Supersymmetric particles in $N=2$ superspace: Phase-space variables and Hamiltonian dynamics

José A. de Azcárraga* and Jerzy Lukierski†
International Centre for Theoretical Physics, Trieste, Italy
(Received 17 January 1983)

We consider a reparametrization-invariant model recently proposed based on the N-extended super-Poincaré group with central charges, which leads to trajectories on the N-extended Salam-Strathdee superspace. The case $N=2$ is discussed in detail. We show that the $N=2$ model is invariant under four real supergauge transformations generated by first-class odd constraints which imply the Dirac equation. We introduce one bosonic (which fixes the reparametrization) and four real spinorial (which fix the supergauges) gauge conditions and calculate the Dirac brackets for the remaining unconstrained variables $(\vec{x},\vec{p},\theta^{\alpha},\bar{\theta}^{\dot{\alpha}})$. The equations of motion are written in Hamiltonian form, with $H\propto \mathrm{Tr}\{Q_{\alpha i},\bar{Q}_{\dot{\beta} i}\}$ and correspond to the Heisenberg equations of the (first) quantized theory.

I. INTRODUCTION

There have been several attempts (see, e.g., Refs. 1–7 and references therein) to describe relativistic classical supersymmetric systems in the framework of a Grassmann variant of classical mechanics.[8] Recently, the present authors proposed a new pseudoclassical particle model.[9] The model, which is based on the action[10]

$$
\begin{aligned}
&I=\int d\tau L\ ,\\
&L=-m(\dot{\omega}_{\mu}\dot{\omega}^{\mu})^{1/2}+i(\theta_{\alpha i}A_{ij}\dot{\theta}_j^{\alpha}+\bar{\theta}_{\dot{\alpha} i}A_{ij}\dot{\bar{\theta}}_j^{\dot{\alpha}})\ ,\\
&\dot{\omega}^{\mu}\equiv d\omega^{\mu}/d\tau,\ \ A_{ij}=-A_{ji},\ \ \dot{\theta}\equiv d\theta/d\tau\ ,\\
&d\omega^{\mu}=dx^{\mu}-i[d\theta_i^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\theta}_i^{\dot{\beta}}-\theta_i^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}d\bar{\theta}_i^{\dot{\beta}}]
\end{aligned}
\tag{1.1}
$$

has the following features.

(a) It leads to trajectories on the N-extended Salam-Strathdee (SS) superspace $(x^{\mu},\theta^{\alpha i},\bar{\theta}^{\dot{\alpha} i})$ $(\alpha=1,2;\ i=1,2,\ldots,N)$. In the case $N=2$ it has been shown[11] that the first-quantized theory is described by the $N=2$ free massive matter multiplet, which describes one Dirac and two Klein-Gordon fields; the same result has been obtained[12] from the group approach to quantization[13] by using the structure of the underlying U(1)-extended $N=2$ super-Poincaré symmetry of the model. Thus, the model is the first pseudoclassical model which after quantization leads to a free SS superfield.[14]

(b) Under the transformations of N-extended Poincaré supersymmetry realized on SS superspace as

$$
\begin{aligned}
&\theta'^{\alpha i}=\theta^{\alpha i}+\epsilon^{\alpha i},\ \ \bar{\theta}'^{\dot{\alpha} i}=\bar{\theta}^{\dot{\alpha} i}+\bar{\epsilon}^{\dot{\alpha} i}\ ,\\
&x'^{\mu}=x^{\mu}+i[\theta_i^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\epsilon}_i^{\dot{\beta}}-\epsilon_i^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\theta}_i^{\dot{\beta}}]\ ,
\end{aligned}
\tag{1.2}
$$

where $\bar{\theta}^{\dot{\alpha} i}\equiv(\theta^{\alpha i})^*$, $\bar{\epsilon}^{\dot{\alpha} i}\equiv(\epsilon^{\alpha i})^*$, the Lagrangian of (1.1) undergoes the change

$$
\delta L=iA_{ij}\frac{d}{d\tau}(\epsilon_{\alpha i}\theta_j^{\alpha})+i\bar{A}_{ij}\frac{d}{dt}(\bar{\epsilon}_{\dot{\alpha} i}\bar{\theta}_j^{\dot{\alpha}})\ . \tag{1.3}
$$

The fact that δL is not zero but a total derivative implies that the $4N$ generators of the transformations (1.2) $Q_{\alpha i}$ and $\bar{Q}_{\dot{\alpha} i}$ have to be supplemented by the so-called "anomalous" terms in order to obtain a set of conserved charges. [These terms are a consequence of a proper application of the Noether theorem to the case δL is a total derivative as in (1.3) instead of being zero.] Explicitly, one obtains[9]

$$
\begin{aligned}
&\tilde{Q}_{\alpha i}=Q_{\alpha i}(\tau)+iA_{ij}\theta_{\alpha j}(\tau)\ ,\\
&\tilde{\bar{Q}}_{\dot{\alpha} i}=\bar{Q}_{\dot{\alpha} i}(\tau)+i\bar{A}_{ij}\bar{\theta}_{\dot{\alpha} j}(\tau)
\end{aligned}
\tag{1.4}
$$

and the conserved charges $\tilde{Q}_{\alpha i},\tilde{\bar{Q}}_{\dot{\alpha} i}$ now satisfy the algebra of the N-extended super-Poincaré symmetry with central charges,[16]

$$
\begin{aligned}
&\{\tilde{Q}_{\alpha i},\tilde{\bar{Q}}_{\dot{\beta} j}\}=-2\delta_{ij}(\sigma p)_{\alpha\dot{\beta}}\ ,\\
&\{\tilde{Q}_{\alpha i},\tilde{Q}_{\beta j}\}=2i\epsilon_{\alpha\beta}A_{ij}\ ,\\
&\{\tilde{\bar{Q}}_{\dot{\alpha} i},\tilde{\bar{Q}}_{\dot{\beta} j}\}=2i\epsilon_{\alpha\beta}\bar{A}_{ij}\ .
\end{aligned}
\tag{1.5}
$$

From dimensional analysis we get $[A_{ij}]=M$; interestingly enough, one can show that the central charges A_{ij} describe the masses of fermionic "Grassmann" partners of "bosonic" point particles. In order to obtain the supersymmetric model we should have the masses of the bosonic and fermionic particles equal; for $N=2$ this means that we should substitute $m\epsilon_{ij}$ for A_{ij} in the above formulas. We therefore see that we are led to a picture already familiar from the description of a Newtonian particle by the Galilean group, in which the mass parameter determines a central extension of the Galilei group.[17,18]

(c) Both the Klein-Gordon and Dirac equations are described by first-class constraints. Because first-class constraints generate local gauge transformations which leave the action invariant (see, e.g., Refs. 19 and 20) it turns out that half of the Grassmann variables $\theta_i^\alpha,\bar{\theta}_i^{\dot{\alpha}}$ $(i=1,\ldots,N=2k)$ can be eliminated by the gauge-fixing condition. Thus, since our model was defined in $N=2k$ extended SS superspace, only $N/2=k$ extended SS superspace describes the unconstrained Grassmann coordinates.

The aim of this paper is to consider in detail the case $N=2$ and, in particular, the problem of reducing the Grassmann sector by means of the gauge-fixing constraints. In such a way one obtains an unconstrained set of quantum-mechanical variables in supersymmetric quantum mechanics.

The paper is organized as follows.

We introduce in Sec. II the Hamiltonian formalism for our model, and calculate the Dirac brackets $\{A,B\}^*$ which take into account the fermionic second-class constraints. In this section we supplement the brackets given in Ref. 11 with the Dirac brackets of the covariant position variables x^μ. At this stage (one-star brackets) we obtain

$$\{x^\mu,x^\nu\}^* = \frac{-i}{2p^2}\bar{\theta}_1^{\dot{\gamma}}[(\sigma p)_{\alpha\dot{\gamma}}(\sigma^{\mu\nu})^\alpha{}_\rho - (\sigma^{\mu\nu})_{\dot{\gamma}}{}^{\dot{\alpha}}(\sigma p)_{\rho\dot{\alpha}}]\theta_1^\rho \equiv -\frac{S^{\mu\nu}}{p^2}\ , \tag{1.6a}$$

where

$$(\sigma^{\mu\nu})^\alpha{}_\rho \equiv \tfrac{1}{2}[(\sigma^\mu)^{\alpha\dot{\beta}}(\sigma^\nu)_{\rho\dot{\beta}} - (\sigma^\nu)^{\alpha\dot{\beta}}(\sigma^\mu)_{\rho\dot{\beta}}]\ , \tag{1.6b}$$

$$(\sigma^{\mu\nu})_{\dot{\gamma}}{}^{\dot{\alpha}} \equiv \tfrac{1}{2}[(\sigma^\nu)_{\beta\dot{\gamma}}(\sigma^\mu)^{\beta\dot{\alpha}} - (\sigma^\mu)_{\beta\dot{\gamma}}(\sigma^\nu)^{\beta\dot{\alpha}}]\ . \tag{1.6c}$$

[An equivalent way of expressing (1.6a) is given in (2.13).]

In Sec. III we define the local supergauges (C_α and $\bar{C}_{\dot{\alpha}}$) and introduce the subsidiary conditions which fix the associated nonphysical degrees of freedom. Using the iterative property of the Dirac brackets (see, e.g., Ref. 19) we introduce the two-star Dirac bracket $\{A,B\}^{**}$ which takes into consideration all the constraints in the odd sector. It is useful to introduce at the stage of two-star Dirac brackets the complexified phase-space coordinates $Z_A=(z_\mu,\theta_\alpha)$, where $z_\mu=x^\mu-ip_u/p^2-i\Sigma'_\mu$ [the definition of Σ'_μ is given by (3.10)]. It appears that

$$\{Z_A,Z_B\}^{**}=\{\bar{Z}_A,\bar{Z}_B\}^{**}=0\ . \tag{1.7}$$

In order to complete the elimination of the constraints we introduce in Sec. IV the three-star Dirac bracket, taking into account the mass-shell condition $p^2-m^2=0$ and the reparametrization-fixing condition $x^0=\tau$. In this way we finally obtain the set of unconstrained variables $(x_i,p_i,\theta_\alpha,\bar{\theta}_{\dot{\alpha}})$. Further in Sec. IV we introduce the supersymmetric Hamiltonian as the sum of squares of supercharges. It turns out that after taking into account the adequate gauge-fixing conditions we obtain both from the equations derived from the Lagrangian (1.1) and from the Hamilton formalism the equations of motion

$$\dot{\vec{x}}=\vec{p}/p_0,\quad \vec{p}=\text{const},\quad p^0=(m^2+\vec{p}^{\,2})^{1/2}\ ,$$
$$\dot{\theta}^\alpha=0,\quad \dot{\bar{\theta}}^{\dot{\alpha}}=0\ . \tag{1.8}$$

It appears that with our choice of gauge the so-called *Zitterbewegung* term is absent[21] and thus only the nonlinear structure of our supersymmetric phase space indicates the coupling between the space-time and Grassmann degrees of freedom.

Some final comments are made in Sec. V.

II. DESCRIPTION OF THE MODEL

The first term in the Lagrangian (1.1) is the Lagrangian of the relativistic G_4 model of Casalbuoni.[1] The novelty of the model (1.1), in comparison with other proposals,[1-6] is the presence of a bilinear fermionic term in the action. This term is nontrivial because $N\geq 2$ and the A's are antisymmetric; indeed, it is through its presence that enters the necessity of using N-extended superspace since for $N=1$, the only bilinear term which can be added becomes a total derivative $[\dot{\theta}^\alpha\theta_\alpha+\dot{\bar{\theta}}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}=\frac{1}{2}(d/d\tau)(\theta^\alpha\theta_\alpha+\theta^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})]$.

We shall take as canonical variables of the model $x^\mu,p_\mu,\theta_i^\alpha,\bar{\theta}_i^{\dot{\alpha}},\pi_{\alpha i},\bar{\pi}_{\dot{\alpha}i}$. The momenta are defined by

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = -m(\dot{\omega}_\nu\dot{\omega}^\nu)^{-1/2}\dot{\omega}_\mu\ , \tag{2.1a}$$

$$\pi_{\alpha i} \equiv \frac{\partial L}{\partial \dot{\theta}_i^\alpha} = -i\theta_{\alpha j}A_{ji} - i(\sigma p)_{\alpha\dot{\beta}}\bar{\theta}_i^{\dot{\beta}}\ , \tag{2.1b}$$

$$\bar{\pi}_{\dot{\alpha} i} \equiv \frac{\partial L}{\partial \dot{\bar{\theta}}_i^{\dot{\alpha}}} = -i\bar{\theta}_{\dot{\alpha}j}\bar{A}_{ji} - i\theta^\beta(\sigma p)_{\beta\dot{\alpha}}\ , \tag{2.1c}$$

and satisfy the canonical Poisson-brackets relations

$$\{x^\mu,p_\nu\}=\delta^\mu_\nu,\ \ \{\theta^\alpha_i,\pi_{\beta j}\}=\delta^\alpha_\beta\delta_{ij}\ ,$$
$$\{\bar\theta^{\dot\alpha}_i,\bar\pi_{\dot\beta j}\}=\delta^{\dot\alpha}_{\dot\beta}\delta_{ij} \tag{2.2}$$

(all others are zero). From the expressions (2.1) of the canonical momenta it is simple to check the vanishing of the covariant Hamiltonian H,

$$H\equiv\dot x^\mu p_\mu+\dot\theta^\alpha_i\pi_{\alpha i}+\dot{\bar\theta}{}^{\dot\alpha}_i\bar\pi_{\dot\alpha i}-L\equiv 0 \tag{2.3}$$

which indicates the presence of constraints in the model. Indeed, it is simple to check that

$$p^\mu p_\mu=m^2 \tag{2.4}$$

(bosonic constraint of first class) and that

$$G_{\alpha i}\equiv\pi_{\alpha i}-iA_{ij}\theta_{\alpha j}+i(\sigma p)_{\alpha\dot\beta}\bar\theta^{\dot\beta}_i=0\ , \tag{2.5a}$$

$$\bar G_{\dot\alpha i}\equiv\bar\pi_{\dot\alpha i}-i\bar A_{ij}\bar\theta_{\dot\alpha j}+i\theta^\beta_i(\sigma p)_{\beta\dot\alpha}=0 \tag{2.5b}$$

($2N$ spinorial fermionic constraints). Equations (2.5) show that $\pi^*_{\alpha i}=-(\bar\pi_{\dot\alpha i})$ in agreement with the Hermiticity of the Hamiltonian (2.3). Assuming

$$A_{il}\bar A_{jl}=m^2\delta_{ij}\ ,$$

we obtain from (2.5) using (2.4) the relations

$$\bar A_{kl}\pi_{\alpha l}=(\sigma p)_\alpha{}^{\dot\beta}\bar\pi_{\dot\beta k} \tag{2.6a}$$

(N first-class spinorial constraints) which may also be written in the form

$$A_{kl}\bar\pi_{\dot\alpha l}=\pi_{\beta k}(\sigma p)^\beta{}_{\dot\alpha}\ . \tag{2.6b}$$

For the case $N=2$, to which we shall restrict ourselves henceforth, we choose the set of constraints in the following form.

(a) Second class:

$$G_{\alpha 1}\equiv\pi_{\alpha 1}-im\theta_{\alpha 2}+i(\sigma p)_{\alpha\dot\beta}\bar\theta^{\dot\beta}_1=0\ , \tag{2.7a}$$

$$\bar G_{\dot\alpha 1}\equiv\bar\pi_{\dot\alpha 1}-im\bar\theta_{\dot\alpha 2}+i\theta^\beta_1(\sigma p)_{\beta\dot\alpha}=0\ . \tag{2.7b}$$

(b) First class:

$$B\equiv p_\mu p^\mu-m^2=0\ , \tag{2.8a}$$

$$C_\alpha\equiv m\pi_{\alpha 1}+(\sigma p)_\alpha{}^{\dot\beta}\bar\pi_{\dot\beta 2}=0\ , \tag{2.8b}$$

$$\bar C_{\dot\alpha}\equiv m\bar\pi_{\dot\alpha 1}+\pi_{\beta 2}(\sigma p)^\beta{}_{\dot\alpha}=0\ , \tag{2.8c}$$

where we have used that, for $N=2$, $A_{ij}\equiv m\epsilon_{ij}$; it is clear that the constraints (2.7a) [(2.8b)] and (2.7b) [(2.8c)] are complex conjugated. To eliminate the second-order constraints from the theory (and two irrelevant variables, say, $\pi_{\alpha 1},\bar\pi_{\dot\alpha 1}$, by solving these constraints) we introduce the corresponding Dirac brackets[19,20]

$$\{A,B\}^*=\{A,B\}-\{A,G_s\}C^{-1}{}_{ss'}\{G_{s'},B\}\ . \tag{2.9}$$

In the present case, the matrix of the constraints is

$$C\equiv\begin{bmatrix}\{G_{\alpha 1},G_{\beta 1}\} & \{G_{\alpha 1},\bar G_{\dot\beta 1}\}\\ \{\bar G_{\dot\alpha 1},G_{\beta 1}\} & \{\bar G_{\dot\alpha 1},\bar G_{\dot\beta 1}\}\end{bmatrix}=2i\begin{bmatrix}0 & (\sigma p)_{\alpha\dot\beta}\\ (\sigma p)_{\beta\dot\alpha} & 0\end{bmatrix} \tag{2.10}$$

and (2.9) is given by

$$\{A,B\}^*=\{A,B\}-\frac{1}{2i}\{A,G_{\alpha 1}\}\frac{(\sigma^Tp)^{\dot\beta\alpha}}{p^2}\{\bar G_{\dot\beta 1},B\}-\frac{1}{2i}\{A,\bar G_{\dot\alpha 1}\}\frac{(\sigma^Tp)^{\dot\alpha\beta}}{p^2}\{G_{\beta 1},B\}\ . \tag{2.11}$$

The nonvanishing Dirac brackets are, explicitly, as follows.

(a) Fermionic sector[22]

$$\begin{aligned}
&\{\theta^\rho_1,\bar\theta^{\dot\sigma}_1\}^*=\frac{i}{2}\frac{(\sigma p)^{\rho\dot\sigma}}{p^2}\ ,\\
&\{\theta^\rho_1,\pi_{\sigma 1}\}^*=\tfrac12\delta^\rho_\sigma,\ \ \{\bar\theta^{\dot\rho}_1,\bar\pi_{\dot\sigma 1}\}^*=\tfrac12\delta^{\dot\rho}_{\dot\sigma}\ ,\\
&\{\theta^\rho_1,\bar\pi_{\dot\sigma 2}\}^*=\frac12\frac{m}{p^2}(\sigma p)^{\rho\dot\gamma}\epsilon_{\dot\gamma\dot\sigma}\ ,\\
&\{\bar\theta^{\dot\rho}_1,\pi_{\sigma 2}\}^*=\frac12\frac{m}{p^2}(\sigma p)^{\gamma\dot\rho}\epsilon_{\gamma\sigma}\ ,\\
&\{\theta^\rho_2,\pi_{\sigma 2}\}^*=\delta^\rho_\sigma,\ \ \{\bar\theta^{\dot\rho}_2,\bar\pi_{\dot\sigma 2}\}^*=\delta^{\dot\rho}_{\dot\sigma}\ ,\\
&\{\pi_{\rho 1},\bar\pi_{\dot\sigma 1}\}^*=-\frac{i}{2}(\sigma p)_{\rho\dot\sigma}\ ,\\
&\{\pi_{\rho 2},\bar\pi_{\dot\sigma 2}\}^*=-\frac{i}{2}\frac{m^2}{p^2}(\sigma p)_{\rho\dot\sigma}\ ,\\
&\{\pi_{\rho i},\pi_{\sigma j}\}^*=\frac{i}{2}m\epsilon_{\rho\sigma}\epsilon_{ij},\ \ \{\bar\pi_{\dot\rho i},\bar\pi_{\dot\sigma j}\}^*=\frac{i}{2}m\epsilon_{\dot\rho\dot\sigma}\epsilon_{ij}\ .
\end{aligned} \tag{2.12}$$

(b) Bosonic sector

Apart from $\{x^\mu,p_\nu\}^*=\delta^\mu_\nu$ we have

$$\{x^\mu,x^\nu\}^*=-\frac{i}{2p^2}\{(\sigma^\mu)_{\rho\dot\alpha}(\sigma p)^{\beta\dot\alpha}(\sigma^\nu)_{\beta\dot\gamma}-(\sigma^\nu)_{\rho\dot\alpha}(\sigma p)^{\beta\dot\alpha}(\sigma^\mu)_{\beta\dot\gamma}\}\bar\theta^{\dot\gamma}_1\theta^\rho_1\ . \tag{2.13}$$

(c) Even-odd sector

$$\begin{aligned}
&\{x^\mu,\theta^\sigma_1\}^*=-\tfrac12\theta^\gamma_1(\sigma^\mu)_{\gamma\dot\alpha}\frac{(\sigma p)^{\sigma\dot\alpha}}{p^2}\ ,\\
&\{x^\mu,\bar\theta^{\dot\sigma}_1\}^*=-\tfrac12(\sigma p)^{\alpha\dot\sigma}(\sigma^\mu)_{\alpha\dot\gamma}\bar\theta^{\dot\gamma}_1\ ,\\
&\{x^\mu,\pi_{\sigma 1}\}^*=-\frac{i}{2}(\sigma^\mu)_{\sigma\dot\gamma}\bar\theta^{\dot\gamma}_1\ ,\\
&\{x^\mu,\bar\pi_{\dot\sigma 1}\}^*=-\frac{i}{2}\theta^\gamma_1(\sigma^\mu)_{\gamma\dot\sigma}\ ,\\
&\{x^\mu,\bar\pi_{\dot\sigma 2}\}^*=-\frac{i}{2}m\theta_{\gamma 1}(\sigma^\mu)^{\gamma\dot\alpha}\frac{(\sigma p)_{\sigma\dot\alpha}}{p^2}\ ,\\
&\{x^\mu,\bar\pi_{\dot\sigma 2}\}^*=-\frac{i}{2}m\frac{(\sigma p)_{\alpha\dot\sigma}}{p^2}(\sigma^\mu)^{\alpha\dot\gamma}\bar\theta_{\dot\gamma 1}\ .
\end{aligned} \tag{2.14}$$

Expressions such as $\{p^\mu$, an odd variable$\}^*$ $\{x^\mu,\theta_2^\alpha\}^*$ are zero. The one-star Dirac bracket satisfies the relations $\{A,G_{\alpha 1}\}^*=0=\{A,\overline{G}_{\dot\alpha 1}\}^*$ for any canonical variable A by construction.

The above Dirac brackets may be used to describe the theory. In particular, the extended supersymmetry transformations are generated by the conserved supercharges[9]

$$\begin{aligned}
\widetilde{Q}_{\alpha 1}&=\pi_{\alpha 1}-i(\sigma p)_{\alpha\dot\beta}\overline{\theta}_1^{\dot\beta}+im\,\theta_{\alpha 2}=2\pi_{\alpha 1}\ ,\\
\widetilde{\overline{Q}}_{\dot\alpha 1}&=\overline{\pi}_{\dot\alpha 1}-i\theta_1^{\beta}(\sigma p)_{\beta\dot\alpha}+im\overline{\theta}_{\dot\alpha 2}=2\overline{\pi}_{\dot\alpha 1}\ ,\\
\widetilde{Q}_{\alpha 2}&=\pi_{\alpha 2}-i(\sigma p)_{\alpha\dot\beta}\overline{\theta}_2^{\dot\beta}-im\,\theta_{\alpha 1}\ ,\\
\widetilde{\overline{Q}}_{\dot\alpha 2}&=\overline{\pi}_{\dot\alpha 2}-i\theta_2^{\beta}(\sigma p)_{\beta\dot\alpha}-im\overline{\theta}_{\dot\alpha 1}\ ,
\end{aligned}\tag{2.15}$$

which, because they all have vanishing Poisson brackets with the constraints $G_{\alpha 1},\overline{G}_{\dot\alpha 1}$, satisfy the same algebra relations (1.5) now expressed in the form

$$\begin{aligned}
&\{\widetilde{Q}_{\alpha i},\widetilde{\overline{Q}}_{\dot\beta j}\}^*=-2\delta_{ij}(\sigma p)_{\alpha\dot\beta}\ ,\\
&\{\widetilde{Q}_{\alpha i},\widetilde{Q}_{\beta j}\}^*=2im\,\epsilon_{\alpha\beta}\epsilon_{ij}\ ,\\
&\{\widetilde{\overline{Q}}_{\dot\alpha i},\widetilde{\overline{Q}}_{\dot\beta j}\}^*=2im\,\epsilon_{\dot\alpha\dot\beta}\epsilon_{ij}\ ,
\end{aligned}\tag{2.16}$$

and produce the adequate transformations on $x^\mu,\theta^{\alpha i},\overline{\theta}^{\dot\alpha i}$

$$\begin{aligned}
&\{x^\mu,\widetilde{Q}_{\alpha i}\}^*=-i(\sigma^\mu)_{\alpha\dot\beta}\overline{\theta}_i^{\dot\beta}\ ,\\
&\{x^\mu,\widetilde{\overline{Q}}_{\dot\alpha i}\}^*=-i\theta_i^{\gamma}(\sigma^\mu)_{\gamma\dot\alpha}\ ,\\
&\{\theta_i^{\sigma},\widetilde{Q}_{\alpha j}\}^*=\delta_{ij}\delta_\alpha^\sigma,\quad \{\theta_j^{\sigma},\widetilde{\overline{Q}}_{\dot\alpha j}\}^*=0\ ,\\
&\{\overline{\theta}_i^{\dot\sigma},\widetilde{\overline{Q}}_{\dot\alpha j}\}^*=\delta_{ij}\delta_{\dot\alpha}^{\dot\sigma},\quad \{\overline{\theta}_i^{\dot\sigma},\widetilde{Q}_{\alpha j}\}^*=0\ ,
\end{aligned}\tag{2.17}$$

in agreement with the transformation laws (1.2).

In the theory defined by the Dirac brackets (2.12) the first-class constraints (2.8) have not yet been taken into account. There are two ways of quantizing a theory with first-order constraints.

(1) After quantization, i.e., the replacement $\{A,B\}^*\to(1/i\hbar)[\,,\,]_\pm$ of Dirac brackets by anticommutators in the fermionic sector and commutators in the remaining ones, one can impose the first-order constraints as subsidiary conditions on the wave function or quantum state of the system. It was shown by two different methods[11,12] that by adding these conditions as differential constraints on $N=2$ SS superfields one obtains the covariant Klein-Gordon and Dirac equations.

(2) The existence of first-class constraints implies the presence of a gauge freedom in the dynamical system.[19] To fix this gauge freedom one supplements the constraints by a new set of relations (the gauge-fixing relations) between the canonical variables. In this new constrained system with gauge-fixing conditions all constraints are now of second class, and one can proceed to express its dynamics in terms of unconstrained canonical variables by introducing a new Dirac bracket. We shall now follow this second method and proceed to calculate such Dirac brackets for our model in the next section.

III. GAUGE FREEDOM, GAUGE-FIXING CONSTRAINTS IN THE FERMIONIC SECTOR, AND TWO-STAR DIRAC BRACKETS $\{A,B\}^{**}$

The first-class constraints are the generators of the gauge freedom. In our model we shall first consider the supergauge transformations generated by the constraints (2.8b) and (2.8c). The reparametrization invariance generated by the mass-shell condition (2.8a) will be considered in Sec. IV.

We calculate first the supergauge transformations for the set of canonical variables. The generator of the supergauge transformations is given by

$$C=\eta^\sigma C_\sigma+\overline{\eta}^{\dot\sigma}\overline{C}_{\dot\sigma}\ ,\tag{3.1}$$

with C_σ and $\overline{C}_{\dot\sigma}$ given by (2.8b) and (2.8c). From

$$\begin{aligned}
&\delta\theta_1^\alpha(\tau)\equiv\eta^\sigma\{\theta_1^\alpha,C_\sigma\}^*=m\eta^\alpha,\quad \delta\theta_2^\alpha=0\ ,\\
&\overline{\delta}\theta_1^\alpha(\tau)\equiv\overline{\eta}^{\dot\sigma}\{\theta_1^\alpha,\overline{C}_{\dot\sigma}\}^*=0,\quad \overline{\delta}\theta_2^\alpha=(\sigma p)^\alpha{}_{\dot\beta}\overline{\eta}^{\dot\beta}\ ,\\
&\delta\overline{\theta}_1^{\dot\alpha}(\tau)\equiv\eta^\sigma\{\overline{\theta}_1^{\dot\alpha},C_\sigma\}^*=0,\quad \delta\overline{\theta}_2^{\dot\alpha}=\eta^\beta(\sigma p)_\beta{}^{\dot\alpha}\ ,\\
&\overline{\delta}\,\overline{\theta}_1^{\dot\alpha}(\tau)\equiv\overline{\eta}^{\dot\sigma}\{\overline{\theta}_1^{\dot\alpha},\overline{C}_{\dot\sigma}\}^*=m\overline{\eta}^{\dot\alpha},\quad \overline{\delta}\,\overline{\theta}_2^{\dot\alpha}=0\ ,\\
&\delta\ (\text{any }\pi_\alpha,\overline{\pi}_{\dot\alpha})=0=\overline{\delta}\ (\text{any }\pi_\alpha,\overline{\pi}_{\dot\alpha})\ ,\\
&\delta x^\mu\equiv\eta^\sigma\{x^\mu,C_\sigma\}^*=\eta^\sigma\{x^\mu,C_\sigma\}=\eta^\alpha(\sigma^\mu)_\alpha{}^{\dot\gamma}\overline{\pi}_{\dot\gamma 2}\ ,\\
&\overline{\delta}x^\mu\equiv\overline{\eta}^{\dot\alpha}\pi_{\beta 2}(\sigma^\mu)^\beta{}_{\dot\alpha}\ ,
\end{aligned}\tag{3.2}$$

where it may be checked that the action (1.1) is invariant under the supergauge transformations (3.1).

Using this supergauge freedom, half of the Grassmann coordinates may be removed by means of the subsidiary conditions, which we choose to be

$$\theta_1^\alpha=0,\quad \overline{\theta}_1^{\dot\alpha}=0\ .\tag{3.3}$$

Adding (3.3) to (2.8a) and (2.8b) we get a set of second-class constraints, and we may proceed to evaluate the Dirac brackets. From

$$C_{ss'}=\begin{array}{c|cccc} & C_\beta & \bar{C}_{\dot\beta} & \theta_1^\beta & \bar\theta_1^{\dot\beta} \\ \hline C_\alpha & 0 & 0 & mI_2 & 0 \\ \bar{C}_{\dot\alpha} & 0 & 0 & 0 & mI_2 \\ \theta_1^\alpha & mI_2 & 0 & 0 & \frac{-1}{2i}\frac{(\sigma p)^{\alpha\dot\beta}}{p^2} \\ \bar\theta_1^{\dot\alpha} & 0 & mI_2 & \frac{-1}{2i}\frac{(\sigma^T p)^{\dot\alpha\beta}}{p^2} & 0 \end{array}\,, \tag{3.4a}$$

where we have indicated in the first row and the first column the elements which determine the one-star Dirac brackets which constitute the elements of the Dirac matrix $C_{ss'}$, and from

$$C^{-1}=\frac{1}{m^2}\begin{bmatrix} 0 & \frac{1}{2i}\frac{(\sigma^T p)^{\dot\gamma\beta}}{p^2} & mI_2 & 0 \\ \frac{1}{2i}\frac{(\sigma^T p)^{\dot\beta\gamma}}{p^2} & 0 & 0 & mI_2 \\ mI_2 & 0 & 0 & 0 \\ 0 & mI_2 & 0 & 0 \end{bmatrix} \tag{3.4b}$$

we obtain

$$\begin{aligned}\{A,B\}^{**}=\{A,B\}^{*}&-\frac{1}{2i}\{A,C_\alpha\}^{*}\frac{(\sigma^T p)^{\dot\beta\alpha}}{m^2p^2}\{\bar C_{\dot\beta},B\}^{*}-\frac{1}{2i}\{A,\bar C_{\dot\alpha}\}^{*}\frac{(\sigma^T p)^{\alpha\dot\beta}}{m^2p^2}\{C_\beta,B\}^{*}\\ &-\{A,C_\alpha\}^{*}\frac{1}{m}\{\theta_1^\alpha,B\}^{*}-\{A,\bar C_{\dot\alpha}\}^{*}\frac{1}{m}\{\bar\theta_1^{\dot\alpha},B\}^{*}-\{A,\theta_1^\alpha\}^{*}\frac{1}{m}\{C_\alpha,B\}^{*}-\{A,\bar\theta_1^{\dot\alpha}\}\frac{1}{m}\{\bar C_{\dot\alpha},B\}^{*}\,.\end{aligned} \tag{3.5}$$

At this stage we see that because θ_1^α and $\bar\theta_1^\alpha$ vanish and *all* the fermionic constraints (2.5a) and (2.5b) are now second class and can be used for the elimination of $\pi_{\alpha i},\bar\pi_{\dot\alpha i}$, we are left only with $\theta_2^\alpha,\bar\theta_2^{\dot\alpha}$ (the θ^α and $\bar\theta^{\dot\alpha}$ of Sec. I) as unconstrained Grassmann variables of the theory.[23] The set of canonical variables $(x^\mu,p_\mu,\theta_2^\alpha,\bar\theta_2^{\dot\alpha})$ satisfy the following new brackets.

(a) Fermionic sector $\{F,F\}^{**}$

$$\{\theta_2^\rho,\bar\theta_2^{\dot\sigma}\}^{**}=\frac{i}{2}\frac{(\sigma p)^{\rho\dot\sigma}}{m^2}\,, \tag{3.6a}$$

$$\{\theta_2^\rho,\theta_2^\sigma\}^{**}=0,\quad \{\bar\theta_2^{\dot\rho},\bar\theta_2^{\dot\sigma}\}^{**}=0\,. \tag{3.6b}$$

Because of the constraints which define the momenta, we now find

$$\begin{aligned}&\pi_{\alpha 1}=im\,\theta_{\alpha 2},\quad \bar\pi_{\dot\alpha 1}=im\,\bar\theta_{\dot\alpha 2}\,,\\ &\pi_{\alpha 2}=i\frac{m^2}{p^2}(\sigma p)_\alpha{}^{\dot\beta}\bar\theta_{\dot\beta 2},\quad \bar\pi_{\dot\alpha 2}=i\frac{m^2}{p^2}\theta_{\beta 2}(\sigma p)^\beta{}_{\dot\alpha}\,,\end{aligned} \tag{3.7}$$

so that we obtain either from (3.6a) and (3.7) or directly from (3.5)

$$\begin{aligned}&\{\theta_2^\rho,\pi_{\sigma 1}\}^{**}=0,\quad \{\theta_2^\rho,\bar\pi_{\dot\sigma 1}\}^{**}=-\frac{1}{2m}(\sigma p)^\rho{}_{\dot\sigma}\,,\\ &\{\theta_2^\rho,\pi_{\sigma 2}\}^{**}=\tfrac{1}{2}\delta_\sigma^\rho,\quad \{\theta_2^\rho,\bar\pi_{\dot\sigma 2}\}^{**}=0\,.\end{aligned} \tag{3.6c}$$

In addition, the two-star brackets among the momenta are the same as the one-star Dirac brackets [see the last four of (2.12)]. Later on we shall use $\pi_{\alpha 2},\bar\pi_{\dot\alpha 2}$ as fundamental Grassmann variables.

(b) Bosonic sector $\{B,B\}^{**}$

$$\begin{aligned}\{x^\mu,p_\mu\}^{**}&=\delta_\nu^\mu\,,\\ \{x^\mu,x^\nu\}^{**}&=\frac{i}{2m^2p^2}\bar\pi_2^{\dot\gamma}[(\sigma p)_{\alpha\dot\gamma}(\sigma^{\mu\nu})^\alpha{}_\rho-(\sigma^{\mu\nu})_{\dot\gamma}{}^{\dot\alpha}(\sigma p)_{\rho\dot\alpha}]\pi_2^\rho\\ &=\frac{1}{2im^2p^2}[(\sigma^\mu)_{\alpha\dot\gamma}(\sigma p)^{\alpha\dot\beta}(\sigma^\nu)_{\rho\dot\beta}-(\sigma^\nu)_{\alpha\dot\gamma}(\sigma p)^{\alpha\dot\beta}(\sigma^\mu)_{\rho\dot\beta}]\bar\pi_2^{\dot\gamma}\pi_2^\rho\\ &\equiv-\frac{S'^{\mu\nu}}{p^2}\,.\end{aligned} \tag{3.8}$$

(c) Mixed sector $\{B,F\}^{**}$. From (3.5) we obtain

$$\{x^\mu,\theta_2^\sigma\}^{**}=\frac{i}{2}\frac{(\sigma^\mu)^{\sigma\dot\gamma}}{m^2}\bar\pi_{\dot\gamma 2},\quad \{x^\mu,\pi_{\sigma 1}\}^{**}=-\frac{1}{2m}(\sigma^\mu)_{\bar\sigma}{}^{\dot\gamma}\bar\pi_{\dot\gamma 2}\,,$$
$$\{x^\mu,\bar\theta_2^{\dot\sigma}\}^{**}=\frac{i}{2}\pi_{\gamma 2}\frac{(\sigma\mu)^{\gamma\dot\sigma}}{m^2},\quad \{x^\mu,\bar\pi_{\dot\sigma 1}\}^{**}=-\frac{1}{2m}\pi_{\gamma 2}(\sigma^\mu)^\gamma{}_{\dot\sigma}\,,$$
$$\{x^\mu,\pi_{\sigma 2}\}^{**}=-\pi_{\gamma 2}(\sigma^\mu)^{\gamma\dot\alpha}\frac{(\sigma p)_{\sigma\dot\alpha}}{2p^2},\quad \{x^\mu,\bar\pi_{\dot\sigma 2}\}^{**}=-\frac{(\sigma p)_{\alpha\dot\sigma}}{2p^2}(\sigma^\mu)^{\alpha\dot\gamma}\bar\pi_{\dot\gamma 2}\,,\tag{3.9}$$
$$\{p^\mu,F\}^{**}=0\,,$$

where $\pi_{\alpha i},\bar\pi_{\dot\alpha i}$ are expressed in terms of $\theta_\alpha\equiv\theta_{\alpha 2},\bar\theta_{\dot\alpha}\equiv\bar\theta_{\dot\alpha 2}$ by formula (3.7) and $\theta_1^\alpha=\bar\theta_1^{\dot\alpha}=0$ in consistency with $\{\theta_1^\alpha, B \text{ or } F\}^{**}=0=\{\bar\theta_1^{\dot\alpha}, B \text{ or } F\}^{**}$.

The choice of variables $(x^\mu,p_\mu,\pi_{\alpha 2},\pi_{\dot\alpha 2})$ does not exhibit fully the structure of the covariant Poisson brackets $\{A,B\}^{**}$ and the geometry of nonlinear phase superspace. We introduce the set of complex phase-space variables

$$z^\mu=x^\mu+i\frac{p^\mu}{p^2}+i\Sigma'^\mu,\quad \bar z^\mu=x^\mu-i\frac{p^\mu}{p^2}-i\Sigma'^\mu,\quad \Sigma'^\mu\equiv\pi_{\alpha 2}\frac{(\sigma^\mu)^{\alpha\dot\beta}}{m^2}\pi_{\dot\beta 2}\,.\tag{3.10}$$

From (3.8) and (3.9) one gets

$$\{\Sigma'^\mu,\Sigma'^\nu\}^{**}=-\frac{S'^{\mu\nu}}{p^2}\,,$$
$$\{z^\mu,z^\nu\}^{**}=0,\quad \{\bar z^\mu,\bar z^\nu\}^{**}=0,\quad \{z^\mu,\pi_{\sigma 2}\}^{**}=0,\quad \{\bar z^\mu,\bar\pi_{\dot\sigma 2}\}^{**}=0\,,\tag{3.11a}$$

and $(p^\mu=[(z^\mu-\bar z^\mu)/2i-\Sigma'^\mu]p^2)$

$$\{z^\mu,\bar z^\nu\}^{**}=\frac{i}{2}g^{\mu\nu}(z^\sigma-\bar z^\sigma)(z_\sigma-\bar z_\sigma)+g^{\mu\nu}(z^\sigma-\bar z^\sigma)\Sigma'_\sigma-i(z^\mu-\bar z^\mu)(z^\nu-\bar z^\nu)$$
$$-[(z^\mu-\bar z^\mu)\Sigma'^\nu+(z^\nu-\bar z^\nu)\Sigma'^\mu]-2S'^{\mu\nu}\left[\frac{z^\sigma-\bar z^\sigma}{2i}-\Sigma'^\sigma\right]^2,\tag{3.11b}$$

$$\{z^\mu,\bar\pi_{\dot\sigma 2}\}^{**}=-(\sigma^\xi)_{\alpha\dot\sigma}(\sigma^\mu)^{\alpha\dot\gamma}\left[\frac{z_\xi-\bar z_\xi}{2i}-\Sigma'_\xi\right]\bar\pi_{\dot\gamma 2}\,,$$
$$\{\bar z^\mu,\pi_{\sigma 2}\}^{**}=-\pi_{\gamma 2}(\sigma^\mu)^{\gamma\dot\alpha}(\sigma^\xi)_{\sigma\dot\alpha}\left[\frac{z_\xi-\bar z_\xi}{2i}-\Sigma'_\xi\right].\tag{3.11c}$$

The above set of relations (3.11) can be written down in compact form as

$$\{Z_A,Z_B\}^{**}=0,\quad \{\bar Z_A,\bar Z_B\}^{**}=0\,,\tag{3.12a}$$
$$\{Z_A,\bar Z_B\}=g_{A\bar B}(z,\bar z)\,,\tag{3.12b}$$

where $Z_A=(z^\mu,\pi_{\alpha 2})$ describes the complex coordinates of Hermitian complex six-dimensional superspace with four bosonic and two fermionic complex coordinates.

Finally we shall express all charges $\tilde Q_{\alpha i},\tilde{\bar Q}_{\dot\alpha i}$ in terms of the independent fermionic variables $\theta_2^\alpha,\bar\theta_2^{\dot\alpha}$; we have [cf. (2.15)]

$$\tilde Q_{\alpha 1}=2\pi_{\alpha 1}=2im\theta_{\alpha 2},\quad \tilde{\bar Q}_{\dot\alpha 1}=2\bar\pi_{\dot\alpha 1}=2im\bar\theta_{\dot\alpha 2}\,,\tag{3.13a}$$

$$\tilde Q_{\alpha 2}=\left[1+\frac{m^2}{p^2}\right]i(\sigma p)_\alpha{}^{\dot\beta}\bar\theta_{\dot\beta 2}\,,$$
$$\tilde{\bar Q}_{\dot\alpha 2}=\left[1+\frac{m^2}{p^2}\right]i\theta_{\beta 2}(\sigma p)^\beta{}_{\dot\alpha}\,.\tag{3.13b}$$

Their Poisson brackets may be evaluated from (3.13a) (3.13b), and (3.6a) or directly from the expression of the charges (2.15) and the complete set (3.6a) and (3.6b) of two-star fermionic Dirac brackets. The result is

$$\{\tilde{Q}_{\alpha 1},\tilde{\bar{Q}}_{\dot\beta 1}\}^{**}=-2i(\sigma p)_{\alpha\dot\beta}\,;$$
$$\{\tilde{Q}_{\alpha 2},\tilde{\bar{Q}}_{\dot\beta 2}\}^{**}=-i(\sigma p)_{\alpha\dot\beta}\left[1+\frac{p^2}{m^2}+\frac{m^2}{p^2}\right], \quad (3.14)$$
$$\{\tilde{Q}_{\alpha 1},\tilde{Q}_{\beta 2}\}^{**}=im\epsilon_{\alpha\beta}\left[1+\frac{p^2}{m^2}\right].$$

The expressions involving $\tilde{Q}_{\alpha 2},\tilde{\bar{Q}}_{\dot\alpha 2}$ do not quite coincide with (2.16) or (1.5). This is due to the fact that $\{\tilde{Q}_{\alpha 2},C_\sigma\}^*$ and $\{\tilde{\bar{Q}}_{\dot\alpha 2},\bar{C}_{\dot\sigma}\}^*$ are not zero, but $i\epsilon_{\alpha\sigma}(p^2-m^2)$. When the mass-shell condition is taken into account, the usual U(1) extended $N=2$ super-Poincaré algebra is recovered.

IV. REPARAMETRIZATION INVARIANCE, $x^0=\tau$ GAUGE CONDITION, AND HAMILTON DYNAMICS

In order to obtain the set of unconstrained canonical variables we introduce the last set of second-class constraints

$$\varphi_1\equiv p^\mu p_\mu-m^2=0,\quad \varphi_2=x^0-\tau=0\,. \quad (4.1)$$

Since

$$C_{ss'}=\{\varphi_s,\varphi_{s'}\}=2p^0\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$
$$C^{-1}=\frac{1}{2p^0}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad (4.2)$$

we find that the "physical" Dirac bracket $\{A,B\}^{***}$ has the form

$$\{A,B\}^{***}=\{A,B\}^{**}-\frac{1}{2p^0}\{A,p^2\}^{**}\{x^0,B\}^{**}+\frac{1}{2p^0}\{A,x^0\}^{**}\{p^2,B\} \quad (4.3)$$

from which we obtain the following brackets for the unconstrained variables $(x^i,p_j,\theta_2^\alpha,\bar\theta_2^{\dot\alpha})$.

(a) *Fermionic sector*

$$\{F,F'\}^{***}=\{F,F'\}^{**}\,. \quad (4.4)$$

(b) *Bosonic sector* $\{B,B'\}^{***}$

$$\{x^i,p_j\}^{***}=\delta^i_j,\quad \{p_i,p_j\}^{***}=0\,,$$
$$\{x^i,x^j\}=-\frac{S^{ij}}{m^2}+\frac{1}{m^2p^2}(p^iS^{0j}+p^jS^{0i})\,. \quad (4.5)$$

(c) *Mixed sector* $\{B,F\}^{***}$

$$\{p_i,\theta_2^\alpha\}=0=\{p_i,\bar\theta_2^{\dot\alpha}\}\,,$$
$$\{x^i,\theta_2^\sigma\}=\frac{1}{2m^2p^0}(p^0\sigma^i-p^i\sigma^0)^{\sigma\dot\gamma}(\sigma p)_{\beta\dot\gamma}\theta_2^\beta\,, \quad (4.6)$$
$$\{x^i,\bar\theta_2^{\dot\sigma}\}=\frac{1}{2m^2p^0}(\sigma p)_{\beta\dot\gamma}(p^0\sigma^i-p^i\sigma^0)^{\beta\dot\sigma}\bar\theta_2^{\dot\gamma}\,.$$

Finally, the physical Poisson brackets for the supercharges $(\tilde{Q}_{\alpha i}=2\pi_{\alpha i},\tilde{\bar{Q}}_{\dot\alpha i}=2\bar\pi_{\dot\alpha i})$ given by (3.13) with $p^2=m^2$ adopt the customary form (1.5) with $A_{ij}=m\epsilon_{ij}$.

In order to describe the motion of the physical degrees of freedom let us observe that the equations of motion which follow from the action (1.1) have the form

$$\frac{d}{dt}p^\mu=0\,, \quad (4.7a)$$
$$\frac{d}{d\tau}[im\theta_j^\alpha\epsilon_{ji}+i(\sigma p)^\alpha{}_{\dot\beta}\bar\theta_i^{\dot\beta}]=0\,, \quad (4.7b)$$
$$\frac{d}{dt}[im\bar\theta_j^{\dot\alpha}\epsilon_{ji}+i\theta_i^\beta(\sigma p)_\beta{}^{\dot\alpha}]=0\,. \quad (4.7c)$$

Using the definition (2.1a) for p_μ and the constraints $x^0=\tau,\theta_1^\alpha=0=\bar\theta_1^{\dot\alpha}$, (4.7a) gives with $x^i\equiv\vec{x}$, $\dot x^i\equiv\dot{\vec{x}}$, $p^i\equiv\vec{p}$, $\theta_2^\alpha\equiv\theta^\alpha,\bar\theta_2^{\dot\alpha}\equiv\bar\theta^{\dot\alpha}$

$$\vec{p}=m\frac{\dot{\vec{x}}-i[\dot\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}]}{([1-i(\dot\theta^\alpha\delta_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha\delta_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta})]^2-\{\dot{\vec{x}}-i[\dot\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}]\}^2)^{1/2}} \quad (4.8a)$$

$$p^0=m\frac{1-i[\dot\theta^\alpha(\sigma^0)_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha(\sigma^0)_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}]}{([1-i(\dot\theta^\alpha\delta_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha\delta_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta})]^2-\{\dot{\vec{x}}-i[\dot\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}]\}^2)^{1/2}}\,, \quad (4.8b)$$

or

$$\dot{\vec{x}}=\frac{\vec{p}}{p^0}\{1-i[\dot\theta^\alpha(\sigma^0)_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha(\sigma^0)_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}]\}+i[\dot\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\bar\theta^{\dot\beta}-\theta^\alpha(\vec\sigma)_{\alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}]\,. \quad (4.9a)$$

Using (4.7b) or (4.7c) we get

$$\dot\theta^\alpha=0,\quad \dot{\bar\theta}^{\alpha}=0 \quad (4.9b)$$

and thus (4.9a) is simply

$$\dot{\vec{x}}=\frac{\vec{p}}{p^0},\quad p^0\equiv(m^2+\vec{p}^{\,2})^{1/2}\,. \quad (4.9c)$$

In order to derive (4.9b) and (4.9c) in the framework of the Hamiltonian dynamics we observe that the Hamiltonian can be expressed as the sum of squares of the supercharges

$$H=\frac{i}{4}\sum_{i=1}^{2}(\sigma^0)^{\alpha\beta}\{\tilde{Q}_{\alpha i},\tilde{\bar{Q}}_{\dot{\beta} i}\}^{***} . \tag{4.10}$$

It is easy to check using the Poisson brackets that

$$H=p^0 \tag{4.11}$$

and that the Hamilton equations

$$\dot{x}^i=\{x^i,H\}^{***} ,$$
$$\dot{\theta}^\alpha=\{\theta^\alpha,H\}^{***} , \tag{4.12}$$
$$\dot{\bar{\theta}}^{\dot{\alpha}}=\{\bar{\theta}^{\dot{\alpha}},H\}^{***} ,$$

have the form (4.9b) and (4.9c).

As remarked in the Introduction, we observe that the equations of motion (4.9b) and (4.9c) do not have the *Zitterbewegung* term, which disappears on-shell due to the choice of the gauge constraints (3.3).[21]

Let us now discuss the quantized form of the Hamiltonian dynamics. As already mentioned, this is obtained in general by replacing the Dirac brackets, obtained by elimination of the second-class constraints of the theory, by commutators or anticommutators. Depending on the number of first-order constraints, or the number of gauge-fixing conditions, we can introduce three quantization schemes

(1) Covariant quantization scheme with mass-shell and fermionic first-class constraints.

Such a scheme is obtained by the replacement

$$\{A,B\}^*\rightarrow\frac{1}{i\hbar}[\hat{A},\hat{B}]_\mp \tag{4.13}$$

and the Klein-Gordon as well as the Dirac equations are obtained as subsidiary conditions on the quantum set of states. As already pointed out, this quantization of our particle model has already been performed.[11,12]

(2) Covariant quantization scheme with only the mass-shell condition.

This scheme follows from the replacement

$$\{A,B\}^{**}\rightarrow\frac{1}{i\hbar}[\hat{A},\hat{B}]_\mp . \tag{4.14}$$

This is the "minimal" manifestly Lorentz-covariant scheme, where only the Klein-Gordon equation is realized as the restriction $p^2=m^2$ for the four-momenta describing the quantum state. The Dirac equation is built-in into the constraints.

(3) Noncovariant quantization with unconstrained set of variables.

This is obtained by means of the correspondence

$$\{A,B\}^{***}\rightarrow\frac{1}{i\hbar}[\hat{A},\hat{B}]_\mp . \tag{4.15}$$

This form of quantization was used in order to obtain the equations of motion (4.9) [see (4.11) and (4.12)]. In such a framework without first-class constraints the Hamiltonian is given by the time-translation generator expressed in terms of unconstrained variables $[p_0=(\vec{p}^2+m^2)^{1/2}$ in our case]. Because the model is supersymmetric, one can use also the formula (4.10), describing the generator of time translations in terms of supercharges.

V. FINAL COMMENTS

In this paper we have described three quantization schemes, corresponding to three choices of the Dirac brackets. The supersymmetric phase space is nonlinear. The covariant structure of the nonlinear phase space can be studied in two directions.

(a) By looking for the choice of phase-space coordinates which simplify the Poisson bracket of fundamental coordinates. However, because the choice of our fundamental Grassmann coordinates is $\theta_\alpha,\bar{\theta}_{\dot{\alpha}}$, there does not exist a constant covariant tensor which could reduce the Dirac bracket $\{\theta_\alpha,\theta_{\dot{\alpha}}\}^{**}$ to a number.

(b) Because $\bar{\theta}_{\dot{\alpha}}=(\theta_\alpha)^*$, by introducing suitable complex four-vector coordinates it is possible to endow the supersymmetric phase space with the structure of a super-Kähler manifold[24] [see, e.g., relations (3.12)]. The closure property of the two-form $\omega_2=g^{A\bar{B}}dZ_A d\bar{Z}_B$ implies the graded Jacobi identities satisfied by Dirac bracket $\{A,B\}^{**}$ by construction (see, e.g., Ref. 27). The Kähler and super-Kähler structures of covariant supersymmetric phase space induced by the two-star Dirac bracket will be considered in detail in a future paper.

ACKNOWLEDGMENTS

Both authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, and financial support. The authors would also like to thank Dr. L. Lusanna for valuable discussions and A. Frydryszak for the participation in the initial stage of this work.

*Permanent address: Departamento de Fisica Teórica, Facultad de Ciencias Físicas, Universidad de Valencia, Burjasot (Valencia), Spain.

†Permanent address: Institute for Theoretical Physics, University of Wrocław, 50-205 Wrocław, Poland.

[1]R. Casalbuoni, Nuovo Cimento 33A, 389 (1976).

[2]F. A. Berezin and M. A. Marinov, Ann. Phys. (N.Y.) 104, 336 (1977).

[3]D. V. Volkov and A. I. Pashnev, Teor. Mat. Fiz. 44, 321 (1980) [Theor. Math. Phys. 44, 770 (1980)].

[4]C. A. P. Galvao and C. Teitelboim, J. Math. Phys. 21, 1863 (1980).

[5]A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cimento 35A, 377 (1976).

[6]L. Brink and J. H. Schwarz, Phys. Lett. 100B, 310 (1981).

[7]D. J. Almond, Phys. Lett. 101B, 315 (1981).

[8]The basic aspects of the Grassmann extension of classical mechanics, which is also called pseudomechanics, are discussed, e.g., in Refs. 1 and 2.

[9]J. A. de Azcárraga and J. Lukierski, Phys. Lett. 113B, 170 (1982).

[10]Our conventions for dotted and undotted indices are those of V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Pergamon, London, 1981). In particular

$$(\sigma p)^{\alpha\dot{\beta}}(\sigma p)_{\gamma\dot{\beta}}=\delta^{\alpha}_{\gamma}p^2\ [(\sigma p)^{\alpha\dot{\beta}}\equiv(\sigma^{\mu})^{\alpha\dot{\beta}}p_{\mu}]\ .$$

[11]A. Frydryszak and J. Lukierski, Phys. Lett. 117B, 51 (1982).

[12]V. Aldaya and J. A. de Azcárraga, Phys. Lett. 121B, 331 (1983); J. A. de Azcárraga, talk at the International Colloquium on Group Theoretical Methods in Physics, Istanbul, 1982 (unpublished) and at the Supersymmetry Workshop, Trieste, 1982 (unpublished).

[13]V. Aldaya and J. A. de Azcárraga, J. Math. Phys. 23, 1297 (1982).

[14]SS superfields are used to describe four-dimensional super-Poincaré-invariant theories (Ref. 15).

[15]For a review of the SS superfield formalism, see, e.g., A. Salam and J. Strathdee, Fortschr. Phys. 26, 57 (1978); P. Fayet and S. Ferrara, Phys. Rep. 32C, 249 (1977).

[16]R. Hagg, J. Łopuszański, and M. Sohnius, Nucl. Phys. B88, 257 (1975).

[17]V. Bargmann, Ann. Math. 59, 1 (1954).

[18]J. M. Lévy-Leblond, in *Group Theory and its Applications,* Vol. 2, edited by E. M. Loebl (Academic, New York, 1971).

[19]A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1976). See also K. Sundermeyer, *Constrained Dynamics* (Springer, Berlin, 1982).

[20]See, e.g., P. A. M. Dirac, *Lectures on Quantum Mechanics,* Yeshiva University-Belfer Graduate School of Science (Academic, New York, 1964). Dirac brackets are also discussed in Ref. 28, Chaps. 8 and 9.

[21]It could be mentioned that also in the formulation with vectorlike Grassmann coordinates it has been shown that there exists a supergauge fixing condition which removes the *Zitterbewegung* term (see Ref. 2).

[22]Although $\pi_{\alpha 1},\overline{\pi}_{\dot{\alpha}1}$ can be expressed by means of relations (2.7) in terms of the other canonical variables we shall include them in the following formulas for completeness.

[23]We could also take $\pi_{\alpha 2},\overline{\pi}_{\dot{\alpha}2}$; see (3.7).

[24]A supersymmetric extension of Kähler manifolds (Ref. 25) was proposed in Ref. 26 by one of the present authors. For the nonlinear bosonic Kähler-Hamilton mechanics see Ref. 27.

[25]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,* Vol. II (Interscience, New York, 1969), Chap. IX.

[26]J. Lukierski, in *Supergravity,* edited by P. van Nieuwenhuizen and D. Z. Freedman (North-Holland, Amsterdam, 1979), p. 301.

[27]F. A. Berezin, Commun. Math. Phys. 63, 131 (1978).

[28]E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, London, 1974).