Shortcut for constructing any Lagrangian from its equations of motion

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We show how to construct the Lagrangians for any regular mechanical system. The Lagrangians are a linear combination of the left-hand sides of their own equations of motion (up to a total derivative). The examples suggest that the same holds even for singular systems in field theory.

In this paper we present an alternative approach for constructing a Lagrangian from its equations of motion (which sometimes is called the inverse problem of the calculus of variations).

The solution of the one-dimensional case was found in 1894 by Darboux.[1] In 1941, Douglas[2] solved the case of two equations for two variables. Recently, much progress has been achieved for the n-dimensional case both for systems of first-order[3] and second-order differential equations.[4,5] Here, we exhibit an explicit solution for the regular n-dimensional second-order case which shows that any Lagrangian may be written (up to a total derivative) as a linear combination of the left-hand side of its own equations of motion.

The inverse problem of the calculus of variations is reduced to the computation of the coefficients in this linear combination for which the solution may be written in terms of the associated constants of motion (when the Lagrangian exists). In spite of the fact that the proof we present here is rigorously valid only for regular mechanical systems, it will be clear from the examples that its realm encompasses any system of equations.

In Ref. 3, it was proved that the most general Lagrangian for the system of $2n$ first-order differential equations

$$\dot{x}^a = f^a(x^b,t)\ ,\quad a,b=1,\ldots,2n \tag{1}$$

is given by

$$L = l_m(C^p)\left[\frac{\partial C^m}{\partial x^a}\dot{x}^a + \frac{\partial C^m}{\partial t}\right]\ , \tag{2}$$

where $C^p = C^p(x^b,t)$ are any set of $2n$ functionally independent constants of motion of Eq. (1) and l_m are arbitrary functions which satisfy

$$\det\left[\frac{\partial l_m}{\partial C^p} - \frac{\partial l_p}{\partial C^m}\right] \neq 0\ . \tag{3}$$

In mechanical systems, Eq. (1) arises from a system of n second-order differential equations,

$$\ddot{q}^i - F^i(q^j,\dot{q}^j,t) = 0\ ,\quad i,j=1,\ldots,n\ , \tag{4}$$

and the definitions

$$x^i = q^i\ ,\quad x^{i+n} = \dot{q}^i\ ,$$
$$f^i = x^{i+n}\ ,\quad f^{i+n} = F^i(x^j, x^{j+n}, t)\ . \tag{5}$$

In what follows we will briefly state the necessary and sufficient conditions allowing us to proceed from any first-order Lagrangian theory to a related second-order Lagrangian formulation (which includes the well-known relationship between Hamiltonian and Lagrangian formalisms). Essentially, half of the first-order equations can always be turned into the definitions of the n variables x^{i+n} as the time derivatives of x^i. The other half of the equations are dynamical equations. The condition

$$\frac{\partial l_m}{\partial x^s}\frac{\partial C^m}{\partial x^r} = \frac{\partial l_m}{\partial x^r}\frac{\partial C^m}{\partial x^s}\ ,$$
$$m=1,\ldots,2n \quad r,s=n+1,\ldots,2n \tag{6}$$

ensures that no third-order derivatives appear in the equations of motion derived from Lagrangian (2) when $\dot{x}^i$ have been substituted for x^{i+n} in L. [No fourth-order derivatives appear anyway because the Lagrangian (2) is linear in $\dot{x}^a$.]

It is known that if $C = C(q^j,\dot{q}^j,t)$ is a constant of motion associated with system (4), then

$$\frac{dC}{dt} \equiv \frac{\partial C}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial C}{\partial q^i}\dot{q}^i + \frac{\partial C}{\partial t} \equiv \frac{\partial C}{\partial \dot{q}^i}(\ddot{q}^i - F^i)\ . \tag{7}$$

Furthermore, it may be proved that any Lagrangian of type (2) may be written as

$$L = D_1\dot{D}_2 + D_3\dot{D}_4 + \cdots + D_{2n-1}\dot{D}_{2n} \tag{8}$$

for suitably chosen functionally independent constants of motion D^p. Lagrangian (8) can then be written as

$$\tilde{L} = \tilde{L}(q^i,\dot{q}^i,\ddot{q}^i,t) = \mu_i[\ddot{q}^i - F^i(q^j,\dot{q}^j,t)]\ , \tag{9}$$

where

$$\mu_i(q^j,\dot{q}^j,t) = D_1\frac{\partial D_2}{\partial \dot{q}^i} + D_3\frac{\partial D_4}{\partial \dot{q}^i} + \cdots + D_{2n-1}\frac{\partial D_{2n}}{\partial \dot{q}^i} \tag{10}$$

and its equations of motion are equivalent to Eq. (4) provided

$$\frac{\partial \mu_i}{\partial \dot{q}^j}=\frac{\partial \mu_j}{\partial \dot{q}^i}\ , \tag{11}$$

$$\frac{\bar{d}}{dt}\left[\frac{\bar{d}}{dt}\mu_i+\mu_j\frac{\partial F^j}{\partial \dot{q}^i}\right]-\mu_j\frac{\partial F^j}{\partial q^i}=0\ , \tag{12}$$

and

$$\det\left[\frac{\partial}{\partial \dot{q}^j}\left[\frac{\bar{d}}{dt}\mu_i+\mu_K\frac{\partial F^K}{\partial \dot{q}^i}\right]+\frac{\partial \mu_j}{\partial q^i}\right]\neq 0\ , \tag{13}$$

where

$$\frac{\bar{d}}{dt}=F^i\frac{\partial}{\partial \dot{q}^i}+\dot{q}^i\frac{\partial}{\partial q^i}+\frac{\partial}{\partial t}\ . \tag{14}$$

As a matter of fact, conditions (11) imply that $\Lambda(q^i,\dot{q}^i,t)$ exists such that

$$L'=\tilde{L}+\frac{d\Lambda}{dt} \tag{15}$$

is a function of q^i, $\dot{q}^i$, and t only, i.e.,

$$\mu_i=-\frac{\partial \Lambda}{\partial \dot{q}^i}\ . \tag{16}$$

A lot of work can be saved by using the appropriate Euler-Lagrange derivative for acceleration-dependent Lagrangians G_i,

$$G_i\equiv-\frac{d^2}{dt^2}\frac{\partial}{\partial \ddot{q}^i}+\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i}-\frac{\partial}{\partial q^i}\ , \tag{17}$$

which is such that

$$G_iL'=\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}^i}-\frac{\partial L'}{\partial q^i}=G_i\tilde{L} \tag{18}$$

because

$$\frac{\partial L'}{\partial \ddot{q}^i}=0\ . \tag{19}$$

Equations of motion are then obtained by writing

$$G_i\tilde{L}=0\ . \tag{20}$$

It may be readily proved that when conditions (11) are met for some D^p, μ_i defined by Eq. (10) satisfies Eqs. (12) and (13). In other words, the necessary and sufficient conditions to solve the inverse problem is the existence of a set of $2n$ functionally independent constants D^p such that Eq. (11) is satisfied. When the constants of motion associated with Eq. (4) are not known, the inverse problem reduces to finding μ_i satisfying Eqs. (11)–(13).

For the one-dimensional problem, Eq. (11) is an identity and there are infinitely many solutions to the inverse problem.[1] In fact,

$$\tilde{L}=C_1\dot{C}_2=C_1\frac{\partial C_2}{\partial \dot{q}}(\ddot{q}-F) \tag{21}$$

are appropriate Lagrangians for Eq. (4) for any C_1,C_2 functionally independent constants of motion related to Eq. (4) for $n=1$.

It is worth noting that condition (12) and Eq. (4) can also be obtained from the Lagrangian (9) by considering μ_i as "Lagrange multipliers" and therefore varying μ_i and q^j independently in (9).

In what follows we give examples of Lagrangians written as linear combinations of their equations of motion for different systems.

(a) Free particle:

$$\ddot{x}=0\ ,\quad L_{\text{usual}}=\tfrac{1}{2}\dot{x}^2\ ,$$
$$\tilde{L}_1=-\tfrac{1}{2}x\ddot{x}\ ,\quad \Lambda_1=\tfrac{1}{2}x\dot{x}\ ,$$
$$\tilde{L}_2=-\dot{x}t\ddot{x}\ ,\quad \Lambda_2=\tfrac{1}{2}\dot{x}^2t\ .$$

(b) Harmonic oscillator:

$$\ddot{x}+x=0\ ,\quad L_{\text{usual}}=\tfrac{1}{2}(\dot{x}^2-x^2)\ ,$$
$$\tilde{L}_3=-\tfrac{1}{2}x(\ddot{x}+x)\ ,\quad \Lambda_3=\tfrac{1}{2}x\dot{x}\ .$$

The equations of motion are obtained using Eq. (20) or by adding $d\Lambda/dt$ to each Lagrangian to recover L_{usual}.

In field theory, examples also exist.

(c) Scalar field:

$$(\partial_\mu\partial^\mu+m^2)\varphi=0\ ,$$
$$L_{\text{usual}}=-\tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi-m^2\varphi^2)\ ,$$
$$\tilde{L}=\tfrac{1}{2}\varphi(\partial_\mu\partial^\mu+m^2)\varphi\ ,\quad \Lambda^\mu=-\tfrac{1}{2}\varphi\partial^\mu\varphi\ .$$

(d) Electron field:

$$(i\partial\!\!\!/-m)\psi=0\ ,$$
$$\tilde{L}=L_{\text{usual}}=\bar{\psi}(i\partial\!\!\!/-m)\psi\ ,\quad \Lambda^\mu\equiv 0\ .$$

(e) Electromagnetic field:

$$(A^{\mu,\nu}-A^{\nu,\mu})_{,\nu}=0\ ,$$
$$L_{\text{usual}}=-\tfrac{1}{4}(A^{\mu,\nu}-A^{\nu,\mu})(A_{\mu,\nu}-A_{\nu,\mu})\ ,$$
$$\tilde{L}=\tfrac{1}{2}A_\mu(A^{\mu,\nu}-A^{\nu,\mu})_{,\nu}\ ,\quad \Lambda^\mu=-\tfrac{1}{2}A_\nu(A^{\nu,\mu}-A^{\mu,\nu})\ .$$

(f) Gravitational field:

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=0\ ,$$
$$\tilde{L}=L_{\text{usual}}=-\sqrt{-g}\,g_{\mu\nu}(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)\ ,\quad \Lambda^\mu\equiv 0\ .$$

As we stated above, even though our proof is, strictly speaking, valid only for regular mechanical systems, the examples eventually reveal the universality of the fact that the Lagrangians of any system are linear combinations of their own equations of motion. This universality is much wider than the usual $T-V$ construction which is valid only when forces are derivable from a potential.

A comment should be added on the relationship of this approach as compared to the ones presented in Refs. 4 and 5. We will briefly describe the relationship of our approach to the one presented in Ref. 5 because a comparison between the work contained in Ref. 5 and the one done in Ref. 4 is already done in Ref. 5.

The variables α_{ij} and β_{ij} defined by Sarlet are related to μ_i through the following relationships:

$$\alpha_{ij} \equiv \frac{\partial}{\partial \dot{q}^j}\left[\frac{\bar{d}}{dt}\mu_i + \mu_k \frac{\partial F^k}{\partial \dot{q}^i}\right] + \frac{\partial \mu_j}{\partial q^i}\,, \tag{22}$$

$$\beta_{ij} = \frac{\partial}{\partial q^i}\left[\frac{\bar{d}}{dt}\mu_j + \mu_k \frac{\partial F^k}{\partial \dot{q}^j}\right] - \frac{\partial}{\partial q^j}\left[\frac{\bar{d}}{dt}\mu_i + \mu_k \frac{\partial F^k}{\partial \dot{q}^i}\right]. \tag{23}$$

The equations

$$\frac{\bar{d}}{dt}\alpha_{ij} = -\frac{1}{2}\left[\alpha_{ik}\frac{\partial F^k}{\partial \dot{q}^j} + \frac{\partial F^k}{\partial \dot{q}^i}\alpha_{kj}\right], \tag{24}$$

$$\frac{\bar{d}}{dt}\beta_{ij} = \alpha_{ik}\frac{\partial F^k}{\partial q^j} - \frac{\partial F^k}{\partial q^i}\alpha_{kj} \tag{25}$$

obtained by Sarlet are obtained by taking partial derivatives of Eq. (12) with respect to $\dot{q}^j$ (q^j) and symmetrizing (antisymmetrizing) with respect to i and j, respectively, and using Eq. (11). The condition

$$\beta_{ij} = -\frac{1}{2}\left[\frac{\partial F^k}{\partial \dot{q}^i}\alpha_{kj} - \alpha_{ik}\frac{\partial F^k}{\partial \dot{q}^j}\right] \tag{26}$$

is obtained by partially differentiating (12) with respect to $\dot{q}^j$ and antisymmetrizing with respect to i and j and using Eq. (11). The other properties of α_{ij} and β_{ij} are easily proved using definitions (22) and (23) and Eq. (11). It is not difficult to prove that if α_{ij} satisfies Eq. (24), then

$$\mu_i = \alpha_{ij}\eta^j \tag{27}$$

satisfies Eq. (12) if η^j satisfies the variation equation

$$\frac{\bar{d}^2}{dt^2}\eta^j - \frac{\partial F^j}{\partial \dot{q}^k}\frac{\bar{d}}{dt}\eta^k - \frac{\partial F^j}{\partial q^k}\eta^k = 0\,, \tag{28}$$

which is related to the symmetries of the equation of motion (4).

If η^j does not depend on $\dot{q}^k$, then μ_i also satisfies Eq. (11).

Equations (24)—(26) are essentially the integrability conditions for Eqs. (11) and (12) satisfied by μ_i.

Finally, we present a more elaborate example, the central-force problem

$$\ddot{q}^i + \frac{\partial V}{\partial q}\frac{q^i}{q} = 0\,,\quad V = V(q)\,,$$
$$q = (q^i q^i)^{1/2}\,,\quad i = 1,2,3\,. \tag{29}$$

The most general Lagrangian for Eq. (29) is given by[6]

$$L = T - V + \tilde{L} \equiv \tfrac{1}{2}\dot{q}^i\dot{q}^i - V + \tilde{L}\,, \tag{30}$$

where $\tilde{L}$ satisfies

$$\tilde{\alpha}_{ij} \equiv \frac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j} + \frac{g(\hat{J})}{J}\left[\delta_{ij} - \frac{\vec{q}^{\,2}\dot{q}_i\dot{q}_j + \dot{\vec{q}}^{\,2} q_i q_j - (\vec{q}\cdot\dot{\vec{q}})(q_i\dot{q}_j + q_j\dot{q}_i)}{J^2}\right],\quad \vec{J} = \vec{q}\times\dot{\vec{q}}\,,\quad J = (\vec{J}\cdot\vec{J})^{1/2}\,,\quad \hat{J} = \vec{J}/J\,, \tag{31}$$

and g is an arbitrary function of its arguments.

Henneaux and Shepley[6] constructed explicitly $\tilde{L}$ for g = constant only, i.e.,

$$\tilde{L} = g\frac{J}{q^2}\,. \tag{32}$$

For the central-forces case it is easy to prove that

$$\vec{\eta} = \vec{n}\times\vec{q}\,,\quad \vec{n} = \text{constant vector}\,, \tag{33}$$

satisfies Eq. (28).

Therefore, for $g \neq$ constant, we get (for $\tilde{L}$), using Eqs. (27), (31), and (33),

$$\tilde{\mu}_i = \frac{h(\hat{J})}{J}\left[\epsilon_{ikl}n^k q^l - \frac{\hat{J}\cdot\vec{n}}{J}[q^2\dot{q}_i - (\vec{q}\cdot\dot{\vec{q}})q_i]\right] = h(\hat{J})\frac{\partial(\hat{J}\cdot\vec{n})}{\partial \dot{q}^i}\,, \tag{34}$$

which is such that

$$\frac{\partial \tilde{\mu}_i}{\partial \dot{q}^j} - \frac{\partial \tilde{\mu}_j}{\partial \dot{q}^i} = 0 \tag{35}$$

and

$$\tilde{\alpha}_{ij} = \frac{g}{J}\left[\delta_{ij} - \frac{\vec{q}^{\,2}\dot{q}_i\dot{q}_j + \dot{\vec{q}}^{\,2} q_i q_j - (\vec{q}\cdot\dot{\vec{q}})(q_i\dot{q}_j + q_j\dot{q}_i)}{J^2}\right], \tag{36}$$

where

$$g = \hat{J}\cdot(\vec{\nabla}h\times\vec{n})\,,\quad (\vec{\nabla}h)_a \equiv \frac{\partial h}{\partial \hat{J}_a}\,. \tag{37}$$

Equations (35) and (36) hold when projected on the basis $\vec{q}$, $\dot{\vec{q}}$, $\vec{J} = \vec{q}\times\dot{\vec{q}}$, which is complete as long as $\vec{q} \nparallel \dot{\vec{q}}$ ($\vec{J} \neq 0$). When $\vec{q} \parallel \dot{\vec{q}}$ ($\vec{J} = 0$) the arguments of the functions g and h are no longer defined. Equation (36) coincides with the result found in Ref. 6 [Eq. (31)]. However, no explicit expression for the Lagrangian when $g \neq$ constant is given there.

The Lagrangian for $g \neq$ constant when $\vec{q} \nparallel \dot{\vec{q}}$ ($\vec{J} \neq 0$) is

$$L = T - V + h(\hat{J})\frac{\partial}{\partial \dot{q}^i}(\hat{J}\cdot\vec{n})\left[\ddot{q}^i + \frac{\partial V}{\partial q}\frac{q^i}{q}\right]. \tag{38}$$

We have not found the Lagrangian as a function of

q^i, $\dot{q}^i$, and t only (in spite of having shown its existence) because the solution of Eq. (35) cannot be written explicitly, in general. Nevertheless, we have found Lagrangian (38) written as a function of accelerations which shows one of the advantages of being able to prove that all Lagrangians have the form (9) (up to a total time derivative).

The most important result of this note is the realization of the fact that any Lagrangian can be expressed as a linear combination of its own Euler-Lagrange derivatives.

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