Symmetry breaking and phase transitions in general statistics

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A Higgs-type symmetry-breaking mechanism with general statistics is described in scalar ϕ^4 theories. It is shown that the minimum of the potential depends upon statistics and the parameter space is subdivided into separate regions, each with a particular statistics as a phase variable. In the model we describe, a phase diagram is constructed in parameter space, suggesting a new type of phase transition between domains of different statistics.

I. INTRODUCTION

In recent papers we have introduced the concept of general statistics,[1] which is a generalization of parastatistics,[2] and constructed a quantum field theory based on general statistics.[3] In Ref. 3, we also suggested the possibility that statistics may be considered as a dynamical or phase variable with the resulting phase transitions between domains of different statistics. In this article we present a model scalar ϕ^4 theory in which the above ideas are realized.

General statistics is defined by either single or double commutation relations in which Heisenberg's equation of motion and the spin-statistics relationship is satisfied. This is a further generalization of Green's parastatistics in that it allows the identification of Green's component indices as the internal-symmetry indices. An important property of general statistics, as opposed to normal statistics or parastatistics, is that it is not invariant under internal-symmetry transformations (double commutation relations are covariant). Because of this property the amplitude relations implied by internal symmetries are violated and replaced by other relationships which we call symmetry transmutation.[3]

Based on this generalization of statistics, we proposed a formulation of a quantum field system in which the generating functional is given by

$$Z[\{J_i,j_\alpha\},[\sigma]]=\int \mathscr{D}(\bar{\psi},\psi,\phi)\exp\left[i\int [\mathscr{L}(\phi,\psi)+\bar{J}_i\psi_i+\bar{\psi}_iJ_i+j_\alpha\phi_\alpha]d^4x\right] . \tag{1.1}$$

In Eq. (1.1), the variable $[\sigma]$ represents the statistics satisfied by the fields $\{\phi_\alpha,\psi_i\}$, external sources $\{J_i,j_\alpha\}$, and their differentiations. Note that the functional integral is defined and corresponds to an allowed physical system with arbitrary statistics $[\sigma]$ provided locality is satisfied. In Ref. 3, we proposed that $[\sigma]$ is chosen by physical conditions such as energy minimalization exactly in the same manner that a phase such as liquid or gas is preferred. The transition between domains of different statistics $[\sigma]$ (phase transition) occurs with variation in parameters of the theory, such as coupling constants, masses, or temperature. We emphasize that, because $[\sigma]$ is determined by relative commutation relations,[3] different $[\sigma]$ phases correspond to different symmetries in the many-body wave function. The identity of each particle, fermion or boson, is the same in all phases due to the spin-statistics relation.

In the following we show that a Higgs type of symmetry-breaking mechanism in a scalar ϕ^4 theory exhibits the phase structure between statistics described above. Note that theories with different relative statistics $[\sigma]$ are not unitarily equivalent. In fact, the examples below show that they may describe different physical states of the same underlying system, i.e., the same internal symmetry. We propose that this is analogous to the Cooper pair and conduction electron phases of superconducting material or the solid-liquid-gas phase structure of matter.

In Sec. II, we introduce the definition of $[\sigma]$ and discuss Higg-type symmetry breaking of an SO(3)-invariant ϕ^4 model. The phase diagram is constructed for this model. Section III contains the mass spectrum of Higgs particles resulting from the symmetry breaking in each phase. A discussion follows in Sec. IV.

II. THE SO(3) ϕ^4 MODEL

As mentioned earlier, general statistics is defined by double or single commutation relations (referred to as Γ_d and Γ_s, respectively, in Ref. 3). For simplicity in this paper we consider only single commutation relations (Γ_s) among real scalar fields ϕ_α, $\alpha=1,2,\ldots,N$, as the allowed values of $[\sigma]$. In this case $[\sigma]$ is defined by a set of numbers[3] $\{\rho_{\alpha\beta}\}$ such that

$$\rho_{\alpha\beta}=\pm 1 , \tag{2.1a}$$

$$\rho_{\alpha\beta}=\rho_{\beta\alpha} , \tag{2.1b}$$

and

$$\rho_{\alpha\alpha}=1 . \tag{2.1c}$$

The fields $\phi_\alpha(x)$, $\alpha=1,2,\ldots,N$, satisfying statistics $[\sigma]$

obey the commutation relations

$$[\phi_\alpha(x)\dot{\phi}_\beta(x') - \rho_{\alpha\beta}\dot{\phi}_\beta(x')\phi_\alpha(x)]_{t=t'} = i\delta(\vec{x}-\vec{x}\,')\delta_{\alpha\beta'} \,, \tag{2.2a}$$

$$[\phi_\alpha(x)\phi_\beta(x') - \rho_{\alpha\beta}\phi_\beta(x')\phi_\alpha(x)]_{t=t'} = 0 \,. \tag{2.2b}$$

Consider the SO(3)-invariant Lagrangian (repeated indices summed)

$$\mathscr{L} = -\tfrac{1}{2}(\partial_\mu\phi_\alpha)(\partial_\mu\phi_\alpha) - V(\phi) \,, \tag{2.3}$$

where

$$V(\phi) = a(\phi_\alpha\phi_\alpha)^2 + \frac{b}{4}(\epsilon_{\alpha\beta\gamma}\phi_\beta\phi_\gamma)(\epsilon_{\alpha\beta'\gamma'}\phi_{\beta'}\phi_{\gamma'}) - c(\phi_\alpha\phi_\alpha) \,. \tag{2.4}$$

From the fact that

$$\epsilon^{\alpha\beta\gamma}\epsilon^{\alpha\beta'\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\gamma\beta'} \,, \tag{2.5}$$

we write the second term (b term) in Eq. (2.4),

$$\frac{b}{4}(\phi_\beta\phi_\gamma\phi_\beta\phi_\gamma - \phi_\beta\phi_\gamma\phi_\gamma\phi_\beta) \,, \tag{2.6a}$$

as

$$\frac{b}{4}(\rho_{\beta\gamma} - 1)\phi_\beta^2\phi_\gamma^2 \,, \tag{2.6b}$$

from Eq. (2.2b). Note that for normal statistics or classical fields ($\rho_{\alpha\beta} = 1$, for all α,β) this term vanishes, while for general statistics it is nonvanishing and gives different values depending on $[\sigma]$.

Denote nonequivalent statistics, σ_{I}, σ_{II}, σ_{III}, σ_{IV} ($\alpha = 1,2,3$):

$$\text{(I)}\ \ [\sigma_{\mathrm{I}}] = \{\rho_{\alpha\beta} = 1, \ \text{for all } \alpha,\beta\} \,, \tag{2.7a}$$

$$\text{(II)}\ \ [\sigma_{\mathrm{II}}] = \{\rho_{12} = -1, \ \rho_{13} = \rho_{23} = 1\} \,, \tag{2.7b}$$

$$\text{(III)}\ \ [\sigma_{\mathrm{III}}] = \{\rho_{12} = 1, \ \rho_{13} = \rho_{23} = -1\} \,, \tag{2.7c}$$

$$\text{(IV)}\ \ [\sigma_{\mathrm{IV}}] = \{\rho_{\alpha\beta} = -1, \ \alpha \neq \beta\} \,. \tag{2.7d}$$

The b term in the Lagrangian which results for each statistics is given by

$$\text{(I)}\ \ 0 \,, \tag{2.8a}$$

$$\text{(II)}\ \ -b\phi_1^2\phi_2^2 \,, \tag{2.8b}$$

$$\text{(III)}\ \ -b(\phi_2^2\phi_3^2 + \phi_3^2\phi_1^2) \,, \tag{2.8c}$$

$$\text{(IV)}\ \ -b(\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2) \,. \tag{2.8d}$$

Regarding the potential, Eqs. (2.4) and (2.8), as a function of mutually commuting objects, ϕ_1^2, ϕ_2^2, and ϕ_3^2, for any statistics, we minimize with respect to ϕ_i^2. The minimum of the potential in each case is given by

$$\text{(I)}\ \ V_{\min} = \frac{-c^2}{4a} \ \text{at } \phi_1^2 + \phi_2^2 + \phi_3^2 = \frac{c}{2a} \,, \tag{2.9a}$$

$$\text{(II)}\ \ V_{\min} = \frac{-c^2}{(4a-b)}$$

$$\text{at } \phi_1^2 = \phi_2^2 = \frac{c}{(4a-b)}, \ \ \phi_3^2 = 0 \ \ (4a > b) \,, \tag{2.9b}$$

$$\text{(III)}\ \ V_{\min} = \frac{-c^2}{(4a-b)}$$

$$\text{at } \phi_1^2 + \phi_2^2 = \phi_3^2 = \frac{c}{(4a-b)} \ \ (4a > b) \,, \tag{2.9c}$$

$$\text{(IV)}\ \ V_{\min} = \frac{-c^2}{4}\left[\frac{1}{(a-b/3)}\right]$$

$$\text{at } \phi_1^2 = \phi_2^2 = \phi_3^2 = \frac{c}{2(3a-b)} \ \ (3a > b) \,. \tag{2.9d}$$

The minimum value of the potential, $V_{\min}$, as a function of b/a is shown in Fig. 1 for each statistics. In this discussion we assume both c and a positive in order to have a Higgs phase with a stable ϕ^4 theory. As is shown in Fig. 1, for $b/a < 0$, normal statistics $[\sigma_{\mathrm{I}}]$ gives the lowest potential minimum $V_{\min}$ and for $0 < b/a < 3$, $[\sigma_{\mathrm{IV}}]$ gives the lowest minimum potential. In the range $3 < b/a < 4$, $[\sigma_{\mathrm{IV}}]$ is not a consistent theory and is excluded as possible statistics. Among well-defined statistics $[\sigma_{\mathrm{II}}]$ and $[\sigma_{\mathrm{III}}]$ are favored in this range as they both give the smallest minimum. They have degenerate $V_{\min}$ and therefore cannot be separated in this model. For $b/a > 4$ only $[\sigma_{\mathrm{I}}]$, normal statistics, is consistent and therefore is the only allowed phase.

In Fig. 2, this situation is exhibited in the form of a phase diagram. In parameter space (a,b) the domain of each statistics is shown. Each domain is designated by statistics number II, III, IV and I_A, I_B, where A and B denote disconnected domains of normal statistics.

Note that the degeneracy of $[\sigma_{\mathrm{II}}]$ and $[\sigma_{\mathrm{III}}]$ may be broken by the introduction of other interactions, such as ϕ^3 or Yukawa $\bar{\psi}\psi\phi$ interactions. For example, allowed SO(3) interactions

$$f\epsilon_{ijk}\phi_i\phi_j\phi_k \tag{2.10a}$$

and

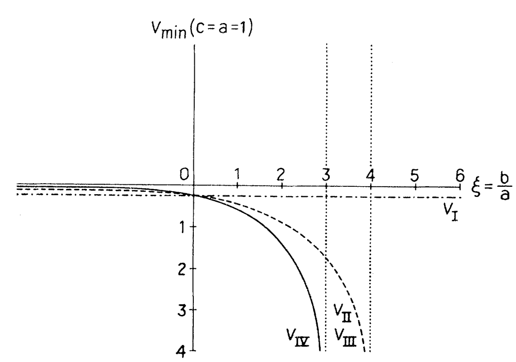

FIG. 1. Dependence of $V_{\min}$ on parameter $\xi = b/a$ for different statistics: The dot-dashed line denotes σ_{I} ($\xi < 0, \xi > 4$). The dashed line denotes $\sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}$ ($3 < \xi < 4$). The solid line denotes σ_{IV} ($0 < \xi < 3$).

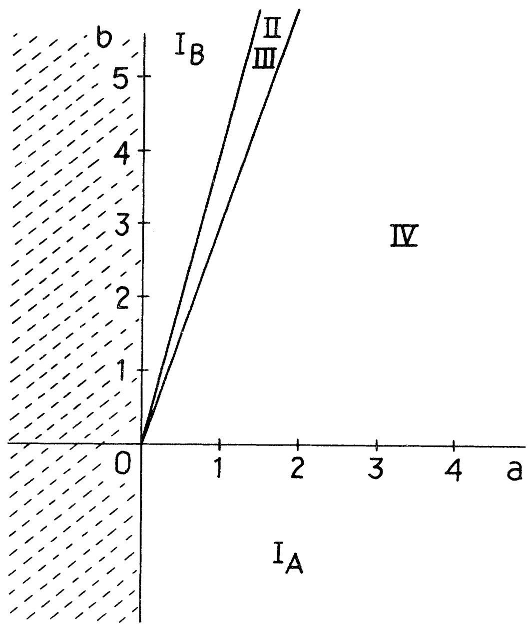

FIG. 2. Phase diagram showing parameter-space domains of different statistics.

$$g\epsilon_{ijk}\bar{\psi}_i\psi_j\phi_k \tag{2.10b}$$

may be introduced. This will modify phase boundaries as well as break degeneracy.

III. MASS SPECTRUM

In this section we analyze the mass spectrum of quantum fluctuations about the minimum in each statistics phase for the SO(3) model discussed in the previous section. We introduce two different methods in order to extract mutually commuting fluctuations which are suitable for describing vacuum expectation values.

A. Method of composite modes

Consider statistics $[\sigma_{\text{IV}}]$ and redefine the fields from Eq. (2.9d),

$$\phi_\alpha{}^2=v^2+\eta_\alpha,\quad \alpha=1,2,3 \tag{3.1}$$

with

$$v^2=\frac{c}{2(3a-b)}\quad (3a>b>0)\ . \tag{3.2}$$

The Lagrangian is then rewritten using

$$(\partial_\mu\phi_\alpha)^2=\frac{1}{4\phi_\alpha{}^2}(\partial_\mu\eta_\alpha)^2$$
$$=\frac{1}{4v^2}(\partial_\mu\eta_\alpha)^2-\frac{1}{4v^4}(\partial_\mu\eta_\alpha)^2\eta_\alpha+\cdots \tag{3.3}$$

and defining

$$\tilde{\eta}_\alpha=\frac{1}{2v}\eta_\alpha\ . \tag{3.4}$$

The resulting potential is

$$V(\tilde{\eta}_i)=\frac{-c^2}{4(a-b/3)}+4v^2a(\tilde{\eta}_1{}^2+\tilde{\eta}_2{}^2+\tilde{\eta}_3{}^2)$$
$$+4v^2(2a-b)(\tilde{\eta}_1\tilde{\eta}_2+\tilde{\eta}_2\tilde{\eta}_3+\tilde{\eta}_3\tilde{\eta}_1)\ . \tag{3.5}$$

Diagonalization of this mass matrix leads to the spectrum

$$m_1{}^2=m_2{}^2=4bv^2=\frac{2bc}{(3a-b)}\ , \tag{3.6a}$$

$$m_3{}^2=8(3a-b)v^2=4c\ , \tag{3.6b}$$

which are positive definite in the range $3a>b$.

For statistics $[\sigma_{\text{III}}]$ in the range $3<b/a<4$, redefine fields from Eq. (2.9c),

$$\phi_1{}^2=v^2\cos^2\theta+2v\cos\theta\tilde{\eta}_1\ , \tag{3.7a}$$

$$\phi_2{}^2=v^2\sin^2\theta+2v\sin\theta\tilde{\eta}_2\ , \tag{3.7b}$$

$$\phi_3{}^2=v^2+2v\tilde{\eta}_3\ , \tag{3.7c}$$

where

$$v^2=c/(4a-b)\ . \tag{3.8}$$

The resulting potential is

$$V(\tilde{\eta}_i)=4v^2[a(\tilde{\eta}_1{}^2\cos^2\theta+\tilde{\eta}_2{}^2\sin^2\theta+\tilde{\eta}_3{}^2)$$
$$+2a\tilde{\eta}_1\tilde{\eta}_2\cos\theta\sin\theta$$
$$+(2a-b)(\tilde{\eta}_1\tilde{\eta}_3\cos\theta+\tilde{\eta}_2\tilde{\eta}_3\sin\theta)]\ . \tag{3.9}$$

Diagonalization of the mass matrix leads to the spectrum

$$m_1{}^2=4bc/(4a-b)\ , \tag{3.10a}$$

$$m_2{}^2=4c\ , \tag{3.10b}$$

$$m_3{}^2=0\ . \tag{3.10c}$$

In the case of statistics $[\sigma_{\text{II}}]$, from Eq. (2.9b), define fluctuations by

$$\phi_1{}^2=v^2+2v\tilde{\eta}_1\ , \tag{3.11a}$$

$$\phi_2{}^2=v^2+2v\tilde{\eta}_2\ , \tag{3.11b}$$

$$\phi_3{}^2=\tilde{\eta}_3{}^2\ , \tag{3.11c}$$

with v given in Eq. (3.8). The resulting potential is given by

$$V(\tilde{\eta}_i)=4v^2a(\tilde{\eta}_1{}^2+\tilde{\eta}_2{}^2)+\frac{bc}{(4a-b)}\tilde{\eta}_3{}^2$$
$$+4v^2(2a-b)\tilde{\eta}_1\tilde{\eta}_2 \tag{3.12}$$

with the resulting masses upon diagonalization

$$m_1{}^2=4bc/(4a-b)\ , \tag{3.13a}$$

$$m_2{}^2=4c\ , \tag{3.13b}$$

and

$$m_3{}^2=2bc/(4a-b)\ . \tag{3.13c}$$

Note that one mass ($m_3{}^2$) is different in cases $[\sigma_{\rm II}]$ and $[\sigma_{\rm III}]$ although they have degenerate minimum energy.

For the normal statistics $[\sigma_{\rm I}]$, define the fluctuations

$$\phi_1{}^2=v^2\cos^2\phi\,\sin^2\theta+2v\,\sin\theta\,\cos\phi\tilde{\eta}_1\ , \tag{3.14a}$$

$$\phi_2{}^2=v^2\sin^2\phi\,\sin^2\theta+2v\,\sin\theta\,\sin\phi\tilde{\eta}_2\ , \tag{3.14b}$$

$$\phi_3{}^2=v^2\cos^2\theta+2v\,\cos\theta\tilde{\eta}_3\ , \tag{3.14c}$$

with

$$v^2=c/2a\ . \tag{3.15}$$

The potential is given by

$$V(\tilde{\eta}_i)=\frac{-c^2}{4a}+4av^2(\tilde{\eta}_1\sin\theta\cos\phi+\tilde{\eta}_2\sin\theta\,\sin\phi+\tilde{\eta}_3\cos\theta)^2\ , \tag{3.16}$$

with the resulting mass spectrum

$$m_1{}^2=4c \tag{3.17a}$$

and

$$m_2{}^2=m_3{}^2=0\ . \tag{3.17b}$$

Obviously, the massless modes are Goldstone bosons, which can be adsorbed in longitudinal modes in gauge theories.

B. Method of pseudoparticle modes

In this subsection, mutually commuting fluctuations are defined by introducing a mutually commuting or anticommuting set of c numbers.

Given $\{\rho_{\alpha\beta}\}$, defined in Eq. (2.1), define an associated set of c numbers $\{K_\alpha\}$ by the equations

$$K_\alpha K_\beta-\rho_{\alpha\beta}K_\beta K_\alpha=0\ , \tag{3.18a}$$

$$K_\alpha\phi_\beta-\rho_{\alpha\beta}\phi_\beta K_\alpha=0 \tag{3.18b}$$

for fields ϕ_β and

$$K_\alpha{}^2=1\ . \tag{3.18c}$$

Consider first statistics $[\sigma_{\rm IV}]$ in the domain $0<b/a<3$ and define ($\alpha=1,2,3$)

$$\phi_\alpha=vK_\alpha+\eta_\alpha\ , \tag{3.19}$$

where

$$v=\left[\frac{c}{2(3a-b)}\right]^{1/2}\ . \tag{3.20}$$

Substitution of Eq. (3.19) into the potential Eqs. (2.4) and (2.8) results in the mass matrix

$$\frac{((M^2))}{2}=2v^2(K_1\eta_1,K_2\eta_2,K_3\eta_3)\begin{bmatrix}2a & 2a-b & 2a-b\\ 2a-b & 2a & 2a-b\\ 2a-b & 2a-b & 2a\end{bmatrix}\begin{bmatrix}K_1\eta_1\\ K_2\eta_2\\ K_3\eta_3\end{bmatrix}\ . \tag{3.21}$$

Note that this is the same matrix as Eq. (3.5) which indicates the same mass spectrum. Diagonalizing Eq. (3.21) yields a commuting basis Φ_1, Φ_2, and Φ_3 related to $(K_i\eta_i)$ by

$$K_1\eta_1=\frac{1}{\sqrt{6}}\Phi_1+\frac{1}{\sqrt{2}}\Phi_2-\frac{1}{\sqrt{3}}\Phi_3\ , \tag{3.22a}$$

$$K_2\eta_2=\frac{1}{\sqrt{6}}\Phi_1-\frac{1}{\sqrt{2}}\Phi_2-\frac{1}{\sqrt{3}}\Phi_3\ , \tag{3.22b}$$

and

$$K_3\eta_3=-(\tfrac{2}{3})^{1/2}\Phi_1-(\tfrac{1}{3})^{1/2}\Phi_3\ . \tag{3.22c}$$

The mass spectrum for the set of mutually commuting fields ϕ_1, ϕ_2, ϕ_3 is given by Eq. (3.6).

For statistics $[\sigma_{\rm III}]$, in the domain $3<b/a<4$, define

$$\phi_1=v\,\cos\theta K_1+\eta_1\ , \tag{3.23a}$$

$$\phi_2=v\,\sin\theta K_2+\eta_2\ , \tag{3.23b}$$

and

$$\phi_3=vK_3+\eta_3\ , \tag{3.23c}$$

where v is given in Eq. (3.8). Substitution of Eqs. (3.23) into the potential Eqs. (2.4) and (2.8) yields the mass matrix

$$\frac{((M^2))}{2}=2v^2(K_1\eta_1,K_2\eta_2,K_3\eta_3)\begin{bmatrix}2a\cos^2\theta & 2a\cos\theta\sin\theta & (2a-b)\cos\theta\\ 2a\cos\theta\sin\theta & 2a\sin^2\theta & (2a-b)\sin\theta\\ (2a-b)\cos\theta & (2a-b)\sin\theta & 2a\end{bmatrix}\begin{bmatrix}K_1\eta_1\\ K_2\eta_2\\ K_3\eta_3\end{bmatrix}\ , \tag{3.24}$$

which is the same mass matrix as Eq. (3.9) and is diagonalized in terms of the commuting basis Φ_1, Φ_2, and Φ_3 defined by

$$K_1\eta_1=\frac{\cos\theta}{\sqrt{2}}(\Phi_1+\Phi_2)+\sin\theta\Phi_3 , \tag{3.25a}$$

$$K_2\eta_2=\frac{\sin\theta}{\sqrt{2}}(\Phi_1+\Phi_2)-\cos\theta\Phi_3 , \tag{3.25b}$$

$$K_3\eta_3=\frac{-1}{\sqrt{2}}(\Phi_1-\Phi_2) . \tag{3.25c}$$

Again by rewriting $K_i\eta_i$ in terms of commuting fields, the mass spectrum derived in Eq. (3.10) is obtained.

For statistics $[\sigma_{\rm II}]$, in the range $3<b/a<4$, define

$$\phi_1=vK_1+\eta_1 , \tag{3.26a}$$

$$\phi_2=vK_2+\eta_2 , \tag{3.26b}$$

and

$$\phi_3=\eta_3 , \tag{3.26c}$$

where v is given in Eq. (3.8). Substitution into the potential, Eqs. (2.4) and (2.8), yields the mass matrix

$$\frac{((M^2))}{2}=v^2(K_1\eta_1,K_2\eta_2,K_3\eta_3)\begin{bmatrix} 4a & 4a-2b & 0 \\ 4a-2b & 4a & 0 \\ 0 & 0 & b \end{bmatrix}\begin{bmatrix} K_1\eta_1 \\ K_2\eta_2 \\ K_3\eta_3 \end{bmatrix} . \tag{3.27}$$

This is diagonalized in terms of commuting fields Φ_1, Φ_2, and Φ_3 by taking

$$K_1\eta_1=\frac{1}{\sqrt{2}}(\Phi_1+\Phi_2) , \tag{3.28a}$$

$$K_2\eta_2=\frac{1}{\sqrt{2}}(-\Phi_1+\Phi_2) , \tag{3.28b}$$

and

$$K_3\eta_3=\Phi_3 . \tag{3.28c}$$

The mass matrix resulting for Φ_1, Φ_2, and Φ_3 is given in Eq. (3.13).

The case of normal statistics $[\sigma_{\rm I}]$ reduces in this formalism to the usual Higgs mechanism ($K_i=1$, for all i).

In this section the masses of quantum fluctuations around potential minima was discussed using composite modes ${\phi_i}^2$ and pseudoparticle modes ϕ_i. The latter method requires an anticommuting number system [Eq. (3.18)] for consistency. While use of this number system is natural in a functional integration formulation, as in the case of anticommuting c-number fields for fermions, it cannot be used in the usual Hilbert-space formulation of a quantum system. This does not permit the ordinary interpretation of the constant term, K_iv, as a vacuum expectation value. However, due to the fact that in both methods the theory is rewritten in terms of commuting fields and gives the same mass spectrum in the Higgs mode, one is led to conclude that the two methods are equivalent. For these reasons we called the second method the pseudoparticle mode, i.e., the η_i fluctuations are not occurring in the final formulation of the second method.

IV. DISCUSSION

In this paper we have showed that different statistics result in different potential minima and different mass spectra for an SO(3) scalar ϕ^4 model in different domains of parameter space. We suggest this is an indication of the existence of phase transitions between domains of different statistics as parameters are changed. Clearly, a detailed description of such a transition requires an extension of the usual formalism of quantum systems. We have hinted at this possibility by the introduction of a new number system, $\{K_i\}$, which most likely would alter the structure of the usual Hilbert-space formulation of physical states.

The above can be easily generalized to SO(N) or SU(N) internal symmetries. In Ref. 3, we showed that for complex fields $\rho_{\alpha\beta}$ may be an arbitrary phase. The SU(N)- or SO(N)-invariant interaction term, analogous to the b term in Eq. (2.4), is given by

$$\sum_{jk}\phi_j^\dagger\phi_k^\dagger\phi_j\phi_k=\sum_{jk}\rho_{jk}\phi_j^\dagger\phi_j\phi_k^\dagger\phi_k . \tag{4.1}$$

The inclusion of fermion fields in a general statistics has been discussed in Ref. 3 and is essentially the same as the boson case. Complications arise, however, in the construction of local interaction terms. In the case of SO(3), the Yukawa interaction

$$\epsilon_{ijk}\bar{\psi}_i\psi_j\phi_k \tag{4.2}$$

is local for any statistics chosen. For any other group an appropriate composite rule[3] must be imposed on ϕ_{ij} where the interaction term is

$$\bar{\psi}_i \psi_j \phi_{ij} \; . \tag{4.3}$$

An SO(3)-invariant model, using interaction term Eq. (4.2) with various statistics, is currently being studied by the authors for application to the generation problem.

ACKNOWLEDGMENT

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