^2\rangle}{2}\lambda^{\gamma_0/\beta_0}[\lambda(q^2/m^2)]^{1-\gamma_0/\beta_0} + \cdots, \tag{1.2}$$

where we have written

$$\begin{aligned}
&\beta(\lambda)=\beta_0\lambda^2+O(\lambda^3)\,,\\
&\gamma_\theta=\gamma_0\lambda+O(\lambda^2)\,,\\
&\gamma_\phi=O(\lambda^2)\,,\\
&s\frac{d\lambda(s)}{ds}=\beta(\lambda)\,,\\
&\lambda(s=1)=\lambda\,,\\
&\lambda(s)=\frac{\lambda}{1-\beta_0\lambda\ln s}\,.
\end{aligned} \tag{1.3}$$

Quinn (in a private communication) has emphasized that this is a key point because it clarifies the origin of the scaling behavior of the subleading term of the propagator. Further, because the term in brackets in (1.2) does not behave like $(q^2/m^2)^{\gamma_\theta/2}$, the term "anomalous dimension" seems a misnomer for γ_θ. We shall refer to γ_θ as the renormalization-group coefficient associated with the operator $\frac{1}{2}\phi^2$. We conclude that, at the level considered by Gupta and Quinn, the naive use of the OPE does indeed predict the correct asymptotic behavior of the propagator.

In Sec. IV we use the effective action to generalize these results to all vertex functions to all orders in perturbation theory. We argue that the operator-product expansion reproduces the perturbation theory in the shifted vacuum. We further argue in Sec. V that renormalization-group improvement of the operator coefficients performed in the unshifted "vacuum" will reproduce the asymptotic behavior of the shifted theory.

II. ONE-LOOP CALCULATIONS

We consider a real scalar field theory defined by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4=\tfrac{1}{2}(\partial_\mu\phi)^2-V\,. \tag{2.1}$$

In particular, we are interested in the case $m^2<0$. In this case, it is usual to shift to the classically stable vacuum defined by

$$\frac{\delta V}{\delta\phi}=0,\quad \frac{\delta^2 V}{\delta\phi^2}>0 \tag{2.2}$$

which implies that ϕ develops a nonvanishing vacuum expectation value

$$\langle\phi\rangle^2=v^2=-\frac{3!m^2}{\lambda}\,. \tag{2.3}$$

If we now shift the field $\phi=\varphi+v$, we have broken the reflection invariance of the original Lagrangian. The new Lagrangian may be written as

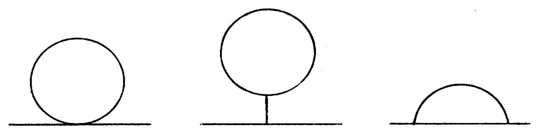
$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\varphi)^2-\frac{1}{2}\left[m^2+\frac{\lambda v^2}{2}\right]\varphi^2-\frac{\lambda v}{3!}\varphi^3-\frac{\lambda}{4!}\varphi^4\,. \tag{2.4}$$

The Lagrangian can also be written in terms of

$$M^2=m^2+\frac{\lambda v^2}{2}=-2m^2>0\,. \tag{2.5}$$

The renormalization of this theory is straightforward, and has been discussed many times.[4] Thus, we shall proceed to the calculation of the lowest-order correction to the two-point function, defining subtractions by those of the unshifted theory at $p=0$. The only contributions at the one-loop level are the three diagrams of Fig. 1. Since only the diagram of Fig. 1(c) contains momentum dependence, $\Gamma^{(2)}$ is given to this order in Euclidean space by

$$\begin{aligned}
\Gamma^{(2)}_\Omega&=p^2+M^2-\frac{\lambda v^2}{2}\int\frac{d^4k}{(2\pi)^4}\left\{\frac{1}{[k^2+M^2]}\frac{1}{[(p+k)^2+M^{2}]}-\frac{1}{[k^2+m^2]^2}\right\}\\
&=p^2+M^2+\frac{\lambda v^2}{32\pi^2}\left\{-2+\left[1+\frac{4M^2}{p^2}\right]^{1/2}\ln\left[\frac{1+(1+4M^2/p^2)^{1/2}}{-1+(1+4M^2/p^2)^{1/2}}\right]+\ln\left[\frac{|m^2|}{M^2}\right]+i\pi\right\},
\end{aligned} \tag{2.6}$$



(a) (b) (c)

FIG. 1. First-order contribution to the propagator in the shifted theory.

where the notation Ω denotes a calculation of the propagator in the physical vacuum. Asymptotically (the region of validity of the operator-product expansion), this yields

$$\Gamma_{\Omega}^{(2)}(p^2,M^2,\lambda)\approx p^2+m^2+\frac{\lambda v^2}{2}+\frac{\lambda v^2}{32\pi^2}\ln p^2/|m^2|+\cdots . \quad (2.7)$$

We now turn to the calculation in the perturbative vacuum. The operator-product expansion tells us that[5]

$$\lim_{x\to y}\phi(x)\phi(y)\sim\sum_{O}C_O(x-y)O\left[\frac{x+y}{2}\right], \quad (2.8)$$

where the $\{O\}$ form a complete set of local renormalized operators. In particular, this implies that

$$\langle\Omega|\phi(p)\phi(-p)|\Omega\rangle\approx C_1(p^2)\langle 1\rangle+C_{\phi^2}(p^2)\langle\phi^2\rangle+\cdots . \quad (2.9)$$

We can calculate the C_O in the perturbative vacuum by inserting (2.8) into the expression for higher-order Green's functions. If we consider, for example, $G^{(4)}(p,p,0,0)$, we find

$$G^{(4)}(p,p0,0)\approx C_1(p^2)G^{(2)}(0)+C_{\phi^2}(p^2)G^{(2)}_{\phi^2(0)}(0)+\cdots , \quad (2.10)$$

where $G^{(2)}_{\phi^2(0)}$ denotes a two-point function with a $\phi^2(0)$ insertion.

To second order, $G^{(4)}(p,p,0,0)$ is given by the five graphs of Fig. 2. If we again subtract at $p^2=0$ to define the divergent integrals, the graph of Fig. 2(c) does not contribute. Finally, recalling that an insertion of $\phi^2(0)$ is equivalent here to differentiation with respect to $-m^2/2$, it is easy to see that

$$C_1(p^2)=\frac{1}{p^2+m^2} \quad (2.11)$$

and

$$C_{\phi^2}(p^2)=\frac{\frac{1}{2}}{[p^2+m^2]^2}\left[-\lambda+\lambda^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{[(p+k)^2+m^2]}\frac{1}{[k^2+m^2]}-\frac{1}{[k^2+m^2]^2}\right]\right] \quad (2.12)$$

to this order. Evaluating the integral and keeping track of the negative mass yields asymptotically

$$C_{\phi^2}(p^2)=\frac{\frac{1}{2}}{[p^2+m^2]^2}\left\{-\lambda-\frac{\lambda^2}{16\pi^2}\left[-2+\left[1-\frac{4m^2}{p^2}\right]^{1/2}\ln\left[\frac{1+(1-4m^2/p^2)^{1/2}}{1-(1-4m^2/p^2)^{1/2}}\right]\right]\right\}. \quad (2.13)$$

Hence,

$$\begin{aligned}G^{(2)}(p^2)&\approx\frac{1}{p^2+m^2}+\frac{\langle\phi^2\rangle}{2}\frac{1}{[p^2+m^2]^2}\left[-\lambda-\frac{\lambda^2}{16\pi^2}\ln\frac{p^2}{|m^2|}\right]\\&\approx\left[p^2+m^2+\frac{\lambda}{2}\langle\phi^2\rangle+\frac{\lambda^2\langle\phi^2\rangle}{32\pi^2}\ln\frac{p^2}{m^2}\right]^{-1}\\&=[\Gamma^{(2)}(p^2)]^{-1} .\end{aligned} \quad (2.14)$$

Comparison with (2.7) and (2.3) reveals that the perturbative evaluation of the expansion has reproduced the results of the shifted theory.[6]

As noted in the Introduction, this direct comparison requires knowledge of the vacuum expectation value $\langle\phi^2\rangle$. In the next section, we shall show that the renormalization group allows us to compare asymptotic behavior even if $\langle\phi^2\rangle$ is unknown.

III. RENORMALIZATION-GROUP IMPROVEMENT

In this section, we carefully review the renormalization-group analysis corresponding to the perturbative study of the previous section. We want to compare the lowest-order renormalization-group-improved Green's function $\Gamma_{\Omega}^{(2)}(p^2)$ in the shifted theory with the lowest-order renormalization-group-improvement of Eq. (2.9). The renormalization-group improvement of (2.9) is to be conducted in an unbroken $\lambda\phi^4$ theory, and continued to the case of spontaneous breaking. $\langle\phi^2\rangle$ is to be understood as an unknown parameter. Thus, the crucial question is whether the "anomalous dimension" associated with the first subleading term of the propagator in the shifted theory agrees with the "anomalous dimension" associated with $C_{\phi^2}(p^2)$ in an unbroken theory. Care must be used in this analysis because the quadratic divergence of Fig. 1(a) associated with mass renormalization in an unbroken $\lambda\phi^4$ theory is independent of external momentum.

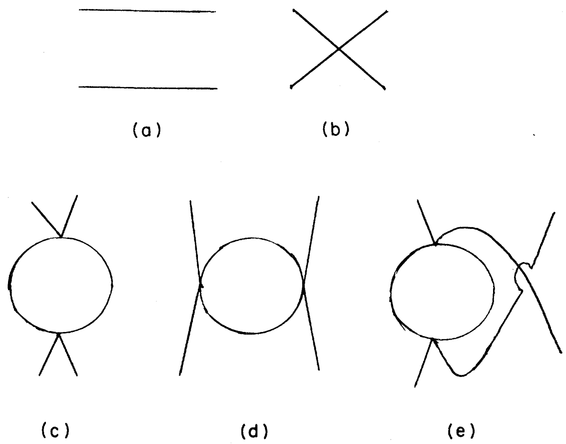


FIG. 2. The four-point function to second order in the unshifted theory.

Since this is a crucial point in the analysis of the problem posed by Gupta and Quinn, we review it in some detail. We follow Callan's derivation closely.[7]

We renormalize the theory defined by the Lagrangian (2.1) by introducing counterterms. The renormalization conditions are chosen to yield the same vertex functions as Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) subtraction. They are

$$\begin{aligned} &\Gamma^{(2)}(0) = -im^2 , \\ &\frac{d}{dp^2}\Gamma^{(2)}(p^2)\Big|_{p^2=0} = i , \\ &\Gamma^{(4)}(0) = -i\lambda , \\ &\Gamma^{(2)}_{\theta(0)}(0) = 1 . \end{aligned} \tag{3.1}$$

$\Gamma^{(2)}_{\theta(q)}(p)$ denotes a vertex with an insertion of the operator $\frac{1}{2}\phi^2(q)$ at momentum q into a two-point function with momentum p. Note that, in particular

$$\frac{d}{dm_0{}^2}\widetilde{\Gamma}^{(2)}(p) = -i\widetilde{\Gamma}^{(2)}_{\theta(0)}(p) , \tag{3.2}$$

where $\widetilde{\Gamma}$ are unrenormalized vertex functions with a cutoff Λ, and the subscript "0" denotes a bare quantity.

The Callan-Symanzik equation is given by

$$\left[m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + n\gamma \right]\Gamma^{(n)}(p) = -im^2\alpha\Gamma^{(n)}_{\theta(0)}(p) , \tag{3.3}$$

where

$$\begin{aligned} &\alpha = 2Z_\theta\left[\frac{\partial m^2}{\partial m_0{}^2}\right]^{-1} , \\ &\beta = 2m^2\left[\frac{\partial m^2}{\partial m_0{}^2}\right]^{-1}\frac{\partial\lambda}{\partial m_0{}^2} , \\ &\gamma = m^2\left[\frac{\partial m^2}{\partial m_0{}^2}\right]^{-1}\frac{\partial}{\partial m_0{}^2}\ln Z , \end{aligned} \tag{3.4}$$

where Z multiplicatively renormalizes ϕ, and Z_θ multiplicatively renormalizes insertions of $\frac{1}{2}\phi^2$.

To first order in λ in an unbroken theory, β and γ vanish, and $\alpha=2$. Also to lowest order, $\Gamma^{(2)}_{\theta(0)}(p)=1$. Hence, the renormalization group tells us that, in an unbroken $\lambda\phi^4$ theory, to first order in λ

$$\Gamma^{(2)} = i(p^2-m^2) . \tag{3.5}$$

As pointed out by Gupta and Quinn there is no nontrivial scaling of the propagator at order λ. Similarly, corrections to the scaling of $C_1^{(2)}(p^2)$ are absent at order λ.

We now review the derivation of the renormalization-group equation for $C_{\phi^2}(p)$. Consider the renormalization-group equation for an $(n+2)$-point vertex and use the operator-product expansion on both sides. We then have a set of equations of the form

$$\left[m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + (n+2)\gamma \right]\left[C_O(p)\Gamma^{(n)}_{O(0)}(k) \right] = -i\alpha m^2[C_O(p)\Gamma^{(n)}_{\theta(0)O(0)}(k)] . \tag{3.6}$$

In this equation, p is the momentum associated with the two points on which the operator-product expansion is used, and k is the momentum associated with the remaining n points. This equation is true in particular for $O=\phi^2$. Now the vertex $\Gamma^{(2)}_{\phi^2(q)}(k)$ obeys a renormalization-group equation

$$\left[m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + 2\gamma + \gamma_{\phi^2} \right]\Gamma^{(2)}_{\phi^2(q)}(k) = -i\alpha m^2\Gamma^{(2)}_{\phi^2(q)\theta(0)}(k) . \tag{3.7}$$

This equation is still valid at the exceptional momentum $q=0$, where, however, the right-hand side cannot be neglected asymptotically. Multiplying from the left by $C_{\phi^2}(p)$, we see that the right-hand side of this equation corresponds to the right-hand side of Eq. (3.6). Hence,

$$\left[m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + 2\gamma - \gamma_{\phi^2} \right]C_{\phi^2}(p) = 0 . \tag{3.8}$$

Now γ_{ϕ^2}, as we have seen, is associated with an insertion of $\phi^2(p)$ at nonvanishing momentum p. We thus except γ_{ϕ^2} to be of order λ because $\Gamma^{(2)}_{\phi^2(p)}(k)$ will be a nontrivial function of p because of the diagram of Fig. 3 with ϕ^2 carrying momentum p. In fact,

$$\gamma_{\phi^2} = 2m^2\left[\frac{\partial m^2}{\partial m_0{}^2}\right]^{-1}\left[\frac{\partial}{\partial m_0{}^2}\ln Z_{\phi^2}\right] = \frac{\lambda}{16\pi^2} . \tag{3.9}$$

Note that this does not contradict the finding that $\Gamma^{(2)}_{\theta(0)}(k)$ in the unbroken theory does not scale at order λ, even though γ_{ϕ^2} enters into the renormalization-group equation (3.7). This is because the right-hand side of (3.7) is not asymptotically negligible. Using the analysis of Symanzik,[8] it can be written in terms of $\gamma_{\phi^2}\Gamma^{(2)}_{\phi^2(0)}$ using the operator-product expansion. The two order-λ terms cancel.

Equation (3.8) can easily be solved. Using the notation of (1.3),

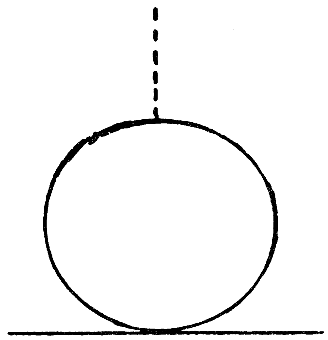


FIG. 3. First-order diagram contributing to ϕ^2 renormalization.

$$C_{\phi^2}(p^2) \sim \frac{\lambda^{\gamma_0/\beta_0}}{2}[\lambda(q^2/m^2)]^{1-\gamma_0/\beta_0} . \quad (3.10)$$

Similar analysis can be carried out to analyze the scaling of the first subleading term of the propagator in the shifted theory. Because this term has order-λ contributions from Fig. 1(c), it scales at order λ in the same way as Eq. (2.9). Asymptotically, this diagram is $\lambda v^2 \frac{1}{2}\gamma_\theta \ln(P^2/m^2)$. At higher orders, diagrams of the same topology, and hence the same large-momentum behavior contribute to both C_{ϕ^2} and $\Gamma_\Omega^{(2)}$ in the shifted theory (see Fig. 4). The series is thus asymptotically

$$\Gamma^{(2)}(p^2) \sim P^2 + \frac{\langle \phi^2 \rangle}{2} \lambda^{\gamma_0/\beta_0}[\lambda(P^2/m^2)]^{1-\gamma_0/\beta_0} . \quad (3.11)$$

Thus, we have reproduced the scaling behavior of the first higher-twist correction to the propagator by means of renormalization-group-improved perturbation theory. The equivalence of the two calculational schemes is ultimately due to the topological equivalence of the diagrams contributing to C_{ϕ^2} and to the subleading term of the physical propagator. That the scaling behavior is nontrivial in both approaches can be summarized by noting that γ_θ is nonzero at order λ, independent of symmetry breaking.

The fact that γ_θ is nonzero to first order in λ is well known. It is related to the trace anomaly of the stress-energy tensor $\theta^{\mu\nu}$. Because $\theta^\mu{}_\mu \propto \phi^2$, the existence of the trace anomaly at this order is equivalent to the operator ϕ^2 having a nonvanishing "anomalous dimension."

Of course, at first order the trace anomaly can be removed by defining an "improved" stress-energy tensor

$$\theta_{\mu\nu} \to \theta_{\mu\nu} + \tfrac{1}{12}\gamma_\theta(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\phi^2 . \quad (3.12)$$

[This is Eq. (3.17) of Ref. 9, where we have identified $\lambda f(0)$ with $\frac{1}{2}\gamma\theta$.] Again, we see that the need for an improvement at this order is due to $\gamma_\theta \neq 0$.

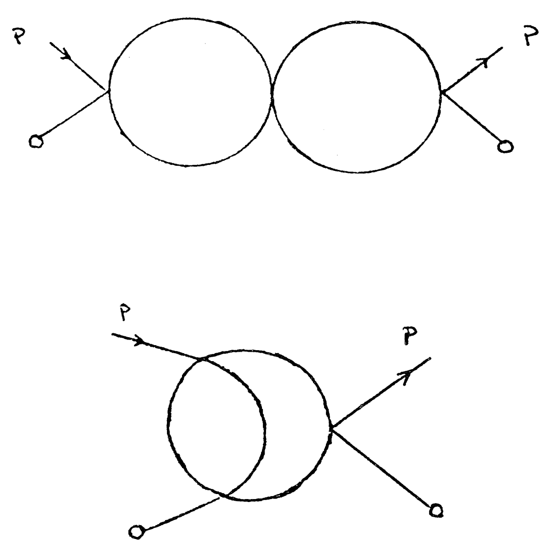


FIG. 4. Two-loop graphs contributing to both C_{ϕ^2} and the subleading term of $\Gamma_\Omega^{(2)}$. The external lines ending in circles carry no momentum.

IV. EXPANSION TO ALL ORDERS

We now turn to the discussion of general n-point functions at an arbitrary order in perturbation theory. We shall use the effective-action formalism and, hence, will discuss one-particle irreducible Green's functions. We are, in particular, interested in checking the validity of the assumption that we may analyze the asymptotic structure of the spontaneously broken theory by using the operator-product expansion about the perturbative (unshifted) vacuum, and then taking matrix elements in the physical vacuum. It is convenient to develop our analysis in the language of the effective action, because it allows us to discuss shifted Green's functions in terms of the unshifted functions.[10]

We begin our analysis be reviewing the effective action. As before, we consider a theory described by the Lagrangian (2.1).

It is convenient to introduce the generating functional

$$Z[J] = \int [d\phi] \exp\left[i \int d^4x [\mathscr{L}(x) + J(x)\phi(x)] \right] . \quad (4.1)$$

Then the generating functional of the connected Green's functions $W[J]$, is defined by

$$Z[J] = \exp\{iW[J]\} . \quad (4.2)$$

The effective action is then defined by the functional Legendre transform of the connected generating functional,

$$\frac{\delta W[J]}{\delta J(x)} = \phi_c(x) ,$$
$$\Gamma[\phi_c] = W[J] - \int d^4x\, \phi_c(x) J(x) . \quad (4.3)$$

Consequently,

$$J(x) = -\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} . \quad (4.4)$$

We can define one-particle irreducible Green's functions by the functional Taylor series expansion

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int \prod_{i=1}^{n} d^4x_i \Gamma^{(n)}(X_1 \cdots X_n)\phi_c(X_1) \cdots \phi_c(\gamma_n) . \tag{4.5}$$

It is important to note that the theory can be renormalized in a manner independent of the vacuum about which the expansion is made: subtractions which make the symmetric theory finite also make the broken theory finite.[4] If we shift

$$\phi_c(x) \to \tilde{\varphi}_c(x) + v , \tag{4.6}$$

where v is given by (2.2), then

$$\Gamma[\tilde{\varphi},v] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^4x_1 \tilde{\varphi}_c(x_1) \cdots \tilde{\varphi}_c(x_n) \times \Gamma^{(n)}(x_1 \cdots x_n; v,m,\lambda) \tag{4.7}$$

and

$$\Gamma^{(n)}(p_1 \cdots p_n; v,m,\lambda) = \sum_{l=0}^{\infty} \frac{1}{l!} v^l \Gamma^{(n+l)}(p_1 \cdots p_n; 0 \cdots 0, \lambda, m) . \tag{4.8}$$

The $\Gamma^{(n+l)}$ of the right-hand side of (3.8) are just the Green's functions of the unbroken theory analytically continued to negative mass. The Euclidean space propagator $\Delta(k)$ now has a tachyon pole at $k^2=m^2$. However, analytic continuation from Minkowski space yields an explicit $i\epsilon$ prescription which is sufficient to define all one-loop integrals:

$$\Delta(k) = (k^2 - m^2 + i\epsilon)^{-1} . \tag{4.9}$$

We are now in a position to study the asymptotic behavior of a $\lambda\phi^4$ theory which is spontaneously broken. In particular, we can examine the predictions of the operator-product expansion. For any vertex function $\Gamma^{(n+l)}(p_1 \cdots p_n, q_1 \cdots q_l)$ of the unbroken theory, we can express the limit $p_i \to \infty$ (with fixed ratios) for fixed q_j as[11]

$$\Gamma^{(n+l)}(p_1 \cdots p_n, q_1 \cdots q_l) \sim \sum_O \hat{C}_O^{(n)}(p_1 \cdots p_n) \Gamma_O^{(l)}(q_1 \cdots q_l) , \tag{4.10}$$

where the operators O form a complete set and $\Gamma_O^{(l)}$ is an insertion of O at zero momentum into the 1PI function $\Gamma_{(q)}^{(l)}$. Note that the $\hat{C}_O$ differ from the C_O of Sec. II because we are expanding proper vertices rather than ordinary Green's functions.

Proceeding as in (4.1)–(4.5), we can express $\Gamma_O^{(n)}$ in terms of the Legendre transform of $W_O[J]$ defined by

$$e^{iW_O[J]} = \int [d\phi] O(q^2=0) \times \exp\left\{ i \int d^4x [\mathscr{L}(x) + J(k)\phi(x)] \right\} . \tag{4.11}$$

Following the arguments of (4.5)–(4.8), we can write the $\Gamma_O^{(l)}$ of the shifted theory in terms of unshifted quantities:

$$\Gamma_O^{(l)}(p,m,\lambda,v) = \sum_n \frac{1}{n!} v^n \Gamma_0^{(l+n)}(p,O;m,\lambda,v=0) . \tag{4.12}$$

In particular,

$$\langle \Omega \mid O \mid \Omega \rangle = \Gamma_O^{(0)}(;m,\lambda,v) = \sum_n \frac{1}{n!} v^n \Gamma_O^{(n)}(0;m,\lambda,v=0) \tag{4.13}$$

represents the expectation value of O in the physical (shifted) vacuum.

Substituting (4.10) into (4.8) yields [with (4.14)]

$$\begin{aligned} \Gamma^{(n)}(p_1 \cdots p_n; v,m,\lambda) &= \sum_{l=0}^{\infty} \frac{v^l}{l!} \sum_O C_O^{(n)}(p,v=0) \Gamma_O^{(l)}(0;\lambda,m,v=0) \\ &= \sum_O C_O^{(n)} \Gamma_O^{(0)} \\ &= \sum_O C_O^{(n)}(p,v=0) \langle \Omega \mid O \mid \Omega \rangle \end{aligned} \tag{4.14}$$

in the large-p^2 limit. Because the left-hand side of Eq. (4.14) has no tachyon poles, the right-hand side must also be free of tachyonic singularities if we perform the sum before evaluating the Feynman integrals. But the naive use of the OPE entails calculating the $C_O^{(n)}(p)$ and treating the $\langle \Omega \mid O \mid \Omega \rangle$ as unknowns. As we saw in the explicit one-loop example, as long as $p^2 \gg \mid m^2 \mid$, we do not expect the tachyonic pole to affect the leading asymptotic behavior of $C_O^{(n)}(p)$. Although the tachyon pole may give rise to problems in nonleading order, this is not the problem found by Gupta and Quinn. It arises because the expansion about the naive ($v=0$) "vacuum" is unstable. There are no analogous problems in gauge theories. The expansion about the perturbative vacuum of QCD is not unstable to symmetry breaking. Since we are concerned with finding possible problems with using the OPE in gauge theories, we will ignore the tachyon pole. Thus, the perturbative operator-product expansion of the unbroken theory (continued to negative mass), taken with the effective-action formalism, implies that the vertex functions of the shifted theory can be expressed by (4.15) in terms of an expansion about the naive ($v=0$) "vacuum". We see explicitly that allowing operators to develop nonvanishing vacuum expectations values incorporates all symmetry-breaking effects in this theory.

V. RENORMALIZATION-GROUP IMPROVEMENT IN GENERAL

The discussion of the preceeding sections should suffice to show that, in perturbation theory, the operator-product expansion in the perturbative vacuum may be used to reproduce the asymptotic behavior in the physical vacuum. We now turn to the question of the renormalization-group improvement of this result, generalizing the arguments of Sec. III.

For purposes of illustration, we adopt a mass-independent renormalization in this section. Such a procedure may be defined by means of dimensional regularization,[12] by subtractions at unphysical momenta,[13] or by an extension of Weinberg's approach[11] in which mass renormalization is defined in terms of an insertion of $\phi^2(p^2)$ at zero mass. In such a scheme, we may write a homogeneous renormalization-group equation, valid in both the perturbative and the physical vacua[13]:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}-(n+l)\gamma_\phi\right]\Gamma^{(n+l)}(p,q;m,\lambda,\mu)\ . \tag{5.1}$$

The coefficients are the same in both vacua since the subtractions are independent of the symmetry-breaking parameter.

The renormalization-group equations for the OPE coefficients follow from expanding $\Gamma^{(n+l)}$ as in (4.10) , and noting that the $\Gamma_O^l(q,m,\lambda,\mu)$ also obey a homogeneous renormalization-group equation for any operator O:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}-l\gamma_\phi\right]\Gamma_O^l(q)=0\ . \tag{5.2}$$

If we consider the particular case $\langle\Omega\,|\,O\,|\,\Omega\rangle=\Gamma_\Omega^{(0)}$, we see that the correct renormalization-group analysis of the $C_O^{(n)}(p)\langle\Omega\,|\,O\,|\,\Omega\rangle$ is such that they obey the same equation as the $\Gamma^{(n)}(p)$ to which they contribute:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}-n\gamma_\phi\right]\{C_O^{(n)}\langle\Omega\,|\,O\,|\,\Omega\rangle\}=0\ . \tag{5.3}$$

Thus, however the renormalization group is used to improve the perturbation theory, it should yield equivalent results for the two approaches. Each term of the operator-product expansion of the $\Gamma_\Omega^{(n)}$ obeys a renormalization-group equation identical to the one $\Gamma_\Omega^{(n)}$ itself obeys. Note, however, that this analysis, as that of the last section, depends on the effective-action formalism. An analogous object is lacking thus far in QCD.

Renormalization-group analysis can also be carried out in a mass-dependent scheme, as in Sec. III. In this case, however, the rather intricate analysis of Symanzik for estimating corrections to the leading behavior must be used.[8] The renormalization-group equations for the coefficients of the operator-product expansion will again be defined by the equation for the corresponding Green's function. The two approaches (OPE and shifted perturbation theory) to spontaneously broken scalar field theory should also yield equivalent answers when the renormalization-group analysis is carried out in a mass-dependent scheme.

VI. CONCLUSIONS

We have examined the operator-product expansion in spontaneously broken scalar field theories. By allowing composite operators to develop nonvanishing vacuum expectation values, the OPE (when calculated in perturbation theory) reproduces all nonperturbative effects resulting from the instability of the naive vacuum, in the asymptotic limit. A careful renormalization-group study of the OPE coefficients (without any use of our knowledge of the nonperturbative structure of the theory) correctly reproduces the corrections to scaling due to symmetry breaking.

By considering the effective action, we have extended this result to all vertex functions at all orders in perturbation theory. Note that the effective-action formalism as used here is a consequence of the simple form of the nonperturbative effects in spontaneously broken scalar field theory. No analogous formalism exists for QCD. Using this formalism, we have argued that no violation of these results will arise from the use of renormalization-group analysis.

It is appropriate to comment on the implications of these results for more general theories. The extension of our analysis to the case of the spontaneous breaking of a continuous internal symmetry is straightforward. It is not clear that these results are of much direct value in theories such as QCD, where the nonperturbative effects arise from more subtle mechanisms. Nevertheless, it should be possible to avoid the particular problems suggested by Gupta and Quinn by a careful application of the renormalization group to the operator-product expansion.

ACKNOWLEDGMENTS

We would like to thank K. Johnson for calling our attention to this problem, and for many illuminating discussions. We would like to thank R. Jaffe, R. Jackiw, and H. Georgi for their interest and encouragement, and H. Quinn for several very useful conversations and for her comments on a preliminary version of this manuscript. We would also like to thank U. Ellwanger for a written communication. This work was supported in part through funds provided by the U.S. Department of Energy under Contract No. DE-AC02-76ER03069.

[1]A. H. Mueller, Phys. Rep. 73, 237 (1981); R. L. Jaffe and M. Soldate, Phys. Rev. D 26, 49 (1982); M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B147, 385 (1979); B147, 448 (1979).

[2]S. Gupta and H. R. Quinn, Phys. Rev. D 26, 499 (1982).

[3]H. R. Quinn and S. Gupta, Phys. Rev. D 27, 980 (1983).

[4]B. W. Lee, Nucl. Phys. B9, 649 (1969); J.-L. Gervais and B. W. Lee, *ibid.* B12, 627 (1969); F. Jegerlehner and B. Schroer, *ibid.* 68B, 461 (1974); C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 543; see also Ref. 12.

[5]K. Wilson, Phys. Rev. 179, 1499 (1969); W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser, M. Grisaru, and H. Pendleton (MIT, Cambridge, Mass. 1971).

[6]R. Chakrabarti, University of Pittsburgh report (unpublished).

After this work was completed, we became aware of this paper in which the one-loop calculation of this section is discussed. His results agree with ours.
[7]C. G. Callan, in *Methods in Field Theory.* edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).
[8]K. Symanzik, Commun. Math. Phys. 23, 49 (1971).
[9]S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).
[10]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
[11]S. Weinberg, Phys. Rev. D 8, 3497 (1973).
[12]G. 't Hooft, Nucl. Phys. B61, 455 (1973).
[13]K. Nishijima and M. Okawa, Prog. Theor. Phys. 61, 1822 (1979).