Classical and quantal supersymmetric Liouville theory

E. D'Hoker
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 4 April 1983)

The classical supersymmetric Liouville theory is shown to be invariant under the supersymmetric extension of the conformal group. Lax pair and Bäcklund transformations are derived and the general classical solution is obtained. The isotropy group for every solution is constant on the solution manifold and equal to an $N=1$ conformal supersymmetry. For the quantum theory, the effective potential is computed exactly. The spectrum of the theory is continuous, bounded from below by zero, but no translationally invariant ground state exists. Translation invariance may be broken without the appearance of a Goldstone boson, and a consistent perturbation theory in the coupling constant is obtained. The constant $N=1$ supersymmetry is also broken to a constant $N=\frac{1}{2}$ supersymmetry, and no Goldstone fermion arises. Space is spontaneously reduced to the half-line. The $N=1$ conformal supersymmetry remains exact to all orders of perturbation theory.

I. INTRODUCTION

The supersymmetric Liouville theory (henceforth abbreviated SLT) is the supersymmetric ($N=1$) extension of the ordinary, purely bosonic, Liouville theory. Both have arisen in the reformulation of the dual string model.[1] While the ordinary Liouville theory has been extensively studied during the last year,[2–4] relatively little progress has been made on its supersymmetric extension. The purpose of this paper is to investigate systematically the SLT model, both at the classical and at the quantum levels.

First, we examine the classical theory. In Sec. II, we construct the invariance group of the action. In addition to the global $N=1$ supersymmetry, the theory is invariant under conformal transformations. Furthermore, the global supersymmetry can be generalized to a local (conformal) supersymmetry. Composition of two local supersymmetry transformations yields a conformal transformation. Both the local supersymmetry and the conformal transformations may be united into the superconformal group. Its algebra may also be realized on the fields using Poisson brackets and we identify the central charges. In Sec. III, a Lax pair is constructed and a Bäcklund transformation is derived. We integrate this Bäcklund transformation and obtain the general solution to the SLT. We show that the superconformal group acts transitively on the solution manifold. Every solution has the same isotropy group OSp(1,1), the simplest graded extension of SO(2,1). An improved energy-momentum tensor is derived in terms of which the superconformal OSp(1,1) symmetry is realized without central charges.

Next, the quantum theory is considered. In Sec. IV, we compute the full effective potential using functional methods, and we check the answer to second order in the loop expansion. We argue that the SLT—exactly as the ordinary Liouville theory—possesses a continuous spectrum, but no translationally invariant ground state. In Sec. V, we show that a consistent perturbation theory in the dimensionless coupling constant may be developed around a classical solution of the SLT. Space translation invariance is broken spontaneously in the same sense as in the ordinary Liouville theory.[3] No Goldstone bosons arise and space is semicompactified. The constant $N=1$ Poincaré supersymmetry is spontaneously broken down to $N=\frac{1}{2}$ supersymmetry without the appearance of Goldstone fermions. The fermionic generator of the $N=\frac{1}{2}$ supersymmetry is both a Majorana *and* a Weyl spinor. However, the OSp(1,1) invariance of the classical background is left unbroken and the $N=1$ *conformal* supersymmetry remains exact. The proof is given first for the one-loop approximation to the propagator and then generalized to all Green's functions and all orders of perturbation theory. Finally, it is indicated that different types of classical solutions give rise to different physical theories. A consistent perturbation theory may be constructed for a theory in which translation invariance is broken, and space compactified to a finite interval. For the latter theory, the symmetry group is still OSp(1,1), however, the global $N=1$ supersymmetry is completely broken down without the appearance of Goldstone fermions and the remaining (nonconstant) $N=1$ supersymmetry is no longer conformal.

II. CLASSICAL THEORY: THE SUPERCONFORMAL GROUP

A. Global supersymmetry

The SLT is a two-dimensional field theory of one real scalar Φ and one Majorana spinor Ψ, whose dynamics is governed by the action

$$S=\int d^2x\left[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi+\frac{i}{2}\bar{\Psi}\partial\!\!\!/\Psi-\frac{m^2}{\beta^2}e^{\beta\Phi}-\frac{m}{2\sqrt{2}}e^{\beta\Phi/2}\bar{\Psi}\Psi\right]. \tag{2.1}$$

We shall use the following representation of the two-dimensional Clifford algebra:

$$\gamma^0=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \gamma^1=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix},\quad \gamma^5=\gamma^0\gamma^1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.2}$$

With this convention, a Majorana spinor is real. Please note that upon setting $\Psi=0$ in (2.1) we recover the action of the ordinary Liouville theory. The action S is invariant under the following supersymmetry transformations δ_ϵ associated with a constant spinor ϵ (Ref. 5),

$$\delta_\epsilon\Phi=\bar{\epsilon}\Psi , \tag{2.3a}$$

$$\delta_\epsilon\Psi=\left[-i\partial\!\!\!/\Phi-\frac{m\sqrt{2}}{\beta}e^{\beta\Phi/2}\right]\epsilon . \tag{2.3b}$$

The theory may equivalently be expressed using the superfield formalism.[6] Let us introduce a real scalar superfield,

$$H=\Phi+\bar{\theta}\Psi-\tfrac{1}{2}\bar{\theta}\theta F . \tag{2.4}$$

We shall also need the supercharges

$$Q_\alpha=\gamma^0_{\alpha\beta}\frac{\partial}{\partial\theta^\beta}+i\gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu \tag{2.5}$$

and their associated superderivatives

$$D_\alpha=\gamma^0_{\alpha\beta}\frac{\partial}{\partial\theta^\beta}-i\gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu . \tag{2.6}$$

Supercharges and superderivatives obey the anticommutation relations

$$\{D_\alpha,D_\beta\}=2i(\gamma^\mu\gamma^0)_{\alpha\beta}\partial_\mu , \tag{2.7a}$$

$$\{Q_\alpha,Q_\beta\}=-2i(\gamma^\mu\gamma^0)_{\alpha\beta}\partial_\mu , \tag{2.7b}$$

$$\{D_\alpha,Q_\beta\}=0 . \tag{2.7c}$$

The superspace action

$$S=\int d^2x\int d^2\theta\left[-\frac{i}{4}\overline{DH}DH-\frac{im2\sqrt{2}}{\beta^2}e^{\beta H/2}\right] \tag{2.8}$$

yields, upon performing the θ integrations,

$$S=\int d^2x\left[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi+\frac{i}{2}\bar{\Psi}\partial\!\!\!/\Psi+\tfrac{1}{2}F^2-mF\frac{\sqrt{2}}{\beta}e^{\beta\Phi/2}-\frac{m}{2\sqrt{2}}e^{\beta\Phi/2}\bar{\Psi}\Psi\right]. \tag{2.9}$$

Using the equation of motion for the auxiliary field

$$F=\frac{\sqrt{2}m}{\beta}e^{\beta\Phi/2} , \tag{2.10}$$

the latter may be eliminated from (2.9), so that the action (2.1) is recovered. With the help of the superfield language, the supersymmetry transformations (2.3) are expressed in terms of the superfield H alone:

$$\delta_\epsilon H=\bar{\epsilon}QH \tag{2.11}$$

or in components

$$\delta_\epsilon\Phi=\bar{\epsilon}\Psi , \tag{2.12a}$$

$$\delta_\epsilon\Psi=(-i\partial\!\!\!/\Phi-F)\epsilon , \tag{2.12b}$$

$$\delta_\epsilon F=i\bar{\epsilon}\partial\!\!\!/\Psi . \tag{2.12c}$$

The field equations derived from (2.1) read

$$\Box\Phi+\frac{m^2}{\beta^2}e^{\beta\Phi}+\frac{m\beta}{4\sqrt{2}}e^{\beta\Phi/2}\bar{\Psi}\Psi=0 , \tag{2.13a}$$

$$i\partial\!\!\!/\Psi-\frac{m}{\sqrt{2}}e^{\beta\Phi/2}\Psi=0 , \tag{2.13b}$$

or in terms of the superfield

$$\bar{D}DH-\frac{2\sqrt{2}m}{\beta}e^{\beta H/2}=0 . \tag{2.14}$$

B. Conformal invariance

We recall that S is invariant under ordinary conformal transformations,

$$x^\mu\rightarrow x^\mu-f^\mu , \tag{2.15}$$

where f^μ is a conformal Killing vector satisfying

$$\partial^\mu f^\nu+\partial^\nu f^\mu-g^{\mu\nu}\partial_\alpha f^\alpha=0$$

or (2.16)

$$f^+=f^+(x^+) ,\quad f^-=f^-(x^-) .$$

Under this transformation, the fields change according to[2,7]

$$\delta\Phi=f^\mu\partial_\mu\Phi+\frac{1}{\beta}\partial_\mu f^\mu ,$$

$$\delta\Psi=f^\mu\partial_\mu\Psi+\tfrac{1}{4}(\partial_\mu f^\mu)\Psi+\tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu f_\nu)\gamma^5\Psi , \tag{2.17}$$

$$\delta F=f^\mu\partial_\mu F+\tfrac{1}{2}\partial_\mu f^\mu F .$$

(We use the definition of ϵ such that $\epsilon^{01}=-\epsilon^{+-}=1$.) If we make the convention that the θ coordinates transforms as a spinor of dimension $-\frac{1}{2}$,

$$\delta\theta=-\tfrac{1}{4}(\partial_\mu f^\mu)\theta+\tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu f_\nu)\gamma^5\theta , \tag{2.18}$$

then the scalar superfield H has the transformation properties of the scalar field Φ:

$$\delta H=f^\mu\partial_\mu H+\frac{1}{\beta}\partial_\mu f^\mu . \tag{2.19}$$

Invariance of the action under the transformations (2.18) and (2.19) is easily verified.

C. Local supersymmetry

Next, we show that the global supersymmetry defined in (2.11) and (2.12) may be generalized to a local invariance transformation. To do so, we first construct the invariance group of the $H\bar{D}DH$ part of the action, and then restrict that group to the elements which also leave the full action invariant. To this end, we write the most general superspace transformation in terms of the basis generators Q and D:

$$X=\bar{\epsilon}Q+\bar{\delta}D . \tag{2.20}$$

Invariance of the operator $\bar{D}D$ is obtained when

$$[\bar{D}D,X]\sim\bar{D}D . \tag{2.21}$$

A straightforward calculation shows that $\delta=0$, and that ϵ must be a conformal Killing spinor satisfying

$$\bar{D}\gamma^0(1\pm\gamma^5)\epsilon=0 . \tag{2.22}$$

This equation is easily solved and we obtain

$$\epsilon=\begin{bmatrix}\epsilon_1(x^+)\\ \epsilon_2(x^-)\end{bmatrix} . \tag{2.23}$$

Thus, the $H\bar{D}DH$ part of the action is invariant under the transformations (2.11) and (2.12), but where ϵ is allowed to have the x dependence specified in (2.23). Furthermore, it is clear that we may add to H any scalar superfield Ω satisfying the free equation

$$\bar{D}D\Omega=0 \tag{2.24}$$

with general solution

$$\Omega(x;\theta)=\Omega_1(x^+;\theta_1)+\Omega_2(x^-;\theta_2) . \tag{2.25}$$

To find the invariance group of the full action, we use the ansatz

$$\delta H=\bar{\epsilon}QH+\Omega \tag{2.26}$$

with ϵ and Ω satisfying (2.22) and (2.24), respectively. Then we can easily check that the local supersymmetry transformation

$$\delta H=\bar{\epsilon}QH+\frac{2}{\beta}\bar{Q}\epsilon \tag{2.27}$$

or in component language

$$\begin{aligned}&\delta_\epsilon\Phi=\bar{\epsilon}\Psi ,\\ &\delta_\epsilon\Psi=(-i\partial\!\!\!/\Phi-F)\epsilon-\frac{2i}{\beta}\partial\!\!\!/\epsilon ,\\ &\delta_\epsilon F=i\bar{\epsilon}\partial\!\!\!/\Psi\end{aligned} \tag{2.28}$$

leaves the action S invariant. We shall call this invariance the local supersymmetry of the SLT. Note that the variation in the fermion field has picked up an inhomogeneous term.

D. The superconformal group

We shall now show that the composition of two local supersymmetry transformations yields a conformal transformation. From (2.20) (with $\delta=0$), it is clear that the vector field X induces the following transformation on the superspace coordinates:

$$\delta x^\mu=-i\bar{\epsilon}\gamma^\mu\theta ,\ \ \delta\theta=\epsilon . \tag{2.29}$$

Composition of transformations ϵ and η yields

$$\begin{aligned}[\delta_\epsilon,\delta_\eta]x^\mu&=\delta_\epsilon(-i\bar{\eta}\gamma^\mu\theta)-\delta_\eta(-i\bar{\epsilon}\gamma^\mu\theta)\\ &=-i\partial_\nu\bar{\eta}\gamma^\mu\theta(-i\bar{\epsilon}\gamma^\nu\theta)-i\bar{\eta}\gamma^\mu\epsilon\\ &\quad+i\partial_\nu\bar{\epsilon}\gamma^\mu\theta(-i\bar{\eta}\gamma^\nu\theta)+i\bar{\epsilon}\gamma^\mu\eta .\end{aligned} \tag{2.30}$$

Using (2.23), it is easily seen that this expression simplifies to give

$$[\delta_\epsilon,\delta_\eta]x^\mu=-K^\mu \tag{2.31}$$

with

$$K^\mu=-2i\bar{\epsilon}\gamma^\mu\eta . \tag{2.32}$$

Writing out the components of K^μ

$$K^+=-2\sqrt{2}i\epsilon_1\eta_1 , \tag{2.33a}$$

$$K^-=-2\sqrt{2}i\epsilon_2\eta_2 , \tag{2.33b}$$

and using (2.23), it is clear that K^+ (K^-) depends only on x^+ (x^-), so that K satisfies (2.16) and is a conformal Killing vector. The conformal transformation is also realized on the fields. So we have, e.g.,

$$[\delta_\epsilon,\delta_\eta]\Phi=K^\mu\partial_\mu\Phi+\frac{1}{\beta}\partial_\mu K^\mu . \tag{2.34}$$

This suggests that the algebra generated by the conformal transformations and implemented with the local supersymmetry transformations should close. The infinitesimal action of a combined conformal and local supersymmetry transformation on the coordinates is given by

$$\begin{aligned}&\delta x^\mu=-f^\mu-i\bar{\epsilon}\gamma^\mu\theta ,\\ &\delta\theta=\epsilon-\tfrac{1}{4}(\partial_\mu f^\mu)\theta+\tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu f_\nu)\gamma^5\theta .\end{aligned} \tag{2.35}$$

It is readily checked that the composition of two such transformations is of the same form:

$$[\delta_{f,\epsilon},\delta_{g,\eta}]=\delta_{h,\vartheta} \tag{2.36}$$

with h and ϑ given by

$$h^\mu=f^\nu\partial_\nu g^\mu-g^\nu\partial_\nu f^\mu-2i\bar{\epsilon}\gamma^\mu\eta , \tag{2.37}$$

$$\begin{aligned}\vartheta=&f^\nu\partial_\nu\eta-g^\nu\partial_\nu\epsilon-\tfrac{1}{4}(\partial_\mu f^\mu)\eta+\tfrac{1}{4}(\partial_\mu g^\mu)\epsilon\\ &+\tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu f_\nu)\gamma^5\eta-\tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu g_\nu)\gamma^5\epsilon .\end{aligned} \tag{2.38}$$

One may verify that h^μ (ϑ) is a conformal Killing vector (spinor). A lengthy calculation shows that the composition law is also realized on the fields, so that the combined algebra closes. Henceforth we shall call the latter the superconformal algebra.

The superconformal algebra defined in (2.35) and (2.36) may be easily integrated and the group action obtained:

$$\begin{aligned}&x^\mu\to y^\mu:\ y^+=y^+(x^+;\theta_1) ,\ \ y^-=y^-(x^-;\theta_2) ,\\ &\theta\to\epsilon:\ \epsilon_1=\epsilon_1(x^+) ,\ \ \epsilon_2=\epsilon_2(x^-) .\end{aligned} \tag{2.39}$$

E. Superconformal current algebra and central charges

The superconformal algebra is generated by the combined conformal and local supersymmetry algebra. For each of these transformations, we shall now construct conserved currents and their associated charges, and show that, under usual Poisson bracketing, the algebra of the charges closes up to a center.

Recall that the equal-time Poisson brackets for the elementary fields are given by

$$\{\Pi(x),\Phi(y)\}=\delta(x-y) , \tag{2.40a}$$

$$\{\Psi_\alpha(x),\Psi_\beta(y)\}=i\delta_{\alpha\beta}\delta(x-y) . \tag{2.40b}$$

The conserved current associated with conformal transformations is constructed out of the conserved, symmetric, and traceless energy-momentum tensor:

$$J_C^\mu = f_\nu \Theta^{\mu\nu} , \tag{2.41a}$$

$$\Theta^{\mu\nu} = \partial^\mu\Phi\partial^\nu\Phi + \frac{i}{2}\bar{\Psi}\gamma^\mu\partial^\nu\Psi - g^{\mu\nu}\left[\tfrac{1}{2}\partial_\kappa\Phi\partial^\kappa\Phi + \frac{i}{2}\bar{\Psi}\partial\Psi - \frac{m^2}{\beta^2}e^{\beta\Phi} - \frac{m}{2\sqrt{2}}e^{\beta\Phi/2}\bar{\Psi}\Psi\right] + \frac{2}{\beta}(\Box g^{\mu\nu} - \partial^\mu\partial^\nu)\Phi . \tag{2.41b}$$

Note that the Belinfante improvement identically vanishes, because Ψ is a Majorana field. The only nonvanishing components of this energy-momentum tensor are

$$\Theta_{++} = (\partial_+\Phi)^2 - \frac{2}{\beta}\partial_+{}^2\Phi + \frac{i}{2}\bar{\Psi}\gamma_+\partial_+\Psi , \tag{2.42a}$$

$$\Theta_{--} = (\partial_-\Phi)^2 - \frac{2}{\beta}\partial_-{}^2\Phi + \frac{i}{2}\bar{\Psi}\gamma_-\partial_-\Psi . \tag{2.42b}$$

Because $\Theta^{\mu\nu}$ is conserved, Θ_{++} (Θ_{--}) only depends on x^+ (x^-). The conformal charges

$$Q_f^C = \int dx\, J_C^0 \tag{2.43}$$

generate the conformal transformations of (2.17).

The current associated with the global supersymmetry transformations

$$J_N^\mu = (\partial\Phi + iF)\gamma^\mu\Psi \tag{2.44}$$

is conserved. Using the Poisson brackets in (2.40), the charge associated with J_N^μ is found not to reproduce transformation laws (2.28) on the fields. The inhomogeneous term in the transformation of Ψ is missing. This difficulty is easily circumvented by adding to J_N^μ a term that is automatically conserved:

$$J_S^\mu = (\partial\Phi + iF)\gamma^\mu\Psi + \frac{4}{\beta}\gamma^5\epsilon^{\mu\nu}\partial_\nu\Psi . \tag{2.45}$$

The improved supercurrent J_S^μ has the following three properties in addition to being conserved:

$$\epsilon_{\mu\nu}\partial^\mu J_S^\nu = 0 , \tag{2.46a}$$

$$\gamma_\mu J_S^\mu = 0 , \tag{2.46b}$$

$$J_{S+} = J_{S+}(x^+) , \quad J_{S-} = J_{S-}(x^-) \tag{2.46c}$$

and its associated charges generate correctly the local supertransformations:

$$Q_\epsilon^S = \int dx\, \bar{\epsilon} J_S^0 , \tag{2.47}$$

$$[Q_\epsilon^S, \Phi] = \bar{\epsilon}\Psi , \tag{2.48a}$$

$$[Q_\epsilon^S, \Psi] = (-i\partial\Phi - F)\epsilon - \frac{2i}{\beta}\partial\epsilon . \tag{2.48b}$$

The charge of a superconformal transformation specified by f^μ and ϵ is

$$Q_{f,\epsilon} = Q_f + Q_\epsilon . \tag{2.49}$$

The algebra of superconformal charges closes under Poisson bracketing:

$$\{Q_f, Q_g\} = -Q_h + \Delta(f,g) , \tag{2.50a}$$

$$\{Q_\epsilon, Q_\eta\} = -Q_k + \Sigma(\epsilon,\eta) , \tag{2.50b}$$

$$\{Q_\epsilon, Q_f\} = -Q_\vartheta . \tag{2.50c}$$

Here we have defined the conformal Killing vectors h and k and the conformal Killing spinor ϑ as in (2.37) and (2.38):

$$h^\mu = f^\nu\partial_\nu g^\mu - g^\nu\partial_\nu f^\mu , \tag{2.51a}$$

$$k^\mu = -2i\bar{\epsilon}\gamma^\mu\eta , \tag{2.51b}$$

$$\vartheta = f^\nu\partial_\nu\epsilon - \tfrac{1}{4}(\partial_\mu f^\mu)\epsilon + \tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu f_\nu)\gamma^5\epsilon . \tag{2.51c}$$

The central elements Δ and Σ are readily calculated:

$$\Delta(f,g) = \frac{4}{\beta}\int dx[g^1(x)\partial_x{}^3 f^0(x) - f^1(x)\partial_x{}^3 g^0(x)] , \tag{2.52a}$$

$$\Sigma(\epsilon,\eta) = -\frac{16i}{\beta^2}\int dx\, \epsilon^T\partial_x{}^2\eta . \tag{2.52b}$$

We have now determined the full invariance group of the classical theory as well as its representation in terms of fields and Poisson brackets.

III. CLASSICAL SOLUTIONS

A. The Lax pair

A Lax pair for Eq. (2.14) may be found starting from the following ansatz for the covariant superderivatives:

$$\mathscr{D}_\alpha = D_\alpha - \frac{\beta}{4}(D_\alpha H)A_\alpha + \sigma B_\alpha e^{\beta H/4} . \tag{3.1}$$

Here the c number σ and the numerical matrices A_α and B_α are determined by the requirement that the system

$$\mathscr{D}_\alpha V = 0 \tag{3.2}$$

be compatible if and only if H obeys the field equations (2.14). It is found that σ, A_α, and B_α must satisfy

$$\begin{gathered} -A_2 = A_1 = A , \\ [A,B_1] = -B_1 , \quad [A,B_2] = B_2 , \\ \{B_1,B_2\} = 2A , \quad \sigma = \left[\frac{im}{2\sqrt{2}}\right]^{1/2} , \end{gathered} \tag{3.3}$$

so that A, B_1, and B_2 span the graded algebra SU(1/1). It is easy to find the lowest dimensional representation of this algebra:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad B_1 = \lambda\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} , \quad B_2 = \frac{1}{\lambda}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} . \tag{3.4}$$

The free parameter λ may be thought of as the spectral parameter in the Lax pair. The inverse scattering problem can be set up with a formalism similar to that developed for the ordinary Liouville equation,[2] but we shall not study it here.

B. The Bäcklund transformation

A Bäcklund transformation can be derived from the Lax pair in the following way. [Henceforth we use the representation (3.4) with $\lambda=1$.] First we write the system (3.2) in components:

$$D_1V_1=\frac{\beta}{4}(D_1H)V_2-\sigma e^{\beta H/4}V_3\ ,$$
$$D_1V_2=\frac{\beta}{4}(D_1H)V_1+\sigma e^{\beta H/4}V_3\ , \tag{3.5}$$
$$D_1V_3=-\sigma e^{\beta H/4}(V_1+V_2)\ ,$$

and similarly for $\mathscr{D}_2V=0$. If V_1 and V_2 are taken to be bosonic superfields, then V_3 must be fermionic. Now, introduce the functions

$$h=\frac{4}{\beta}\text{argtanh}\frac{V_2}{V_1}\ ,\quad \chi=\frac{V_3}{(V_1{}^2-V_2{}^2)^{1/2}}\ . \tag{3.6}$$

Then h and χ satisfy the following equations:

$$D_1h-D_1H=\frac{4\sigma}{\beta}\chi\exp[\tfrac{1}{4}\beta(H+h)]\ , \tag{3.7a}$$
$$D_2h+D_2H=-\frac{4\sigma}{\beta}\chi\exp[\tfrac{1}{4}\beta(H-h)]\ , \tag{3.7b}$$
$$D_1\chi=-\sigma\exp[\tfrac{1}{4}\beta(H+h)]\ , \tag{3.7c}$$
$$D_2\chi=\sigma\exp[\tfrac{1}{4}\beta(H-h)]\ . \tag{3.7d}$$

This first-order system is precisely a Bäcklund transformation.[8] It is clear the H must satisfy the supersymmetric Liouville equation, whereas h must be free ($\bar{D}Dh=0$). The fermionic field χ does not seem to obey any equation all by itself, and may be viewed as a subsidiary field in the Bäcklund transformation.

C. Integration of the Bäcklund

The Bäcklund transformation may be integrated explicitly using quadratures only. First, we recall that the free field h is of the form

$$h(x;\theta)=h_1(x^-;\theta_2)-h_2(x^+;\theta_1)\ . \tag{3.8}$$

Then, we may eliminate χ between (3.7a) and (3.7c) and between (3.7b) and (3.7d), so as to obtain two ordinary differential equations for H alone:

$$D_1(e^{-\beta h/2}D_1e^{-\beta(H-h)/4})=-\sigma^2e^{\beta(H+h)/4}\ , \tag{3.9a}$$
$$D_2(e^{\beta h/2}D_2e^{-\beta(H+h)/4})=\sigma^2e^{\beta(H-h)/4}\ . \tag{3.9b}$$

In terms of the field

$$Z=\exp[-\tfrac{1}{2}\beta(H-h_1-h_2)]\ , \tag{3.10}$$

Eq. (3.9) becomes, after some simplifications,

$$D_1(e^{-\beta h_1/2}D_1Z)=-2\sigma^2e^{\beta h_1/2}\ , \tag{3.11a}$$
$$D_2(e^{-\beta h_2/2}D_2Z)=2\sigma^2e^{\beta h_2/2}\ . \tag{3.11b}$$

Next, we define the Green's functions

$$G_1(x^-,y^-;\theta_2,\theta_2')=\frac{1}{\sqrt{2}}g(x^--y^-)-i\delta(x^--y^-)\theta_2\theta_2'\ , \tag{3.12a}$$
$$G_2(x^+,y^+;\theta_1,\theta_1')=-\frac{1}{\sqrt{2}}g(x^+-y^+)+i\delta(x^+-y^+)\theta_1\theta_1' \tag{3.12b}$$

obeying

$$D_1G_1(x^-,y^-;\theta_2,\theta_2')=\delta(x^--y^-)(\theta_2'-\theta_2)\ , \tag{3.13a}$$
$$D_2G_2(x^+,y^+;\theta_1,\theta_1')=\delta(x^+-y^+)(\theta_1'-\theta_1)\ . \tag{3.13b}$$

Here the function g satisfies

$$\partial_\pm g(x^\pm-y^\pm)=\delta(x^\pm-y^\pm)\ . \tag{3.14}$$

With the help of the definition of Z, we find the expression for the general solution to the SLT field equations[9]

$$H=h_1+h_2-\frac{2}{\beta}\ln\left[-2\sigma^2\int G_1e^{\beta h_1/2}\int G_1e^{\beta h_2/2}+2\sigma^2\int G_2e^{\beta h_2/2}\int G_2e^{\beta h_2/2}+2\sigma^2\int G_1e^{\beta h_1/2}\int G_2e^{\beta h_2/2}\right]\ . \tag{3.15}$$

It is easily checked that when we restrict h_1 and h_2 to have vanishing fermionic part, the scalar component Φ of H precisely gives the general solution to the ordinary Liouville theory[2]:

$$\Phi=h_1+h_2-\frac{2}{\beta}\ln\frac{m}{2}\left[\frac{1}{\partial_+}e^{\beta h_2}-\frac{1}{\partial_-}e^{\beta h_1}\right]\ . \tag{3.16}$$

From (3.15), it is clear that any solution to the SLT may be obtained by applying some superconformal transformation [of Eq. (2.39)] to a given, fixed, solution of the SLT. Let $\mathscr{A}$ be the solution manifold and $\mathscr{G}$ the superconformal group, then for all fixed $x_0\in\mathscr{A}$ we have

$$\mathscr{G}\cdot x_0=\mathscr{A}\ . \tag{3.17}$$

In other words, the action of the superconformal group is transitive on the solution manifold. There is, however, a subtlety, which we discuss in Appendix A.

D. The isotropy group[11]

Since the group action is transitive, the isotropy group $\mathscr{H}_x$ at a point $x\in\mathscr{A}$ is obtained from the isotropy group at a point x_0 by conjugation. If $g(x)\in\mathscr{G}$ and $x_0,x\in\mathscr{A}$ and

$$g(x)x_0=x\ ,\quad \mathscr{H}_x=g(x)\mathscr{H}_{x_0}g^{-1}(x)\ . \tag{3.18}$$

In other words, the isotropy group is the same for all solutions, up to a conjugation. We shall explicitly determine the isotropy group for the general solution to the ordinary Liouville theory,

$$\Phi_c=\frac{1}{\beta}\ln\frac{4F'G'}{m^2(F-G)^2}\ ,\quad F=F(x^+),\ G=G(x^-) \tag{3.19}$$

and check that it is constant. The isotropy group is that subgroup of $\mathscr{G}$ that leaves the solution invariant, or $\delta\Phi=0$ and $\delta\Psi=0$. For a purely bosonic solution, this requirement becomes, with the help of (2.28),

$$\delta\Phi=f^\mu\partial_\mu\Phi_c+\frac{1}{\beta}\partial_\mu f^\mu=0\ , \tag{3.20a}$$

$$\delta\Psi=\left[-i\partial\Phi_c-\frac{m\sqrt{2}}{\beta}e^{\beta\Phi_c/2}\right]\epsilon-\frac{2i}{\beta}\partial\epsilon=0\ . \tag{3.20b}$$

Equations (3.20) are easily solved for f and ϵ, and we find

$$f^+=\frac{1}{F'}(\tfrac{1}{2}AF^2+BF+C)\ , \tag{3.21a}$$

$$f^-=\frac{1}{G'}(\tfrac{1}{2}AG^2+BG+C)\ , \tag{3.21b}$$

$$\epsilon_1=\frac{F}{\sqrt{F'}}+\frac{1}{\sqrt{F'}}\chi_1\ , \tag{3.22a}$$

$$\epsilon_2=-\frac{G}{\sqrt{G'}}\chi_0-\frac{1}{\sqrt{G'}}\chi_1\ . \tag{3.22b}$$

Only in the case where $F=x^+$, $G=x^-$ or $F=1/x^+$, $G=1/x^-$ and

$$\Phi_c=-\frac{1}{\beta}\ln\frac{m^2x^2}{2}$$

is the classical solution invariant under a single *constant* supersymmetry transformation. In all other cases the supersymmetry is realized in a fully nonlinear fashion. It is clear that the composition of two transformations of (3.21) and (3.22) closes according to the rules of (2.37) and (2.38). This graded group is called OSp(1,1) and has three elements of even and two of odd grading.

E. Linear action of the isotropy group on the fluctuation field

Let Φ_c be a classical solution and $\mathscr{H}$ its isotropy group. Then the action of $\mathscr{H}$ on the fluctuation field $\phi=\Phi-\Phi_c$ is linear, and the algebra $\mathscr{H}$ closes without central charges. Indeed, combination of (2.17) and (3.20) yields the transformation laws of ϕ and Ψ which are clearly linear:

$$\delta\phi=f^\mu\partial_\mu\phi+\bar{\epsilon}\Psi\ , \tag{3.23}$$

$$\delta\Psi=f^\mu\partial_\mu\Psi+\tfrac{1}{4}(\partial_\mu f^\mu)\Psi+\tfrac{1}{4}(\epsilon^{\mu\nu}\partial_\mu f_\nu)\gamma^5\Psi+[-i\partial\phi-F_c(e^{\beta\phi/2}-1)]\epsilon\ .$$

Here $F_c=(\sqrt{2}/\beta)me^{\beta\Phi_c/2}$. To see that the $\mathscr{H}$ algebra closes without center, let us express the centers Δ and Σ in (2.52) in terms of the background Φ_c and the killing vectors h_μ and k_μ alone. We find

$$\Delta(f,g)=\int dx\,\Theta^c_{0\mu}h^\mu\ , \tag{3.24a}$$

$$\Sigma(\epsilon,\eta)=\int dx\,\Theta^c_{0\mu}k^\mu\ . \tag{3.24b}$$

Here $\Theta^c_{\mu\nu}$ is the value of the energy-momentum tensor of (2.14), evaluated at the classical solution Φ_c, and h and k are defined by

$$h^\mu=f^\nu\partial_\nu g^\mu-g^\nu\partial_\nu f^\mu\ , \tag{3.25a}$$

$$k^\mu=-2i\bar{\epsilon}\gamma^\mu\eta\ . \tag{3.25b}$$

Thus, for f, g, ϵ, and η restricted to the isotropy group, (2.50) becomes

$$\{Q_f,Q_g\}=-\int dx(\Theta^{0\mu}-\Theta^{0\mu}_c)h_\mu\ , \tag{3.26a}$$

$$\{Q_\epsilon,Q_\eta\}=-\int dx(\Theta^{0\mu}-\Theta^{0\mu}_c)k_\mu\ . \tag{3.26b}$$

Here $\Theta^{0\mu}$ is the energy-momentum tensor evaluated on the complete field Φ. We may define an improved energy-momentum tensor

$$\Theta^{\mu\nu}_I=\Theta^{\mu\nu}-\Theta^{\mu\nu}_c \tag{3.27}$$

in terms of which the algebra closes without central charges. Choosing Φ_c to be a time-independent solution, the isotropy group will always contain time translations, associated with a constant timelike Killing vector $h^\mu_t=\delta_{\mu 0}$. Then we can define a Hamiltonian[12]

$$H=\int dx\,\Theta^{00}_I$$

$$=\int dx\left[\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2+\frac{i}{2}\bar{\Psi}\gamma^1\partial^1\Psi+\frac{m^2}{\beta^2}e^{\beta\Phi_c}(e^{\beta\phi}-\beta\phi-1)+\frac{m}{2\sqrt{2}}e^{\beta\Phi_c/2}e^{\beta\phi/2}\bar{\Psi}\Psi\right]\ , \tag{3.28}$$

which is positive definite and vanishes only when $\phi=0$. Consequently, there is a *unique* (classical) ground state at $\phi=\Psi=0$ which has the full OSp(1,1) superconformal symmetry.

IV. QUANTUM THEORY. THE EFFECTIVE POTENTIAL

Quantization of the SLT presents the same fundamental obstacles as the quantization of the ordinary Liouville theory. The calculation of the usual Green's functions with the help of a naive perturbative expansion in β is impossible since terms linear in Φ cannot be removed by shifting the field with a constant. Mass perturbation theory exhibits increasingly singular contributions.

Still, the theory is rendered ultraviolet finite by mass renormalization alone, and it is easy to show that the bare mass is given in terms of the cutoff Λ and some finite mass by the relation[13]

$$m_0{}^2(\Lambda,\mu)=\mu^2\left[\frac{\mu^2}{\Lambda^2}\right]^{\hbar\beta^2/16\pi}\ . \tag{4.1}$$

Hence, there is no reason to believe that the theory itself does not make sense; rather, the usual calculational methods fail to be applicable. In fact, calculation of the effective potential and in principle also of the effective action *can* be performed in a perturbative expansion in β. So the complication inherent to this system seems to lie in the choice of a ground state, as calculation of neither the effective potential nor the effective action require the choice of such a state *a priori.*

Goldstone[14] has evaluated the effective potential for the ordinary Liouville theory with the help of certain properties of the normal ordering of exponentials. For our supersymmetric theory, normal ordering may be suspect, so we shall rather use direct functional methods alone. We shall evaluate the effective potential for the bosonic field only. The generating functional

$$Z_\mu(J)=\int \mathscr{D}\Phi\,\mathscr{D}\Psi \exp\left[\frac{i}{\hbar}S_\mu+\frac{i}{\hbar}\int J\Phi\right], \tag{4.2}$$

with

$$S_\mu=\int d^2x\left[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi+\frac{i}{2}\bar{\Psi}\partial\Psi-\frac{m_0{}^2(\Lambda,\mu)}{\beta^2}e^{\beta\Phi}-\frac{m_0(\Lambda,\mu)}{2\sqrt{2}}e^{\beta\Phi/2}\bar{\Psi}\Psi\right], \tag{4.3}$$

is clearly finite in the limit where $\Lambda\to\infty$, and as a consequence does not actually depend on Λ in that limit. (All renormalized Green's functions are cutoff independent.) Henceforth we restrict ourselves to constant, space-time independent J, and define the function

$$W_\mu(J)=\lim_{V\to\infty}\frac{\hbar}{Vi}\ln Z_\mu(J)\ . \tag{4.4}$$

This limit exists for $J>0$ since the theory effectively has a nonzero mass gap. Since $W_\mu(J)$ is of dimension 2 in mass, and only depends on μ, J, and β, a scaling argument tells us that

$$W_\mu(J)=\mu^2 w(J\mu^{-2})\ , \tag{4.5}$$

where w is a dimensionless function that only depends on the combination $J\mu^{-2}$. Recall that the vacuum expectation value $\bar{\Phi}$ in the presence of the source J is given by

$$\bar{\Phi}=\frac{\partial W_\mu(J)}{\partial J}\bigg|_{J=J(\bar{\Phi})}\ . \tag{4.6}$$

Making a shift of the field Φ by a constant Φ_0 in the functional integrals leads to the relation

$$W_{\mu_0}(J)=W_{\mu(\Phi_0)}(J)+J\Phi_0\ , \tag{4.7}$$

where the function $\mu(\Phi_0)$ is defined by

$$m_0(\Lambda,\mu_0)=m_0(\Lambda,\mu(\Phi_0))e^{-\beta\Phi_0/2}\ . \tag{4.8}$$

Upon taking the derivative of (4.7) with respect to J, holding Φ_0 fixed, yields

$$\bar{\Phi}=\frac{\partial W_{\mu_0}(J)}{\partial J}\bigg|_{J=J(\bar{\Phi})}=\frac{\partial W_{\mu(\Phi_0)}(J)}{\partial J}\bigg|_{J=J(\bar{\Phi})}+\Phi_0\ . \tag{4.9}$$

For $\bar{\Phi}=\Phi_0$, this relation reduces to

$$\frac{\partial W_{\mu(\Phi_0)}(J)}{\partial J}\bigg|_{J=J(\Phi_0)}=0\ . \tag{4.10}$$

With the help of (4.5), we can rewrite (4.10) in the form

$$\frac{\partial W_\mu(J)}{\partial J}\bigg|_{J=J(\Phi_0)\mu^{-2}(\Phi_0)\mu_0{}^2}=0\ . \tag{4.11}$$

Now, on the other hand, we have by definition that

$$\frac{\partial W_\mu(J)}{\partial J}\bigg|_{J=J(0)}=0\ . \tag{4.12}$$

Provided $J(\phi)$ is a monotonic function, we find

$$J(\Phi_0)=J(0)\mu^2(\Phi_0)\mu_0^{-2}\ . \tag{4.13}$$

Using the expression for $\mu(\Phi)$ deduced from (4.8),

$$\mu^2(\Phi_0)=\mu_0{}^2e^{\hat{\beta}\Phi_0}\ , \tag{4.14}$$

where

$$\hat{\beta}=\frac{\beta}{1+\hbar\beta^2/16\pi}\ , \tag{4.15}$$

we find that

$$J(\Phi_0)=J(0)e^{\hat{\beta}\Phi_0}\ , \tag{4.16}$$

which is indeed monotonic and positive. From the relation

$$\frac{\partial V_{\text{eff}}(\Phi)}{\partial\Phi}=J(\Phi) \tag{4.17}$$

we extract the effective potential using (4.16):

$$V_{\text{eff}}(\Phi)=\gamma(\beta)e^{\hat{\beta}\Phi}+\gamma'(\beta)\ . \tag{4.18}$$

Here γ and γ' are functions of β only.

The effective potential may also be evaluated in the loop expansion. The contribution of the purely bosonic Feynman diagrams can be taken directly from a previous calculation[2]:

$$V_{\text{eff}}^{\text{bosonic}}(\Phi)=\frac{m_0{}^2}{\beta^2}e^{\beta\Phi}\left\{1+\frac{\hbar\beta^2}{8\pi}\left[\ln\frac{\Lambda^2}{m_0{}^2}-\beta\Phi+1\right]+\frac{1}{2}\left[\frac{\hbar\beta^2}{8\pi}\right]^2\left[\left[\ln\frac{\Lambda^2}{m_0^2}-\beta\Phi\right]^2+C\right]\right\}+O(\hbar^3)\ ,\quad C=3.05208 \tag{4.19}$$

and the fermionic contribution is easily evaluated:

$$V_{\rm eff}^{\rm fermionic}(\Phi)=\frac{m_0{}^2}{\beta^2}e^{\beta\Phi}\left\{-\frac{\hbar\beta^2}{16\pi}\left[\ln\frac{\Lambda^2}{m_0{}^2}-\beta\Phi+1+\ln2\right]\right.$$
$$\left.+\frac{1}{2}\left[\frac{\hbar\beta^2}{16\pi}\right]^2\left[-3\left[\ln\frac{\Lambda^2}{m_0^2}-\beta\Phi\right]^2-2\ln2\left[\ln\frac{\Lambda^2}{m_0{}^2}-\beta\Phi\right]+(\ln2)^2-8G\right]\right\}+O(\hbar^3)\ . \quad (4.20)$$

Here G is Catalan's constant ($G=0.915961\cdots$). Upon carrying out the renormalization prescribed in (4.1) to this order, we find

$$V_{\rm eff}(\Phi)=\frac{\mu^2}{\beta^2}e^{\beta\Phi}\left[1+\frac{\hbar\beta^2}{16\pi}(1-\beta\Phi+\ln2)+\frac{1}{2}\left[\frac{\hbar\beta^2}{16\pi}\right]^2[\beta^2\Phi^2+2\ln2\beta\Phi+C+(\ln2)^2-8G]\right]+O(\hbar^3)\ . \quad (4.21)$$

To the same order of approximation in $\hbar$, this expression is equivalent to

$$V_{\rm eff}(\Phi)=\gamma(\beta)e^{\hat\beta\Phi}\ ,$$
$$\gamma(\beta)=1+\frac{\hbar\beta^2}{16\pi}(1-\ln2)+\frac{1}{2}\left[\frac{\hbar\beta^2}{16\pi}\right]^2[C+(\ln2)^2-8G]\ . \quad (4.22)$$

Here $\hat\beta$ is the function defined in (4.15), so that (4.18) completely agrees with (4.22). Furthermore, inspection of higher-order contributions shows that $\gamma'(\beta)=0$ in (4.18). Thus our final expression for the effective potential is

$$V_{\rm eff}(\Phi)=\frac{m_r{}^2}{\beta^2}e^{\hat\beta\Phi}\ . \quad (4.23)$$

The effective potential has no minimum, except of course at the singular configuration $\Phi=-\infty$. Thus, the spectrum, in agreement with conformal invariance, is continuous, bounded from below by zero, and zero is not attained by any translationally invariant eigenstate. In other words, the theory possesses no translationally invariant ground state, a situation familiar from the ordinary Liouville theory. Two plausible alternatives then remain. The theory is perfectly well defined, has a continuous spectrum that extends down to zero energy, but no state of zero energy actually belongs to the spectrum. The second alternative is that a ground state exists that is not translation invariant, and this possibility is investigated in the next section.

V. SPONTANEOUS BREAKDOWN OF SPACE TRANSLATIONS AND $N=1$ POINCARÉ SUPERSYMMETRY

In the previous section, we have shown that the SLT cannot possess a translationally invariant ground state. For the ordinary Liouville theory, it has been argued[3] that a consistent quantum field theory may nevertheless be constructed order by order in perturbation theory, if it is assumed that space translation invariance is broken by the ground state. This mechanism was possible because the Goldstone zero mode that always appears as a result of the symmetry breaking does not possess a normalizable wave function, so that no physical Goldstone boson was produced. As a consequence, the hypothesis of spontaneous breakdown of translation invariance did not violate any physical principles.

For the SLT, we shall show that the same mechanism is applicable and that space translation invariance can be broken without Goldstone bosons. As in the case of the purely bosonic theory, space semicompactifies. We shall establish these results first to one-loop order, then to all orders in perturbation theory.

We start with a time-independent, purely bosonic classical solution. For simplicity, we take the one that has also been studied in the purely bosonic case[3] (we shall indicate the results for other time-independent solutions at the end of the section):

$$\Phi_c(x)=-\frac{1}{\beta}\ln\frac{m^2x^2}{2},\quad \Psi_c=0\ . \quad (5.1)$$

The Killing vectors of this solution are

$$f_H^\mu=\delta^{\mu0}\ ,\quad f_D^\mu=x^\mu\ ,\quad f_K^\mu=2x^\mu x^0-x^2\delta^{\mu0}\ , \quad (5.2a)$$

and the Killing spinors are

$$\epsilon_1=\begin{bmatrix}x^+\\-x^-\end{bmatrix}\chi_0\ ,\quad \epsilon_2=\begin{bmatrix}1\\-1\end{bmatrix}\chi_1\ . \quad (5.2b)$$

Together, they span the timelike superconformal algebra OSp(1.1).

From (2.1), we easily find the bare boson and fermion inverse propagators in the Φ_c background:

$$i\mathscr{D}^{-1}(\Phi_c;x,y)=(-\Box_x-m^2e^{\beta\Phi_c(x)})\delta^2(x-y)\ , \quad (5.3a)$$

$$iS^{-1}(\Phi_c;x,y)=\left[i\partial_x-\frac{m}{\sqrt2}e^{\beta\Phi_c(x)/2}\right]\delta^2(x-y)\ . \quad (5.3b)$$

The small oscillations around the background Φ_c,

$$\Phi=\Phi_c+e^{-i\omega t}\phi_\omega\ , \quad (5.4a)$$

$$\Psi=e^{-i\omega t}\psi_\omega\ , \quad (5.4b)$$

satisfy the equations

$$-\phi_\omega''+\frac{2}{(x^1)^2}\phi_\omega=\omega^2\phi_\omega\ , \quad (5.5a)$$

$$\left[\gamma^0\omega+i\gamma^1\partial_1-\frac{1}{x}\right]\psi_\omega=0\ . \quad (5.5b)$$

The solutions regular at $x=0$ are given by[3]

$$\phi_\omega(x^1)=\phi_\omega^1(x^1)\ , \quad (5.6a)$$

$$\psi_\omega(x^1)=\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\phi_\omega^1(x^1)+\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\phi_\omega^2(x^1)\ , \quad (5.6b)$$

where

$$\phi^1_\omega(x^1)=\left[\frac{2}{\pi}\right]^{1/2}\left[\frac{\sin\omega x^1}{\omega x^1}-\cos\omega x^1\right], \tag{5.6c}$$

$$\phi^2_\omega(x^1)=i\left[\frac{2}{\pi}\right]^{1/2}\sin\omega x^1 . \tag{5.6d}$$

All solutions ϕ_ω,ψ_ω have $\omega^2>0$, so that Φ_c is stable. Please note that the singularity at $x^1=0$ in the background field, responsible for the elimination of the Goldstone boson, is also responsible for the absence of a Goldstone fermion. Both ϕ_ω and ψ_ω are complete only on the half-line $r=x^1>0$, and their normalizations are such that

$$\int_0^\infty d\omega\,\phi_\omega(r)\phi_\omega(r')=\delta(r-r') , \tag{5.7a}$$

$$\int_{-\infty}^\infty d\omega\,\psi_\omega(r)\psi^+_\omega(r')=\mathbb{1}\delta(r-r') \tag{5.7b}$$

with $\mathbb{1}$ the identity matrix in spinor space. On the half-line the Feynman boson and fermion propagators are given by[3]

$$\mathscr{D}(\Phi_c;x,x')=\tfrac{1}{2}\int_0^\infty\frac{d\omega}{\omega}e^{-i\omega|t-t'|}\phi_\omega(r)\phi_\omega(r'), \tag{5.8a}$$

$$S_F(\Phi_c;x,x')=\int_0^\infty d\omega\, e^{-i\omega|t-t'|}\times[\theta(t-t')\psi_{-\omega}(r)\psi^+_{-\omega}(r')-\theta(t'-t)\psi_\omega(r)\psi^+_\omega(r')]\gamma^0 . \tag{5.8b}$$

These are easily computed and we find

$$\mathscr{D}(\Phi_c;x,x')=-\frac{1}{2\pi}+\frac{1}{4\pi}\frac{T^2-r^2-r'^2}{2rr'}\ln\frac{T^2-(r-r')^2-i0}{T^2-(r+r')^2-i0} , \tag{5.9a}$$

$$\begin{aligned}S(\Phi_c;x,x')=&(\gamma^0T-\gamma^1(r-r'))\left[\frac{i}{8\pi rr'}\ln\frac{T^2-(r-r')^2-i0}{T^2-(r+r')^2-i0}-\frac{i}{2\pi}\frac{1}{T^2-(r-r')^2-i0}\right]\\&+(\gamma^0T-\gamma^1(r+r'))i\gamma^1\left[\frac{i}{8\pi rr'}\ln\frac{T^2-(r-r')^2-i0}{T^2-(r+r')^2-i0}-\frac{i}{2\pi}\frac{1}{T^2-(r+r')^2-i0}\right] .\end{aligned} \tag{5.9b}$$

Here we have used the notation $x=(r,t)$, $x'=(r',t')$ and $T=t-t'$.

Both propagators are infrared regular, suggesting that perturbation theory should be infrared finite. Furthermore, these propagators together are invariant under the OSp(1,1) superconformal group, the invariance group of Φ_c.

A. One-loop perturbation theory

First, we compute the correction to the background field by minimizing the one-loop effective action. To this order, we have

$$\Gamma(\Phi,\Psi)=S(\Phi,\Psi)+\tfrac{1}{2}i\hbar\ln\operatorname{Det}(i\mathscr{D}^{-1})-\tfrac{1}{2}i\hbar\ln\operatorname{Det}(iS^{-1}) . \tag{5.10}$$

The shift $\delta\Phi$ is then determined by the equation

$$\begin{aligned}0=&\left.\frac{\delta\Gamma(\Phi,\Psi)}{\delta\Phi(x)}\right|_{\substack{\Phi=\Phi_c+\delta\Phi\\ \Psi=0}}\\=&-\Box\Phi_c-\Box\delta\Phi-\frac{m^2}{\beta}e^{\beta\Phi_c}\left[1+\frac{\delta m^2}{m^2}+\beta\delta\Phi+\frac{\hbar\beta^2}{2}\mathscr{D}(\Phi_c;x,x)-\frac{\hbar\beta^2}{4\sqrt{2}m}e^{-\beta\Phi_c/2}\operatorname{tr}S(\Phi_c;x,x)\right] .\end{aligned} \tag{5.11}$$

Here δm^2 is the mass renormalization ($m_0{}^2=m^2+\delta m^2$). The singularity at coincident points in the boson and fermion propagators must be regularized. This regularization is not unique. The fact that we started with a supersymmetric bare Lagrangian, however, requires that the regularization prescription be invariant under supersymmetry transformations. Still, this leaves a large class of possible schemes, and we shall specialize to the following one:

$$t'=t\ ,\quad r'=r+\lambda(r)\epsilon . \tag{5.12}$$

Here ϵ is a dimensionless infinitesimal cutoff and λ is a function of r with dimensions of length. The function λ further determines the regularization, and taking $\lambda(r)=2r$ for the boson propagator and $\lambda(r)=2re$ for the fermion propagator, the procedure is actually supersymmetric and SO(2,1) invariant. [This combines into OSp(1,1) invariance.] This regularization is clearly preferred, since the theory is OSp(1,1) invariant and we shall henceforth adopt it. Then the regularized propagators take the form

$$\mathscr{D}(\Phi_c;x,x)=-\frac{1}{4\pi}(2+\ln\epsilon^2) , \tag{5.13a}$$

$$S_F(\Phi_c;x,x)=-\mathbb{1}\frac{m}{4\pi\sqrt{2}}e^{\beta\Phi_c/2}(2+\ln\epsilon^2) . \tag{5.13b}$$

The appropriate mass renormalization is

$$\delta m^2=m^2\frac{\hbar\beta^2}{16\pi}\ln\epsilon^2 . \tag{5.14}$$

The solution to (5.11) is now easily obtained:

$$\Phi_c+\delta\Phi=-\frac{1}{\beta_R}\ln\frac{\mu^2r^2}{2}\ . \tag{5.15}$$

Here μ is a finite, renormalized mass:

$$\mu^2=m^2\left[1-\frac{\hbar\beta^2}{8\pi}\right] \tag{5.16}$$

and β_R a renormalized coupling constant, given by[15] $\beta_R=\beta$.

Next, we compute the one-loop inverse boson propagator:

$$\frac{\delta^2\Gamma(\Phi,\Psi)}{\delta\Phi(x)\delta\Phi(y)}\Big|_{\substack{\Phi=\Phi_c+\delta\Phi\\ \Psi=0}}=\left[-\Box_x-m_0{}^2e^{\beta(\Phi_c+\delta\Phi)(x)}-\tfrac{1}{2}\hbar\beta^2m^2e^{\beta\Phi_c(x)}\mathscr{D}(\Phi_c;x,x)+\frac{m\beta^2\hbar}{8\sqrt{2}}e^{\beta\Phi_c(x)/2}\mathrm{tr}S(\Phi_c;x,x)\right]\delta^2(x-y)$$
$$+2i\hbar m^2\beta^2e^{\beta\Phi_c(x)}\mathscr{D}^2(\Phi_c;x,y)-\frac{im^2\beta^2\hbar}{16}e^{\beta\Phi_c(x)/2}e^{\beta\Phi_c(y)/2}\mathrm{tr}S(\Phi_c;x,y)S(\Phi_c;y,x)\ . \tag{5.17}$$

The square of the boson propagator is finite, but the double-fermion propagator still contains an ultraviolet divergence, which must be extracted. From (5.9b), we find

$$\tfrac{1}{2}\,\mathrm{tr}S(\Phi_c;x,y)S(\Phi_c;y,x)=\frac{1}{16\pi^2rr'}\left[\ln\frac{T^2-(r-r')^2-i0}{T^2-(r+r')^2-i0}\right]^2+\frac{1}{4\pi^2}\left[\frac{1}{T^2-(r-r')^2-i0}-\frac{1}{T^2-(r+r')^2-i0}\right]$$
$$=-\frac{i}{4\pi}\left[\ln\frac{\tilde{\lambda}^2(r)}{4r^2}+\ln\epsilon^2\right]\delta^2(x-y)+\text{regular}\ . \tag{5.18}$$

The function $\tilde{\lambda}(x)$ again depends on how the product of the distributions S has been regularized and we choose it to be $\tilde{\lambda}(r)=2er$ so that OSp(1,1) invariance is maintained. This implies

$$\tfrac{1}{2}\mathrm{tr}(\Phi_c;x,y)S(\Phi_c;y,x)=\tfrac{1}{2}[\mathrm{tr}S(\Phi_c;x,y)S(\Phi_c;y,x)]_{\mathrm{reg}}-\frac{i}{4\pi}(2+\ln\epsilon^2)\delta^2(x-y)\ . \tag{5.19}$$

The one-loop inverse boson propagator thus becomes[17]

$$\frac{\delta^2\Gamma(\Phi,\Psi)}{\delta\Phi(x)\delta\Phi(y)}\Big|_{\substack{\Phi=\Phi_c+\delta\Phi\\ \Psi=0}}=\left[-\Box_x+\frac{2}{r^2}\frac{\beta}{\beta_R}\right]\delta^2(x-y)+\frac{4i\hbar\beta^2}{r^2r'^2}\mathscr{D}^2(\Phi_c;x,y)-\frac{i\beta^2\hbar}{8rr'}\left[\mathrm{tr}S(\Phi_c;x,y)S(\Phi_c;y,x)\right]_{\mathrm{reg}}, \tag{5.20}$$

whereas the fermion inverse propagator becomes

$$\frac{\delta^2\Gamma(\Phi,\Psi)}{\delta\Psi(y)\delta\overline{\Psi}(x)}=\left[i\partial_x-\left[\frac{\beta}{\beta_R}\right]^{1/2}\frac{1}{r}\right]\delta^2(x-y)+\frac{i\hbar\beta^2}{4rr'}S(\Phi_c;x,y)\mathscr{D}(\Phi_c;x,y)\ . \tag{5.21}$$

Here again, we find that[15] $\beta=\beta_R$. The one-loop inverse propagators are easily shown to be invariant under the OSp(1,1) conformal supersymmetry as announced. Clearly, translation and $N=1$ Poincaré invariance remain broken.

B. OSp(1,1) invariance to all orders in perturbation theory

We now wish to prove ultraviolet finiteness and OSp(1,1) invariance to all orders of perturbation theory. We shall again restrict ourselves to the use of the OSp(1,1)-invariant regularization scheme. Unfortunately, calculations in component language to higher orders become extremely cumbersome, and the simplicity of the proof of ultraviolet finiteness and invariance is obscured by technical difficulties of imposing the supersymmetry order by order. Thus we have chosen to switch to superfield language and use superpropagators. We start with the superfield action given in (2.8). We expand about the same classical solution

$$H=H_c+h\ ,\quad H_c=\Phi_c-\tfrac{1}{2}\bar{\theta}\theta F_c\ , \tag{5.22a}$$

$$F_c=\frac{m\sqrt{2}}{\beta}e^{\beta\Phi_c/2}\ , \tag{5.22b}$$

and rearrange the quadratic terms

$$S(H)=S(H_c)+\int d^2x\int d^2\theta\left[-\frac{i}{4}\overline{Dh}\,Dh-\frac{im}{2\sqrt{2}}e^{\beta H_c/2}h^2+\mathscr{L}_I\right], \tag{5.23a}$$

$$\mathscr{L}_I(H)=-\frac{i2\sqrt{2}m_0}{\beta^2}e^{\beta H_c/2}e^{\beta h/2}+\frac{i\sqrt{2}m}{\beta}e^{\beta H_c/2}h+\frac{im}{2\sqrt{2}}e^{\beta H_c/2}h^2+\frac{i2\sqrt{2}m}{\beta^2}e^{\beta H_c}\ . \tag{5.23b}$$

With the help of the results established in Eqs. (5.3)—(5.9), it is easily seen that the propagator of h is given by

$$\mathscr{D}_S(\Phi_c;x,y)=\mathscr{D}(\Phi_c;x,y)+\bar{\theta}_x S(\Phi_c;x,y)\theta_y+\frac{i}{4}\bar{\theta}_x\theta_x\bar{\theta}_y\theta_y\delta^2(x-y)\ . \tag{5.24}$$

Now, we see that the only divergent graph in the theory is the tadpole, exactly as in the purely bosonic case.[13] We perform all tadpole contractions on the vertices:

$$e^{\beta h/2}=:e^{\beta h/2}:\exp[\tfrac{1}{8}\beta^2\hbar\mathscr{D}_S(\Phi_c;x,x)] \tag{5.25}$$

and we denote by colons the convention that no two legs of the vertex should be contracted. Thus, the interaction Lagrangian becomes

$$\mathscr{L}_I(h)=-\frac{i2\sqrt{2}m_0}{\beta^2}e^{\beta H_c/2}\exp[\tfrac{1}{8}\beta^2\hbar\mathscr{D}_S(\Phi_c;x,x)]:e^{\beta h/2}:+\frac{i\sqrt{2}m}{\beta}e^{\beta H_c/2}h+\frac{im}{2\sqrt{2}}e^{\beta H_c}h^2+\frac{i2\sqrt{2}m}{\beta^2}e^{\beta H_c}\ . \tag{5.26}$$

Next, to this order, we eliminate the term linear in h by renormalizing m_0 in the following way:

$$m_0\exp[\tfrac{1}{8}\beta^2\hbar\mathscr{D}_S(\Phi_c;x,x)]=m\ . \tag{5.27}$$

Finally, the renormalized interaction Lagrangian becomes

$$\mathscr{L}_I(h)=-\frac{i2\sqrt{2}m}{\beta^2}e^{\beta H_c/2}:(e^{\beta h/2}-\tfrac{1}{8}\beta^2h^2-\tfrac{1}{2}\beta h-1): \tag{5.28}$$

and further generates only finite graphs.[13] Of course, there are still further finite contributions to $\langle\phi\rangle$. However, using time translation invariance as well as the fact that the amplitude is now finite, $\langle\phi\rangle$ must be time independent. Dilation invariance then implies that $\langle\phi\rangle$ is actually constant, and this constant can always be set to zero by an appropriate choice of the renormalized mass μ in (5.16) to higher orders. Since the formalism and the regularization procedure are explicitly OSp(1,1) invariant, and since no further divergences occur, all Green's functions are finite and OSp(1,1) invariant. Translation invariance remains broken to all orders, and no Goldstone particles are produced.

C. Spontaneous breaking of translation invariance and complete compactification of space

Instead of the classical, static solution (5.1), one could have used any other static solution to define the translation-noninvariant vacuum. One might worry that the energy density is higher than for solution (5.1). We shall, however, defend the point of view that there is a preferred expression of the energy, defined through the Hamiltonian (3.28). With this definition of energy, the conformal algebra closes without center, and the energy of the solution under consideration vanishes (by definition). Fluctuations about each of these solutions always have positive energy. Thus the solutions are stable on the basis of this energy criterion. Its invariance group is still OSp(1,1) and still contains time translation, however, no constant supersymmetry is present and the supersymmetry is no longer closing on the conformal group. Still, invariance of the quantum theory to all orders may be proven, as well as its breaking of translation invariance. Of particular interest are the solutions with periodically spaced singularities:

$$\Phi_S(x^1)=-\frac{1}{\beta}\ln\frac{m^2}{2\epsilon}\sin^2\sqrt{\epsilon}(x^1-x_0^1)\ . \tag{5.29}$$

The small-fluctuation functions are then complete only on the interval of length $\pi/\sqrt{\epsilon}$, so that space spontaneously compactifies. Even though the interval is finite, the Goldstone zero mode is not normalizable, and no Goldstone boson appears.

Note added. After this work was completed, two papers by I. F. Arvis have appeared, presenting some of the properties of the superconformal invariance group and classical solutions [Nucl. Phys. B212, 151 (1983) and Report No. LPTENS 82/32 (unpublished)].

ACKNOWLEDGMENTS

I am grateful to Professor Roman Jackiw and Dr. Luc Vinet for several stimulating discussions on the superconformal algebra and for their interest in this work. I have also benefited from discussions with Dr. Baha Balantekin, Professor Daniel Freedman, Professor David Gross, and Dr. Yvan Saint-Aubin. This work was supported in part through funds provided by the U. S. Department of Energy under Contract No. DE-AC02-76ER03069.

APPENDIX A

There is, however, a subtlety, which we examine here only for the case of the ordinary Liouville theory. The result is easily generalized to the case of the SLT. Let F and G parametrize the general solution:

$$\Phi=\frac{1}{\beta}\ln\frac{4F'G'}{m^2(F-G)^2}\ . \tag{A1}$$

This parametrization is unique up to a common projective transformation on F and G:

$$F\to\frac{aF+b}{cF+d}\ ,\quad G\to\frac{aG+b}{cG+d}\ . \tag{A2}$$

A given real solution Φ is not necessarily parametrized by real F and G. In general, the complex numbers F and G must lie on a common circle, defined by

$$X^*=\frac{aX+b}{cX+d} \tag{A3}$$

for some set of *complex* $a,b,c,$ and d, characteristic of the solution Φ. For $c=0$, the circle degenerates to a straight line. Real conformal transformations preserve formula (A3) and thus also preserve the curve on which F and G lie. As a consequence real conformal transformations do not act transitively. However, the set of all complex conformal transformations that obey a relation (A3) for *some* constants a, b, c, and d does act transitively and also closes on itself. The isotropy group is the group of projective transformations that leaves (A3) invariant. For $c=0$, it acts on a noncompact manifold, whereas for $c\neq 0$, the manifold is compact.

---

[1]A. Polyakov, Phys. Lett. 103B, 207 (1981); 103B, 211 (1981); and R. Omnès, Nucl. Phys. B149, 269 (1977).

[2]E. D'Hoker and R. Jackiw, Phys. Rev. D 26, 3517 (1982).

[3]E. D'Hoker and R. Jackiw, Phys. Rev. Lett. 50, 1719 (1983).

[4]T. Curtright and C. Thorn, Phys. Rev. Lett. 48, 1309 (1982); 48, 1768(E) (1982); E. Braaten, T. Curtright, and C. Thorn, Florida University report, 1982 (unpublished); J. L. Gervais and A. Neveu, Nucl. Phys. B199, 59 (1982).

[5]L. Alvarez-Gaumé, D. Z. Freedman, and M. T. Grisaru, Harvard University Report No. HUTMP 81/B111 (unpublished).

[6]L. Alvarez-Gaumé, D. Z. Freedman, and S. Mukhi, Ann. Phys. (N.Y.) 134, 85 (1981).

[7]S. Fubini, A. Hanson, and R. Jackiw, Phys. Rev. D 7, 1732 (1973).

[8]M. Chaichian and P. Kulish, Phys. Lett. 78B, 413 (1978).

[9]The supersymmetric Liouville theory has also been integrated in Ref. 10. The method used here is different in that it starts from the Bäcklund transformation. It is also interesting to note that the superalgebra SU(1/1) encountered in our construction is smaller than the B(0,1) algebra found in Ref. 10.

[10]A. W. Leznov, D. A. Leĭtes, and M. V. Savel'ev, Pis'ma Zh. Eksp. Teor. Fiz. 32, 85 (1980) [JETP Lett. 32, 79 (1980)]; Phys. Lett. 96B, 37 (1980).

[11]Physicists use the name "little group" instead of isotropy group.

[12]We have dropped possible boundary terms in (3.28).

[13]T. Marinucci and S. Sciuto, Nucl. Phys. B156, 144 (1979); Y. Goldschmidt, results presented at the Workshop on String Theory, Niels Bohr Institute and Nordita, Copenhagen, 1981 (unpublished).

[14]J. Goldstone, 1981 (unpublished).

[15]Within the class of supersymmetric regularization prescriptions, a more general choice than the one adopted in Sec. V could be used, like, e.g., $\lambda(r)=2r(r\rho)^{\alpha}\epsilon$, where ρ is some mass scale. It is not hard to see that the theory remains OSp(1,1) invariant, but that β_R is modified and takes the value $\beta_R=\beta(1-\alpha\beta^2\hbar/16\pi)^{-1}$. So the algebra is the same, but its generators have received nontrival quantum corrections. For the ordinary Liouville theory, where the same problem arises, it is shown in Ref. 16 that physical states are classified as unitary representations of SO(2,1) with Casimir value $2\beta_R/\beta$. Theories for different values of α are most likely physically equivalent.

[16]E. D'Hoker, D. Z. Freedman, and R. Jackiw, MIT Report No. CTP 1072, 1983 (unpublished).

[17]The question could be raised as to whether $\mathrm{tr}(SS)_{\mathrm{reg}}$ still contains a contribution proportional to $\delta^2(x-y)$. Indeed, from Eq. (5.18) this statement could not be derived, as the separation of only the divergent part is uniquely specified. It is the supersymmetry invariance that determines the finite part uniquely in terms of the boson tadpole graphs. This issue is most properly treated in the superfield formalism where the resolution to such subtleties is built in (cf. Sec. IV B).