Quantum mechanics of predictive Poincaré-invariant systems. II. Mixed spin ($\frac{1}{2}$,0) two-particle systems

Luis Bel
Equipe de Recherche Associée au C.N.R.S. n° 533, Institut Henri Poincaré, 11 rue Pierre et Marie Curie 75231 Paris Cedex 05, France
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We discuss the problem of quantizing mixed spin (0 and $\frac{1}{2}$) two-particle systems in the framework of predictive relativistic mechanics. We propose a general scheme for quantization, but a detailed analysis is given only for the linearized scalar or vector interactions (with short or long ranges). Approximate wave equations are derived for the electromagnetic interaction in the cases where the particles are slowly moving or have very different masses.

I. INTRODUCTION

In a preceding paper[1] we discussed the problem of quantizing two-particle systems in the framework of predictive relativistic mechanics. We assumed there that both particles had spin 0. In this paper, we consider the case where one particle (say particle 1) has spin $\frac{1}{2}$ and the other one (particle 2) has spin 0. We propose a general scheme for quantization, but as in Ref. 1 a rather detailed analysis is given only for the linearized scalar or vector interactions (with short or long ranges). This scheme is rather complicated in the short-range case because it forces the introduction of, so to speak, point-dependent Dirac matrices. This complication is not present in the long-range case. The situation here is therefore different from that of Ref. 1 where all these interactions could be treated on the same basis.

From the point of view of possible physical applications we have concentrated our attention on the case of the electromagnetic interaction. Initially, the formalism that we present is only suitable for dealing with scattering problems, but an induction procedure permits widening the scope of the theory. In general this procedure leads to the consideration of rather complicated nonlocal operators; to simplify the problem we have considered three approximation methods to obtain an induced wave equation at the center of mass. These equations summarize partially the physical content of the theory.

We have considered first the "slow-motion approximation method" which consists of using formal expansions in terms of inverse powers of the speed of light in vacuum (c). Neglecting terms of order higher than $1/c^2$ in the final result the wave equation which we obtain coincides, except for the Darwin-type term (term which contains a Dirac δ), with the equation which could be naively derived from Breit's[2] equation for two spin-$\frac{1}{2}$ particles by assuming that only particle 1 has spin $\frac{1}{2}$, i.e., by dropping the terms which contain Pauli matrices which refer to particle 2.

The second approximation method which we have considered is the "heavy-spin approximation method" which consists of using formal expansions in powers of $1/m_1$. Neglecting in the final result all terms of order higher than $1/m_1$ the wave equation which we obtain still contains nonlocal operators. These operators are not present though in the limit of m_1 going to infinity in which case the wave equation reduces to two uncoupled Klein-Gordon equations modified by a Coulomb field. This last result could have been anticipated from the fact that a spin-$\frac{1}{2}$ particle is endowed with a magnetic moment inversely proportional to its mass and therefore is zero if its mass goes to infinity.

And finally we have considered the "light-spin approximation method" which consists of using formal expansions in powers of $1/m_2$ and neglecting all terms of order higher than $1/m_2$. This approximation method is simpler than the preceding ones. The wave equation which we obtain does not contain nonlocal operators and when m_2 tends to infinity it reduces to the Dirac equation for a spin-$\frac{1}{2}$ particle in a Coulomb field.

Most often the Dirac equation is inadequate to deal with mixed spin-($\frac{1}{2}$,0) systems because it does not take into account the recoil of the heavier particle[3,4] when $m_1 \ll m_2$, or because it is not the appropriate equation when the masses m_1 and m_2 are comparable or when $m_1 \gg m_2$. The above-mentioned results provide a complete framework to deal with all these cases.[5,6]

II. PREDICTIVE POINCARÉ-INVARIANT SYSTEMS

Let M_4 be Minkowski space-time and let TM_4 be the manifold of pairs[7] (x^α,π^β), where $x^\alpha \in M_4$ and

π^β is a timelike ($\pi^\beta\pi_\beta<0$) future-oriented ($\pi^0>0$) tangent vector to M_4 at the point x^α. In predictive relativistic mechanics the phase space of a system of two pointlike structureless particles is $(TM_4)^2$ whose points will be designated by[8] (x_a^α,π_b^β). The generators of the space-time translations and of the Lorentz group acting on $(TM_4)^2$ are[9]

$$\vec{P}_\lambda=-\epsilon_a\frac{\partial}{\partial x_a^\lambda}\ , \tag{2.1}$$

$$\vec{J}_{\lambda\mu}=(\delta_\lambda^\alpha\eta_{\beta\mu}-\delta_\mu^\alpha\eta_{\beta\lambda})\left[x_a^\beta\frac{\partial}{\partial x_a^\alpha}+\pi_a^\beta\frac{\partial}{\partial\pi_a^\alpha}\right],$$

where $\eta_{\lambda\mu}$ is the metric tensor of M_4. The Lie brackets of these vector fields satisfy the commutation relations of the Lie algebra $\underline{P}$ of the Poincaré group:

$$[\vec{P}_\lambda,\vec{P}_\mu]=0\ ,$$

$$[\vec{P}_\lambda,\vec{J}_{\mu\nu}]=\eta_{\lambda\nu}\vec{P}_\mu-\eta_{\lambda\mu}\vec{P}_\nu\ , \tag{2.2}$$

$$[\vec{J}_{\mu\nu},\vec{J}_{\rho\sigma}]=\eta_{\mu\rho}\vec{J}_{\nu\sigma}+\eta_{\nu\sigma}\vec{J}_{\mu\rho}-\eta_{\nu\rho}\vec{J}_{\mu\sigma}-\eta_{\mu\sigma}\vec{J}_{\nu\rho}\ .$$

The equations of motion of a predictive Poincaré-invariant system (PIS) are

$$\frac{dx_a^\alpha}{d\tau}=\pi_a^\alpha,\ \frac{d\pi_a^\alpha}{d\tau}=\theta_a^\alpha(x_b^\beta,\pi_c^\gamma)\ , \tag{2.3}$$

where the functions θ_a^α satisfy the following equations[10]:

$$\pi_{a'}^\rho\frac{\partial\theta_a^\alpha}{\partial x^{a'\rho}}+\theta_{a'}^\rho\frac{\partial\theta_a^\alpha}{\partial\pi^{a'\rho}}=0\ , \tag{2.4}$$

$$\mathscr{L}(\vec{P}_\lambda)\theta_a^\alpha=0\ ,\qquad \mathscr{L}(\vec{J}_{\lambda\mu})\theta_a^\alpha=\delta_\lambda^\alpha\theta_{a\mu}-\delta_\mu^\alpha\theta_{a\lambda}\ , \tag{2.5}$$

where $\mathscr{L}$ is the Lie derivative operator, and

$$\theta_a^\alpha\pi_{a\alpha}=0\ . \tag{2.6}$$

From these last equations it follows, in particular, that[11]

$$\pi_a{}^2\equiv-\pi_a^\alpha\pi_{a\alpha}=m_a{}^2 \tag{2.7}$$

are first integrals of Eqs. (2.3) which we consider as constraints that allow to specify the masses m_a of the particles.

As is well known[12] Eqs. (2.4) make the system of ordinary differential equations (2.3) equivalent to the following completely integrable system of partial differential equations:

$$\frac{\partial x_a^\alpha}{\partial\tau^a}=\pi_a^\alpha,\ \frac{\partial\pi_a^\alpha}{\partial\tau^a}=\theta_a^\alpha(x_b^\beta,\pi_c^\gamma)\ ,\qquad \frac{\partial x_a^\alpha}{\partial\tau^{a'}}=0,\ \frac{\partial\pi_a^\alpha}{\partial\tau^{a'}}=0\ . \tag{2.8}$$

It follows from this result that any PIS can be equivalently defined by two vector fields of $(TM_4)^2$:

$$\vec{H}_a=\delta_{ab}\left[\pi_b^\alpha\frac{\partial}{\partial x_b^\alpha}+\theta_b^\alpha\frac{\partial}{\partial\pi_b^\alpha}\right] \tag{2.9}$$

satisfying the Lie bracket conditions

$$[\vec{H}_a,\vec{H}_{a'}]=0\ , \tag{2.10}$$

$$[\vec{H}_a,\vec{P}_\lambda]=0,\ \ [\vec{H}_a,\vec{J}_{\lambda\mu}]=0\ , \tag{2.11}$$

and the mass constraints

$$\mathscr{L}(\vec{H}_a)(\pi_a^\alpha\pi_{a\alpha})=0\ . \tag{2.12}$$

From Eqs. (2.2), (2.10), and (2.11) we see that the vector fields $\vec{P}_\alpha$, $\vec{J}_{\lambda\mu}$, and $\vec{H}_a$ can be considered as a basis of a Lie algebra $\underline{G}$ of dimension 12 which is the direct sum of the Lie algebra $\underline{P}$ and an Abelian algebra $\underline{A}_2$ of dimension 2:

$$\underline{G}=\underline{A}_2+\underline{P}\ . \tag{2.13}$$

We call $\underline{G}$ the Lie algebra of the complete symmetry group. This algebra is a central concept in the theory of PIS both at the classical level and at the quantum level.

A PIS is said to be separable if[13]

$$\lim_{\lambda\to\infty}\theta_a^\alpha(x_1^\rho,x_2^\sigma=x_1^\sigma+\lambda n^\sigma,\pi_b^\mu)=0 \tag{2.14}$$

for any unit spacelike vector n^σ. And we say that the separability index is s if s is the supremum of the real members p for which

$$\lim_{\lambda\to\infty}\lambda^p\theta_a^\alpha(x_1^\rho,x_2^\sigma=x_1^\sigma+\lambda n^\sigma,\pi_b^\mu)=0\ . \tag{2.15}$$

Unless otherwise stated from now on we shall assume that the PIS being considered has a separability index $s>2$.

Let $f(x_a^\alpha,\pi_b^\beta)$ be a scalar or tensor function. And let us define the shift operators

$$R_a(\lambda)f(x_a^\alpha,x_{a'}^\beta,\pi_b^\gamma)=f(x_a^\alpha+\lambda\pi_a^\alpha,x_{a'}^\beta,\pi_b^\gamma)\ . \tag{2.16}$$

We shall say that f tends to zero at the infinite past (future) and we shall write

$$\lim_{x^2\to\infty_{p(f)}}f=0\ , \tag{2.17}$$

if we have[14]

$$\lim_{\lambda\to-\infty(+\infty)}R_1(\lambda)R_2(\lambda)f=0\ . \tag{2.18}$$

III. CANONICAL REALIZATION OF $\underline{G}$

Let us consider a separable PIS. The Hamilton form in the past is by definition[12,15] the symplectic form Ω of $(TM_4)^2$:

$$d\Omega=0,\quad \Omega^6\neq 0\ , \tag{3.1}$$

which is invariant by the complete symmetry group

$$\mathscr{L}(\vec{\Lambda})\Omega=0,\quad \vec{\Lambda}\in \underline{G} \tag{3.2}$$

and which satisfies the following asymptotic conditions:

$$\lim_{x^2\to\infty_p}(\Omega_{AB}-\Omega_{AB}^{(0)})=0\quad (A,B,\ldots=1,2,\ldots,12)\ , \tag{3.3}$$

where Ω_{AB} are the components of Ω and $\Omega_{AB}^{(0)}$ the components of

$$\Omega^{(0)}=dx_a^\alpha\wedge d\pi_\alpha^a\ , \tag{3.4}$$

with respect to the cobasis $(dx_a^\alpha, d\pi_b^\beta)$.

A PIS is said to be conservative if moreover

$$\lim_{x^2\to\infty_f}(\Omega_{AB}-\Omega_{AB}^{(0)})=0\ . \tag{3.5}$$

From now on we shall consider conservative PIS's only.

To each $\vec{\Lambda}\in\underline{G}$ it corresponds to a function Λ which is defined, up to an additive constant, by the formula

$$i(\vec{\Lambda})\cdot\Omega=-d\Lambda\ , \tag{3.6}$$

where $i(\vec{\Lambda})$ is the interior product operator. We have proved elsewhere[15] that under very general assumptions we have

$$H_a=\tfrac{1}{2}\pi_a{}^2\ . \tag{3.7}$$

Let $P_\alpha^{(p)}$ and $J_{\lambda\mu}^{(p)}$ be the particular solutions of Eq. (3.6) corresponding to the generators $\vec{\mathrm{P}}_\alpha$ and $\vec{\mathrm{J}}_{\lambda\mu}$ which satisfy the asymptotic conditions

$$\begin{aligned}&\lim_{x^2\to\infty_p}(P_\alpha^{(p)}-\epsilon_a\pi_\alpha^a)=0\ ,\\ &\lim_{x^2\to\infty_p}(J_{\lambda\mu}^{(p)}-x_a\lambda\pi_\mu^a+x_{a\mu}\pi_\lambda^a)=0\ .\end{aligned} \tag{3.8}$$

The general solution is then

$$P_\alpha=P_\alpha^{(p)}+a_\alpha,\quad J_{\lambda\mu}=J_{\lambda\mu}^{(p)}+s_{\lambda\mu}\ , \tag{3.9}$$

where a_α and $s_{\lambda\mu}$ are arbitrary constants.

Let $[\Lambda_1,\Lambda_2]$ be the Poisson bracket, in the sense of two functions Λ_1 and Λ_2. From Eqs. (3.7)—(3.9) and from

$$\begin{aligned}&\mathscr{L}(\vec{\Lambda}_1)\Lambda_2=[\Lambda_1,\Lambda_2]\ ,\\ &i([\vec{\Lambda}_1,\vec{\Lambda}_2])\Omega=-d[\Lambda_1,\Lambda_2]\ ,\end{aligned} \tag{3.10}$$

it follows that

$$\begin{aligned}&[H_1,H_2]=0\ ,\\ &[P_\mu,H_a]=0,\quad [J_{\lambda\mu},H_a]=0\ ,\\ &[P_\lambda,P_\mu]=0\ ,\\ &[P_\lambda,J_{\mu\nu}]=\eta_{\lambda\nu}(P_\mu-a_\mu)-\eta_{\lambda\mu}(P_\nu-a_\nu)\ ,\\ &[J_{\mu\nu},J_{\rho\sigma}]=\eta_{\mu\rho}(J_{\nu\sigma}-s_{\nu\sigma})+\eta_{\nu\sigma}(J_{\mu\rho}-s_{\mu\rho})\\ &\qquad -\eta_{\mu\sigma}(J_{\nu\rho}-s_{\nu\rho})-\eta_{\nu\rho}(J_{\mu\sigma}-s_{\mu\sigma})\ .\end{aligned} \tag{3.11}$$

Let us write Eqs. (2.2), (2.10), and (2.11) in the compact form

$$[\vec{\Lambda}_I,\vec{\Lambda}_J]=C_{IJ}^K\vec{\Lambda}_K,\quad I,J,K,\ldots=1,2,\ldots,12\ . \tag{3.12}$$

It is easy to see from Eqs. (3.11) that the linear mapping $C:\vec{\Lambda}_I\in\underline{G}\to\Lambda_I$ is an isomorphism of $\underline{G}$ onto the finite-dimensional Poisson algebra generated by Λ_I:

$$[\Lambda_I,\Lambda_J]=C_{IJ}^K\Lambda_K\ , \tag{3.13}$$

if and only if $a_\alpha=s_{\lambda\mu}=0$. When this choice of the arbitrary constants is made C is by definition the canonical realization of $\underline{G}$. P_α is then interpreted as the total energy-momentum and $J_{\lambda\mu}$ as the generalized angular momentum of the PIS.

It is important to notice that the Lie algebra generated by P_α, $J_{\lambda\mu}$, and H_a, and the Lie algebra generated by P_α, $J_{\lambda\mu}$, and $F_a(H_a)$, where F_a are arbitrary functions of their argument, are both the same abstract Lie algebra. In the next section we shall consider the case where H_1 is substituted by

$$S_1=\pi_1{}^{-1}\pi_{1\rho}\pi_1^\rho=\pi_{1\rho}u_1^\rho=-\sqrt{2H_1}\ , \tag{3.14}$$

u_1^ρ being the unit future-pointing vector collinear to π_1^ρ. From Eqs. (2.7) it follows that S_1 is the first integral of the equations of motion (2.3) which is associated with the constraint

$$S_1=-m_1\ . \tag{3.15}$$

We shall say that a canonical coordinate system, i.e., a system of coordinates (q_a^α,p_β^b) of $(TM_4)^2$ for which Ω takes the form

$$\Omega=dq_a^\alpha\wedge dp_\alpha^a\ , \tag{3.16}$$

is an adapted canonical coordinated system if[16] the functions (q_a^α,p_β^b) of (x_a^α,π_b^β) are smooth enough and satisfy the equations

$$\begin{aligned}&\mathscr{L}(\vec{\mathrm{P}}_\alpha)(q_a^\alpha-x_a^\alpha)=0\ ,\\ &\mathscr{L}(\vec{\mathrm{J}}_{\lambda\mu})q_a^\alpha=\delta_\lambda^\alpha q_{a\mu}-\delta_\mu^\alpha q_{a\lambda}\ ,\\ &\mathscr{L}(\vec{\mathrm{P}}_\alpha)p_\beta^b=0,\quad \mathscr{L}(\vec{\mathrm{J}}_{\lambda\mu})p_\beta^b=\eta_{\lambda\beta}p_\mu^b-\eta_{\mu\beta}p_\lambda^b\ ,\end{aligned} \tag{3.17}$$

and the asymptotic conditions

$$\lim_{x^2\to\infty_p \text{ and } f} (q_a^\alpha - x_a^\alpha)=0\ ,$$
$$\lim_{x_2\to\infty_p \text{ and } f} (p_\beta^b - \pi_\beta^b)=0\ . \tag{3.18}$$

For each canonical coordinate system satisfying Eqs. (3.17), and therefore in particular for each adapted coordinate system, P_α and $J_{\lambda\mu}$ take the very simple form

$$P_\alpha=\epsilon_a p_\alpha^a,\ \ J_{\lambda\mu}=q_{a\lambda}p_\mu^a - q_{a\mu}p_\lambda^a\ . \tag{3.19}$$

Instead the functions H_a or S_1 which are very simple functions of π_a^α will become in general complicated expressions in terms of an adapted canonical coordinate system.

IV. QUANTIZATION OF MIXED SPIN-($\frac{1}{2}$,0) TWO-PARTICLE PIS

In Ref. 1 we considered the problem of quantizing two-particle PIS's assuming that both particles had spin 0. We consider here the case where one particle, say particle 1, has spin $\frac{1}{2}$ and particle 2 has spin 0. We take for granted that this implies that the state vectors can be described by two-point four-component spinors:

$$\psi(x_1^\alpha,x_2^\beta)=\begin{bmatrix}\psi^1(x_1,x_2)\\ \psi^2(x_1,x_2)\\ \psi^3(x_1,x_2)\\ \psi^4(x_1,x_2)\end{bmatrix}=(\psi^A(x_1,x_2))\ ,\qquad A,B=1,2,3,4\ . \tag{4.1}$$

As a departure point we shall take as fundamental vector space the space $\mathscr{E}$ of all spinors ψ which have a Fourier transform $\Psi(k_1^\alpha,k_2^\beta)\equiv(\Psi^A(k_1,k_2))$:

$$\psi^A(x_a) = \frac{1}{(2\pi\hbar)^4}\int_{R^4\times R^4}\eta(k_b)\exp\left[\frac{i}{\hbar}x_c^\alpha k_\alpha^c\right]\Psi^A(k_a)\ , \tag{4.2}$$

$h=2\pi\hbar$ being the Planck constant and

$$\eta(k_b)=dk_1^0\wedge\cdots\wedge dk_2^3\ .$$

This formula can also be written as

$$\psi=\frac{1}{(2\pi\hbar)^4}\int_{R^4\times R^4}\eta E_A\Psi^A \tag{4.3}$$

with

$$E_A=(\delta_A^B)\exp\left[\frac{i}{\hbar}x_a^\alpha k_\alpha^a\right]\ . \tag{4.4}$$

Let $\hat F$ be a linear operator acting on $\mathscr{E}$. And let $\tilde F_B^A(x_a,k_b)$ be the components of the 4×4 matrix defined by

$$\hat F E_A(x_a,k_b)\equiv\tilde F_A^B(x_a,k_b)E_B(x_c,k_d)\ . \tag{4.5}$$

We shall call $\tilde F(x,k)$ the indicial matrix of $\hat F$. This operator being linear we have

$$\hat F\psi=\frac{1}{(2\pi\hbar)^4}\int_{R^4\times R^4}\eta(k_a)\tilde F_A^B(x_b,k_c)E_B(x_a,k_c)\Psi^A(k_f) \tag{4.6}$$

and therefore the knowledge of the indicial matrix is equivalent to the knowledge of $\hat F$ itself. We shall write

$$\tilde F=\theta(\hat F),\ \ \hat F=\theta^{-1}(\tilde F)\ . \tag{4.7}$$

Two linear operators $\hat F$ and $\hat G$ being given, let us set by definition

$$(\tilde F\circ\tilde G)_B^A\equiv[\theta(\hat F\hat G)]_B^A\ . \tag{4.8}$$

A straightforward calculation then yields

$$(\tilde F\circ\tilde G)=\frac{1}{(2\pi\hbar)^8}\int_{R^4}\eta(\lambda_a)\int_{R^4}\eta(k_b')\tilde F_D^A(x_c,k_d')\tilde G_B^D(x_e+\lambda_e,k_f)\exp\left[\frac{i}{\hbar}(k_\alpha^g-k_\alpha'^g)\lambda_g^\alpha\right] \tag{4.9}$$

or

$$(\tilde F\circ\tilde G)_B^A=\tilde F_D^A\circ\tilde G_B^D\ , \tag{4.10}$$

the composition law in the right-hand term being the quantum composition of two functions which we defined in Ref. 1. The composition law given by formulas (4.9) or (4.10), which we call the quantum composition law of two matrices, is linear and associative, and as for functions, it has some other simple properties easy to derive.

The quantum bracket of two 4×4 matrices will be by definition the expression

$$\{\tilde{F},\tilde{G}\}=\tilde{F}\circ\tilde{G}-\tilde{G}\circ\tilde{F}\ . \tag{4.11}$$

This bracket defines on the set of 4×4 matrices whose elements are functions of (x_a^α,k_b^β) a structure of Lie algebra. Moreover, it is clear that

$$\theta([\hat{F},\hat{G}])=\{\tilde{F},\tilde{G}\}\ , \tag{4.12}$$

where $[\hat{F},\hat{G}]$ is the commutator of the operators $\hat{F}$ and $\hat{G}$.

Let us now proceed to quantize PIS's for which particle 1 has spin $\frac{1}{2}$ and particle 2 has spin 0. We consider as fundamental for this case the Lie algebra generated by P_α and $J_{\lambda\mu}$ as defined by Eqs. (3.9), where again we consider a_α and $s_{\lambda\mu}$ as arbitrary constants, by S_1 as defined in (3.14) and by H_2 as defined by Eq. (3.7). Any of these functions will be generally designated by Λ_I and we shall refer to the Lie algebra they generate as $\underline{G}$. By definition we shall say that φ is a quantizer, or that φ defines a quantization, if φ is a linear mapping of $\underline{G}$ onto a finite-dimensional vector space generated by 12 linear operators $\hat{\Lambda}_I$ acting on $\mathscr{E}$:

$$\varphi:\Lambda_I\rightarrow\hat{\Lambda}_I=\varphi(\Lambda_I) \tag{4.13}$$

such that

$$[\hat{\Lambda}_I,\hat{\Lambda}_J]=C_{IJ}^K\hat{\Lambda}_K\ , \tag{4.14}$$

C_{IJ}^K being the structure constants of the Lie algebra as defined by (3.12). Since any φ can always be written in a unique way as

$$\varphi=\theta^{-1}\circ\rho\ , \tag{4.15}$$

ρ being a linear mapping of $\underline{G}$ onto a finite-dimensional vector space generated by 12 4×4 matrices $\tilde{\Lambda}_I(x,k)$:

$$\rho:\Lambda_I\rightarrow\tilde{\Lambda}_I=\rho(\Lambda_I)\ , \tag{4.16}$$

it follows from Eq. (4.12) that we can equivalently say that φ is a quantizer if ρ is such that

$$\{\tilde{\Lambda}_I,\tilde{\Lambda}_J\}=i\hbar C_{IJ}^K\tilde{\Lambda}_K\ . \tag{4.17}$$

It is obvious that the definition of a quantizer that we have just given is too general to be of any physical interest because it does not bear any information about the dynamics of any particular PIS. Therefore we are going to select a large class of quantizers by giving restrictive constructive prescriptions to define ρ.

A general expression which includes every function Λ^I is the following:

$$\Lambda=F^{(p)}(x,\pi)+\epsilon u_1^\alpha\pi_{1\alpha}+e\ , \tag{4.18}$$

where $F^{(p)}$ is $P_\alpha^{(p)}$, $J_{\lambda\mu}^{(p)}$, or H_2; ϵ is either 1 or 0, and e is a_α or $s_{\lambda\mu}$. We shall assume that $\tilde{\Lambda}$ has the following general expression:

$$\tilde{\Lambda}=\tilde{F}^{(p)}(x,k)\cdot I+\epsilon\tilde{\lambda}^\alpha(x,k)\tilde{\pi}_{1\alpha}+\tilde{e}\ , \tag{4.19}$$

I being the unit 4×4 matrix, and $\tilde{\pi}_{1\alpha},\tilde{F}^{(p)}$ being functions, $\tilde{\lambda}^\alpha$ 4×4 point-dependent matrices, and $\tilde{e}$ constant 4×4 matrices defined below.

(i) Let $[q_a^\alpha(x,\pi),p_\beta^b(x,\pi)]$ be an adapted canonical coordinate system of the Hamilton form Ω. We shall refer to it as $\tilde{\phi}$. Let $[f_a^\alpha(q,p),g_\beta^b(q,p)]$ be the inverted functions. And let us define the functions of (x,k):

$$\tilde{x}_a^\alpha=f_a^\alpha(x,k),\ \ \tilde{\pi}_\beta^b=g_\beta^b(x,k)\ . \tag{4.20}$$

We say then that $\tilde{\pi}_1^\alpha$ in Eq. (4.19) is the corresponding function (4.20) and that

$$\tilde{F}^{(p)}(x,k)\equiv F^{(p)}[\tilde{x}(x,k),\tilde{\pi}(x,k)]\ . \tag{4.21}$$

The functions $\tilde{\pi}_1^\alpha$ and $\tilde{F}^{(p)}$ are therefore associated with $\tilde{\Phi}$. We shall see in a moment that not all coordinates $\tilde{\Phi}$ will be admissible.

(ii) The matrices $\tilde{\lambda}^\alpha$ are 4×4 matrices with the following properties:

(1) They satisfy the equations

$$\tilde{\lambda}^\alpha\tilde{\lambda}^\beta+\tilde{\lambda}^\beta\tilde{\lambda}^\alpha=-2\eta^{\alpha\beta}I\ . \tag{4.22}$$

(2) Under a Poincaré transformation

$$x'^\alpha=L_\beta^\alpha(x^\beta-A^\beta),\ \ k'^\alpha=L_\beta^\alpha k^\beta \tag{4.23}$$

they behave as follows:

$$\tilde{\lambda}^\alpha[L(x-A),Lk]=L_\beta^\alpha\Gamma\tilde{\lambda}^\beta(x,k)\Gamma^{-1}\ , \tag{4.24}$$

where $\Gamma(L)$ are the matrices satisfying the equations

$$L_\beta^\alpha\gamma^\beta=\Gamma^{-1}\gamma^\alpha\Gamma \tag{4.25}$$

and defining the spinor representation of the Poincaré group.

(3) They satisfy the asymptotic conditions[17]

$$\lim_{x^2\rightarrow\infty_p\text{ and }f}(\tilde{\lambda}^\alpha-\gamma^\alpha)=0 \tag{4.26}$$

(γ^α being the usual Dirac matrices).

From the fundamental theorem on Dirac matrices it follows that equivalently we can say that $\tilde{\lambda}^\alpha$ have to be matrices associated to a matrix $\tilde{U}(x,k)$ by the formula

$$\tilde{\lambda}^\alpha=\tilde{U}^{-1}\gamma^\alpha\tilde{U}(x,k) \tag{4.27}$$

with $\tilde{U}$ behaving under a Poincaré transformation as follows:

$$\tilde{U}[L(x-A),Lk]=\Gamma\tilde{U}(x,k)\Gamma^{-1} \tag{4.28}$$

and satisfying the asymptotic condition

$$\lim_{x^2\rightarrow\infty_p\text{ and }f}(\tilde{U}-I)=0\ . \tag{4.28'}$$

(iii) $\tilde{e}$ are the matrices

$$\tilde{a}_\alpha = 0, \quad \tilde{s}_{\lambda\mu} = \frac{i\hbar}{4}[\gamma_\lambda, \gamma_\mu] \; . \tag{4.29}$$

Whichever choice of $\tilde{\Phi}$ and $\tilde{U}$ we make, satisfying the prescribed properties, it follows from Eqs. (3.19) that Eq. (4.19) gives in particular

$$\begin{aligned} \tilde{P}_\alpha &= \epsilon_a k_\alpha^a I \; , \\ \tilde{J}_{\lambda\mu} &= (x_{a\lambda} k_\mu^a - x_{a\mu} k_\lambda^a) I + \frac{i\hbar}{4}[\gamma_\lambda, \gamma_\mu] \; , \end{aligned} \tag{4.30}$$

which are the well-known expressions for the indicial matrices of the generators of the spinor representation of the Poincaré group. Equations (4.30) are thus consistent with our initial assumption by which we assumed that the state vector had to be a two-point four-component spinor. The necessity of having this consistency is part of the logic lying behind the proposition of the properties (i), (ii), and (iii) above. Moreover these conditions imply that Eq. (4.17) will all be satisfied except

$$\{\tilde{S}_1, \tilde{H}_2\} = 0 \; . \tag{4.31}$$

This leads us to the following, and final, definition of a physically admissible quantizer. We shall say that the linear mapping φ is a quantizer (for short) if the linear mapping ρ defined by Eqs. (4.19)–(4.26) and (4.29) is such that Eq. (4.31) is satisfied.[18]

We have seen in Sec. II that the concept of a PIS contained two basic ingredients. The Lie algebra $\underline{G}$ (2.13) and the mass constraints (2.7). For a mixed spin $(\frac{1}{2},0)$ we have considered as fundamental the canonical realization of $\underline{G}$ generated by S_1, H_2, P_α, and $J_{\lambda\mu}$ and as fundamental constraints (2.7), with $a=2$ and (3.15). The idea of representing $\underline{G}$ by commutators as in (4.14) has led us to the concept of a quantization. We use now the mass constraints to prescribe the evolution equations for the quantized system:

$$\hat{S}_1 \Psi = -m_1 \Psi, \quad 2\hat{H}_2 \Psi = {m_2}^2 \Psi \; , \tag{4.32}$$

where $\hat{S}_1$ and $\hat{H}_2$ are the operators with indicial matrices:

$$\tilde{S}_1 = \tilde{\lambda}_1^\alpha \tilde{\pi}_{1\alpha}, \quad \tilde{H}_2 = -\tfrac{1}{2} \tilde{\pi}_2^\alpha \tilde{\pi}_{2\alpha} I \tag{4.33}$$

in accordance with the preceding rules based on and following Eq. (4.19). The evolution equations must be supplemented with some regularity conditions. We shall require here the two conditions

$$\begin{aligned} &\sup_{x^2 \to \infty} \psi\psi^\dagger(\vec{x}_1, \vec{x}_2, t) < \infty \; , \\ &\int_{\mathscr{D}_1 \times \mathscr{D}_2} \psi\psi^\dagger(\vec{x}_1, \vec{x}_2, t) d^3\vec{x}_1 d^3\vec{x}_2 < \infty \; , \end{aligned} \tag{4.34}$$

$\mathscr{D}_a$ being any two bounded domains of R^3, ψ the restriction of Ψ to equal times

$$\psi(\vec{x}_1, \vec{x}_2, t) \equiv \Psi(\vec{x}_1, \vec{x}_2; x_1^0 = x_2^0 = t) \tag{4.35}$$

and $\psi^\dagger$ the Hermitian conjugate of $\psi[\vec{x}_a = (x_a^i)]$.

Let E be the vector space generated by the solutions of Eqs. (4.32) which satisfy conditions (4.34) and let us consider a regular linear operator $\hat{F}$ acting on $\mathscr{E}$, which commutes with $\hat{P}_\alpha$ and $\hat{J}_{\lambda\mu}$:

$$[\hat{P}_\alpha, \hat{F}] = 0, \quad [\hat{J}_{\lambda\mu}, \hat{F}] = 0 \tag{4.36}$$

and which is such that for $\Psi \in E$

$$\Psi^t = \hat{F}\Psi \tag{4.37}$$

satisfies again conditions (4.34). It is obvious then that the operators

$$\hat{\Lambda}_I^t = \hat{F}\hat{\Lambda}_I \hat{F}^{-1} \tag{4.38}$$

will satisfy Eq. (4.14) and that $E^t = \hat{F}E$ will be the vector space of admissible solutions of equations:

$$\hat{S}_1^t \Psi^t = -m_1 \Psi^t, \quad \hat{H}_2^t \Psi^t = {m_2}^2 \Psi^t \; . \tag{4.39}$$

We shall say that each operator $\hat{F}$ provides a quantization equivalent to the preceding one. Notice that the operators $\hat{\Lambda}_I^t$ associated to a given operator $\hat{F}$ may not correspond to a quantization defined by a quantizer $(\tilde{\Phi}^t, \tilde{U}^t)$. Notice also that the quantizations defined by two distinct quantizers $(\tilde{\Phi}, \tilde{U})$ and $(\tilde{\Phi}^t, \tilde{U}^t)$ may, *a priori,* be inequivalent, i.e., an operator $\hat{F}$ with the required properties may not exist. We shall reexamine this point in the more restricted framework of the next section.

To be complete, the general scheme of quantization that we have presented here should include the definition of a Poincaré-invariant product $\langle \mid \rangle$ on E. Let us assume for a moment that this scalar product has been defined. It is clear then that the scalar product on E^t,

$$\langle \Psi_{(1)}^t \mid \Psi_{(2)}^t \rangle^t \equiv \langle \Psi_{(1)} \mid \Psi_{(2)} \rangle \; , \tag{4.40}$$

would be Poincaré invariant because of Eqs. (4.36). Moreover this choice automatically makes the operator $\hat{F}$ unitary. At the present time we do not have any general proposition to make to define $\langle \mid \rangle$. Nevertheless, as we shall see in Sec. VII, this is not an obstacle for some applications of the theory based on approximations. The particular problems we shall study there will help in guessing the scalar product that one should consider.

In the free-particle case, i.e., when $\theta_a^\alpha = 0$, $\Omega = \Omega^{(0)}$, and the simplest choice for $(\tilde{x}_a^\alpha, \tilde{\pi}_b^\beta)$ and $\tilde{\lambda}^\rho$ satisfying the required properties is

$$\tilde{x}_a^\alpha = x_a^\alpha, \quad \tilde{\pi}_b^\beta = k_b^\beta, \quad \tilde{\lambda}^\rho = \gamma^\rho \; . \tag{4.41}$$

For this quantization the evolution equations (4.32) are

$$(-i\hbar\partial_1 + m_1)\Psi = 0, \quad (\hbar^2 \Box_2 - {m_2}^2)\Psi = 0 \tag{4.42}$$

with

$$\partial_1 \equiv \gamma^\rho \frac{\partial}{\partial x_1^\rho}, \quad \Box_2 \equiv \eta^{\alpha\beta} \frac{\partial^2}{\partial x_2^\alpha \partial x_2^\beta} . \tag{4.43}$$

To each pair of solutions $(\Psi_{(1)}, \Psi_{(2)})$ we can associate the conserved bicurrent

$$J^{\alpha_1\alpha_2} = \eta^{\alpha_2\beta_2}\left[\bar{\Psi}_{(1)}\gamma^{\alpha_1}\frac{\partial}{\partial x_2^\beta}\Psi_{(2)} - \left[\frac{\partial}{\partial x_2^\beta}\bar{\Psi}_{(1)}\right]\gamma^{\alpha_1}\Psi_{(2)}\right], \quad \bar{\Psi} = \Psi^\dagger\gamma^0 , \tag{4.44}$$

from which one can construct, as usual, a Poincaré-invariant sesquilinear form, and a scalar product in the reduced space of positive-positive-energy solutions.

V. LINEARIZED PIS's

Let us consider functions $\theta_a^{(n)\alpha}$ and formal power series of a coupling constant g of the following form:

$$\theta_a^\alpha = g\theta_a^{(1)\alpha} + g^2\theta_a^{(2)\alpha} + \cdots . \tag{5.1}$$

These series θ_a^α define a formal PIS if their coefficients are solutions of Eqs. (2.4)—(2.6) considered order by order. From now on we shall consider linearized PIS's, i.e., we shall neglect all powers of g greater than or equal to g^2:

$$\theta_a^\alpha = g\theta_a^{(1)\alpha} . \tag{5.2}$$

To this approximation the Hamilton form can be written as

$$\Omega = \Omega^{(0)} + g\Omega^{(1)} \tag{5.3}$$

and any adapted canonical coordinate system of Ω as

$$p_a^\alpha = \pi_a^\alpha + gp_a^{(1)\alpha}(x,\pi) , \qquad q_a^\alpha = x_a^\alpha + gq_a^{(1)\alpha}(x,\pi) \tag{5.4}$$

the general expressions for $(p_a^{(1)\alpha}, q_b^{(1)\beta})$ being (Ref. 1)

$$p_a^{(1)\alpha} = \int_{-\infty}^{0} d\lambda\, R_1(\lambda)R_2(\lambda)\left[\frac{\partial\Phi^{(1)}}{\partial x_\alpha^a} - \theta_a^{(1)\alpha}\right] , \tag{5.5}$$

$$q_a^{(1)\alpha} = -\int_{-\infty}^{0} d\lambda\, R_1(\lambda)R_2(\lambda)\left[\frac{\partial\Phi^{(1)}}{\partial \pi_\alpha^a} - p_a^{(1)\alpha}\right] , \tag{5.6}$$

where $\Phi^{(1)}$ is any function which makes the functions $(q_a^{(1)\alpha}, p_b^{(1)\beta})$ smooth enough, which satisfies the equations

$$\mathscr{L}(\vec{\mathrm{P}}_\alpha)\Phi^{(1)} = 0, \quad \mathscr{L}(\vec{\mathrm{J}}_{\lambda\mu})\Phi^{(1)} = 0 \tag{5.7}$$

and the asymptotic conditions, equivalent to Eqs. (3.18),

$$\int_{-\infty}^{+\infty} d\lambda\, R_1(\lambda)R_2(\lambda)\left[\frac{\partial\Phi^{(1)}}{\partial x_\alpha^a} - \theta_a^{(1)\alpha}\right] = 0 , \tag{5.8}$$

$$\int_{-\infty}^{+\infty} d\lambda\, R_1(\lambda)R_2(\lambda)\left[\frac{\partial\Phi^{(1)}}{\partial \pi_\alpha^a} - p_a^{(1)\alpha}\right] = 0 . \tag{5.9}$$

From Eqs. (4.20) it follows that the general expression for the functions $(\tilde{x}_a^\alpha, \tilde{\pi}_b^\beta)$ which might lead to a quantization of the PIS is

$$\tilde{x}_a^\alpha = x_a^\alpha + g\tilde{x}_a^{(1)\alpha}, \quad \tilde{\pi}_a^\alpha = k_a^\alpha + g\tilde{\pi}_a^{(1)\alpha} \tag{5.10}$$

with

$$\tilde{x}_a^{(1)\alpha} = -\tilde{q}_a^{(1)\alpha}(x,k), \quad \tilde{\pi}_a^{(1)\alpha} = -\tilde{p}_a^{(1)\alpha}(x,k) , \tag{5.11}$$

$(\tilde{q}_a^{(1)\alpha}, \tilde{p}_b^{(1)\beta})$ being the functions $(q_a^{(1)\alpha}, p_b^{(1)\beta})$ where π_b^β is substituted by k_b^β. We shall always use the tilde to indicate this substitution.

The indicial matrix $\tilde{H}_2$ given by Eq. (4.33) calculated using expressions (5.10) is

$$\tilde{H}_2 = (-\tfrac{1}{2}k_2^\alpha k_{2\alpha} + g\tilde{\Phi}_2^{(1)})\cdot I$$

with

$$\tilde{\Phi}_2^{(1)} = -k_2^\alpha\tilde{\pi}_a^{(1)\alpha} . \tag{5.12}$$

Writing the matrices $\tilde{\lambda}^\alpha$ introduced in the preceding section as

$$\tilde{\lambda}^\alpha = \gamma^\alpha + g\tilde{\lambda}^{(1)\alpha} , \tag{5.13}$$

it follows from Eqs. (4.27)—(4.28′) that the general expression for $\tilde{\lambda}^{(1)\alpha}$ will be

$$\tilde{\lambda}^{(1)\alpha} = [\gamma^\alpha, \tilde{U}^{(1)}] , \tag{5.14}$$

$\tilde{U}^{(1)}$ being a matrix which satisfies Eqs. (4.28) and the asymptotic condition

$$\lim_{x^2\to\infty_{p \text{ and } f}} \tilde{U}^{(1)} = 0 . \tag{5.15}$$

The indicial matrix $\tilde{S}_1$, given by Eq. (4.33), calculated using expressions (5.10) and (5.13) is

$$\tilde{S}_1 = \gamma^\alpha k_{1\alpha} + g\tilde{S}_1^{(1)}$$

with

$$\tilde{S}_1^{(1)} = \tilde{\lambda}^{(1)\alpha}k_{1\alpha} + \gamma^\alpha\tilde{\pi}_{1\alpha}^{(1)} . \tag{5.16}$$

The couple $(\tilde{\Phi}^{(1)}, \tilde{U}^{(1)})$ will define a first-order quantization of the PIS if Eq. (4.31) is satisfied.

Taking into account the fact that because of Eqs. (3.17) and (4.24) both $\tilde{\pi}_a^{(1)\alpha}$ and $\tilde{\lambda}^{(1)\beta}$ depend on x_b^β through $x^\alpha \equiv x_1^\alpha - x_2^\alpha$ only, Eq. (4.31) can be written as[19]

$$D_2\tilde{\lambda}^{(1)\alpha}k_{1\alpha} - \frac{i\hbar}{2}\Box(\tilde{p}_{1\alpha}^{(1)}\gamma^\alpha + \tilde{\lambda}^{(1)\alpha}k_{1\alpha}) = 0 \ , \tag{5.17}$$

where

$$D_2 \equiv \pi_2^\beta \frac{\partial}{\partial x_2^\beta}, \quad \Box \equiv \eta^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta} \ . \tag{5.18}$$

Let us assume that $(\tilde{\Phi}^{(1)}, \tilde{U}^{(1)})$ is a quantizer. The evolution equations (4.32) will then be

$$\begin{aligned} &(-i\hbar\partial_1 + g\hat{S}_1^{(1)})\Psi = -m_1\Psi \ , \\ &(\hbar^2\Box_2 + 2g\hat{\Phi}_2^{(1)})\Psi = m_2{}^2\Psi \ . \end{aligned} \tag{5.19}$$

These two equations, if we consider them as exact equations, will not be compatible in general because Eq. (5.17) guarantees the commutation of the operators in the left-hand terms only when all powers of g are neglected. Therefore they have to be considered as equations for the coefficients of a formal expansion of ψ itself:

$$\Psi = \Psi^{(0)} + g\Psi^{(1)} \tag{5.20}$$

and in this sense they are equivalent to the following set of equations:

$$\begin{aligned} &(-i\hbar\partial_1 + m_1)\Psi^{(0)} = 0, \\ &(\hbar^2\Box_2 - m_2{}^2)\Psi^{(0)} = 0 \ , \end{aligned} \tag{5.21}$$

$$(-i\hbar\partial_1 + m_1)\Psi^{(1)} = -\hat{S}_1^{(1)}\Psi^{(0)} \ , \tag{5.22}$$

$$(\hbar^2\Box_2 - m_2{}^2)\Psi^{(1)} = -2\hat{\Phi}_2^{(1)}\Psi^{(0)} \ . \tag{5.23}$$

If we assume in particular that $\psi^{(0)}$ is a plane wave[20]

$$\Psi^{(0)} = A(k_1,k_2)\exp\left[\frac{i}{\hbar}x_a^\alpha k_\alpha^a\right] \ , \tag{5.24}$$

then Eqs. (5.21) give

$$(\gamma^\alpha k_{1\alpha} + m_1)A = 0, \quad k_1{}^2 = m_1{}^2, \quad k_2{}^2 = m_2{}^2 \quad (k_a{}^2 \equiv -k_a^\alpha k_{a\alpha}) \tag{5.25}$$

and Eqs. (5.22) and (5.23) become

$$\begin{aligned} &(-i\hbar\partial_1 + m_1)\Psi^{(1)} = -\tilde{S}_1^{(1)}\Psi^{(0)} \ , \\ &(\hbar^2\Box_2 - m_2{}^2)\Psi^{(1)} = -2\tilde{\Phi}_2^{(1)}\Psi^{(0)} \ . \end{aligned} \tag{5.26}$$

The system of Eqs. (5.25) and (5.26) is well adapted to discuss first-order scattering problems, and for the moment this is the system of equations we are going to study. Later on, in Sec. VII, we shall see how the scope of the theory can be enlarged to discuss more general problems.

Let us examine now the problem of the equivalence of different quantizers that we mentioned in the preceding section. Let us consider two different quantizers $(\tilde{\Phi}^{(1)}, \tilde{U}^{(1)})$ and $(\tilde{\Phi}^{t(1)}, \tilde{U}^{t(1)})$. The substitution of $\tilde{\Phi}^{(1)}$ by $\tilde{\Phi}^{t(1)}$ in Eqs. (5.5) and (5.6) generates the canonical transformation

$$q_a^{t\alpha} = q_a^\alpha - g\frac{\partial T^{(1)}}{\partial \pi_\alpha^a} \ , \quad p_a^{t\alpha} = p_a^\alpha + g\frac{\partial T^{(1)}}{\partial x_\alpha^a} \ , \tag{5.27}$$

where

$$T^{(1)} = \int_{-\infty}^0 d\lambda\, R_1(\lambda)R_2(\lambda)(\Phi^{t(1)} - \Phi^{(1)}) \ . \tag{5.28}$$

Therefore the functions (5.11) will be changed into

$$\tilde{x}_a^{t\alpha} = x_a^\alpha + g\frac{\partial\tilde{T}^{(1)}}{\partial k_\alpha^a}, \quad \tilde{\pi}_a^{t\alpha} = \tilde{\pi}_a^\alpha - g\frac{\partial\tilde{T}^{(1)}}{\partial x_\alpha^a} \tag{5.29}$$

and the corresponding change of $\tilde{H}_2$ [Eq. (5.12)] is

$$\tilde{H}_2^t = \tilde{H}_2 + gD_2\tilde{T}^{(1)}\cdot I \ . \tag{5.30}$$

The substitution of $\tilde{U}^{(1)}$ by $\tilde{U}^{t(1)}$ in Eq. (5.14) gives

$$\tilde{\lambda}^{t\alpha} = \tilde{\lambda}^\alpha + g[\gamma^\alpha, \tilde{N}^{(1)}]$$

with

$$\tilde{N}^{(1)} = \tilde{U}^{t(1)} - \tilde{U}^{(1)} \tag{5.31}$$

and this, together with Eqs. (5.29), induces the following modification of $\tilde{S}_1$ [Eq. (5.16)]:

$$\tilde{S}_1^t = \tilde{S}_1 + g\left[[\gamma^\alpha, \tilde{N}^{(1)}]k_{1\alpha} - \gamma^\alpha\frac{\partial\tilde{T}^{(1)}}{\partial x^\alpha}\right] \ . \tag{5.32}$$

And finally because of Eqs. (4.30), it is clear that $\tilde{P}_\alpha$ and $\tilde{J}_{\lambda\mu}$ will remain unchanged:

$$\tilde{P}_\alpha^t = \tilde{P}_\alpha, \quad \tilde{J}_{\lambda\mu}^t = \tilde{J}_{\lambda\mu} \ . \tag{5.33}$$

Since we are assuming that both $(\tilde{\Phi}^{(1)}, \tilde{U}^{(1)})$ and $(\tilde{\Phi}^{t(1)}, \tilde{U}^{t(1)})$ are quantizers, Eq. (5.17) will be satisfied for both $(\tilde{\pi}_{1\alpha}^{(1)}, \tilde{\lambda}^{(1)\beta})$ and $(\tilde{\pi}_{1\alpha}^{t(1)}, \tilde{\lambda}^{t(1)\beta})$. Therefore we shall have

$$D_2[\gamma^\alpha, \tilde{N}^{(1)}]k_{1\alpha} - \frac{i\hbar}{2}\Box\left[-\frac{\partial\tilde{T}^{(1)}}{\partial x^\alpha}\gamma^\alpha + [\gamma^\alpha, \tilde{N}^{(1)}]k_{1\alpha}\right] = 0 \ . \tag{5.34}$$

According to our definition of equivalent quantizations that we gave in the preceding section, $(\tilde{\Phi}^{(1)}, \tilde{U}^{(1)})$ and $(\tilde{\Phi}^{t(1)}, \tilde{U}^{t(1)})$ will be first-order equivalent quantizers if there exists a matrix $\tilde{F}^{(1)}$ such that

$$\tilde{H}_2^t=(I+g\tilde{F}^{(1)})\circ(-\tfrac{1}{2}k_2^\rho k_{2\rho}+g\tilde{H}_2^{(1)})\circ(I-g\tilde{F}^{(1)})$$
$$=\tilde{H}_2-g\{-\tfrac{1}{2}k_2^\rho k_{2\rho},\tilde{F}^{(1)}\}$$
$$=\tilde{H}_2-g\left[\frac{\hbar^2}{2}\Box_2\tilde{F}^{(1)}+i\hbar D_2\tilde{F}^{(1)}\right],$$
$$\tilde{S}_1^t=(I+g\tilde{F}^{(1)})\circ(\gamma^\rho k_{1\rho}+g\tilde{S}_1^{(1)})\circ(I-g\tilde{F}^{(1)})$$
$$=\tilde{S}_1-g\{\gamma^\rho k_{1\rho},\tilde{F}^{(1)}\}$$
$$=\tilde{S}_1+g\left[i\hbar\gamma^\alpha\frac{\partial}{\partial x_1^\alpha}\tilde{F}^{(1)}+[\tilde{F}^{(1)},\gamma^\alpha]k_{1\alpha}\right],$$
$$\tilde{P}_\alpha^t=(I+g\tilde{F}^{(1)})\circ\tilde{P}_\alpha\circ(I-g\tilde{F}^{(1)})$$
$$=\tilde{P}_\alpha-i\hbar g\mathscr{L}(\vec{\mathrm{P}}_\alpha)\tilde{F}^{(1)}, \tag{5.35}$$
$$\tilde{J}_{\lambda\mu}^t=(I+g\tilde{F}^{(1)})\circ\tilde{J}_{\lambda\mu}\circ(I-g\tilde{F}^{(1)})$$
$$=\tilde{J}_{\lambda\mu}+i\hbar g\{-\mathscr{L}(\vec{\mathrm{J}}_{\lambda\mu})\tilde{F}^{(1)}$$
$$+\tfrac{1}{4}[\tilde{F}^{(1)},[\gamma_\lambda,\gamma_\mu]]\},$$

and such that for each admissible solution Ψ of Eqs. (5.21), (5.22), and (5.23)

$$\psi^t=\Psi+g\hat{F}^{(1)}\Psi^{(0)} \tag{5.36}$$

is again admissible, i.e., satisfies conditions (4.34). From Eqs. (5.30), (5.32), (5.33), and (5.35) we see that as necessary conditions for the two quantizers to be equivalent there must exist a matrix $\tilde{F}^{(1)}$ satisfying the following equations:

$$D_2\tilde{T}^{(1)}=-i\hbar D_2\tilde{F}^{(1)}-\frac{\hbar^2}{2}\Box_2\tilde{F}^{(1)}, \tag{5.37}$$
$$[\gamma^\alpha,\tilde{N}^{(1)}]k_{1\alpha}-\gamma^\alpha\frac{\partial\tilde{T}^{(1)}}{\partial x^\alpha}$$
$$=i\hbar\gamma^\alpha\frac{\partial}{\partial x_1^\alpha}\tilde{F}^{(1)}+[\tilde{F}^{(1)},\gamma^\alpha]k_{1\alpha},$$
$$\mathscr{L}(\vec{\mathrm{P}}_\alpha)\tilde{F}^{(1)}=0, \tag{5.38}$$
$$\mathscr{L}(\vec{\mathrm{J}}_{\lambda\mu})\tilde{F}^{(1)}=\tfrac{1}{4}[\tilde{F}^{(1)},[\gamma_\lambda,\gamma_\mu]]. \tag{5.39}$$

We are going to prove that Eqs. (5.37) and (5.38) always have a solution satisfying the asymptotic condition

$$\lim_{x^2\to\infty_p}\tilde{F}^{(1)}=0. \tag{5.40}$$

These equations being linear in $(\tilde{T}^{(1)},\tilde{N}^{(1)})$ we can decompose the proof in two parts by assuming first that $\tilde{N}^{(1)}$ is zero and then that $\tilde{T}^{(1)}$ is zero. If $\tilde{N}^{(1)}$ is zero then Eqs. (5.34) and (5.37) become

$$\frac{\partial}{\partial x^\alpha}\Box\tilde{T}^{(1)}=0, \tag{5.41}$$
$$D_2\tilde{T}^{(1)}=-i\hbar D_2\tilde{F}^{(1)}-\hbar^2\Box_2\tilde{F}^{(1)},$$
$$\gamma^\alpha\frac{\partial\tilde{T}^{(1)}}{\partial x^\alpha}=-i\hbar\gamma^\alpha\frac{\partial}{\partial x_1^\alpha}\tilde{F}^{(1)}. \tag{5.42}$$

Equations (5.41) together with the asymptotic conditions

$$\lim_{x^2\to\infty_p\text{ and }f}\frac{\partial T^{(1)}}{\partial x^\alpha}=0 \tag{5.43}$$

which follow from the fact that both (q_a^α,p_β^b) and $(q_a^{t\alpha},p_\beta^{tb})$ satisfy the asymptotic conditions (3.18), imply that

$$\Box\tilde{T}^{(1)}=0 \tag{5.44}$$

and therefore

$$\tilde{F}^{(1)}=\frac{i}{\hbar}\left[\tilde{T}^{(1)}-\lim_{x^2\to\infty_p}\tilde{T}^{(1)}\right]\cdot I \tag{5.45}$$

is a solution of Eqs. (5.42), (5.38), and (5.39) satisfying the asymptotic condition (5.40).

Let us assume now that $\tilde{T}^{(1)}$ is zero. Then Eqs. (5.34) and (5.37) become

$$D_2[\gamma^\alpha,\tilde{N}^{(1)}]k_{1\alpha}-\frac{i\hbar}{2}\Box[\gamma^\alpha,\tilde{N}^{(1)}]k_{1\alpha}=0, \tag{5.46}$$
$$D_2\tilde{F}^{(1)}=\frac{i\hbar}{2}\Box_2\tilde{F}^{(1)}, \tag{5.47}$$
$$\tilde{G}^{(1)}\equiv i\hbar\gamma^\alpha\frac{\partial}{\partial x_1^\alpha}\tilde{F}^{(1)}+[\tilde{F}^{(1)}-\tilde{N}^{(1)},\gamma^\alpha]k_{1\alpha}=0. \tag{5.48}$$

The operator $i\hbar\partial_1$ acting on this last equation gives

$$D_1\tilde{F}^{(1)}=\frac{i\hbar}{2}\Box_1\tilde{F}^{(1)}+\tfrac{1}{2}\gamma^\rho\frac{\partial}{\partial x^\rho}[\tilde{N}^{(1)},\gamma^\alpha]k_{1\alpha}$$
$$\left[D_1\equiv k_1^\rho\frac{\partial}{\partial x_1^\rho}\right], \tag{5.49}$$

which can be equivalently written as

$$D_1(\tilde{F}^{(1)}-\tilde{N}^{(1)})$$
$$=\frac{i\hbar}{2}\Box_1\tilde{F}^{(1)}+\tfrac{1}{2}\left[\gamma^\rho\frac{\partial}{\partial x^\rho}\tilde{N}^{(1)}\gamma^\alpha+\gamma^\alpha\gamma^\rho\frac{\partial}{\partial x^\rho}\tilde{N}^{(1)}\right]k_{1\alpha}. \tag{5.50}$$

Taking into account now the asymptotic condition (5.40) and the "integrability equation (5.46)" from what we proved elsewhere[12] it follows that the system of Eqs. (5.47) and (5.49) has the same solutions as that of the integral equation

$$\tilde{F}^{(1)}=\tfrac{1}{2}\int_{-\infty}^0 d\lambda\,R_1(\lambda)R_2(\lambda)\left[i\hbar\epsilon^a\Box_a\tilde{F}^{(1)}+\gamma^\rho\frac{\partial}{\partial x^\rho}[\tilde{N}^{(1)},\gamma^\alpha]k_{1\alpha}\right]. \tag{5.51}$$

Assuming that both $\widetilde{N}^{(1)}$ and $\widetilde{F}^{(1)}$ are analytic functions of $\hbar$ allows us to solve this equation order by order in $\hbar$. The solution thus obtained is actually a solution also of Eqs. (5.48). To prove it let us consider the operators D_a acting on the left-hand term of Eqs. (5.48). From Eqs. (5.47) and (5.49), or (5.50), we get, using Eq. (5.46),

$$D_a\widetilde{G}^{(1)}=\frac{i\hbar}{2}\Box_a\widetilde{G}^{(1)} . \tag{5.52}$$

It follows then from this system of differential equations and from the asymptotic condition (5.40) that

$$\widetilde{G}^{(1)}=\frac{i\hbar}{2}\int_{-\infty}^{0}d\lambda\, R_1(\lambda)R_2(\lambda)\epsilon^a\Box_a\widetilde{G}^{(1)} . \tag{5.53}$$

Therefore Eqs. (5.48) will be satisfied order by order in $\hbar$. A similar calculation would prove that Eqs. (5.38) and (5.39) are also satisfied. The existence of a solution of Eqs. (5.37)–(5.39) is a necessary condition for two quantizers $(\widetilde{\Phi}^{(1)},\widetilde{U}^{(1)})$ and $(\widetilde{\Phi}^{t(1)},\widetilde{U}^{t(1)})$ to be equivalent. The existence of this solution and its uniqueness has been proved under some mild additional conditions. But these two quantizers will be only actually equivalent if the spinors (5.36) satisfy the supplementary conditions (4.34). The discussion we have presented does not prove that all quantizers are equivalent but may serve as a starting point to prove more precise results for particular examples. From now on we shall choose particular quantizers. This choice will have to be considered on the grounds of its simplicity. But we will have to keep in mind that the generality will be lost, except if it is true that all quantizers are equivalent.

Let us consider the matrices

$$\gamma^{\rho_1\cdots\rho_r}=\frac{1}{r!}\delta^{\rho_1\cdots\rho_r}_{\sigma_1\cdots\sigma_r}\gamma^{\sigma_1}\cdots\gamma^{\sigma_r},\quad 1\le r\le 4 , \tag{5.54}$$

where $\delta^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_r}$ is the Kronecker tensor, and let us write the matrices $\widetilde{\lambda}^{(1)\alpha}$ as

$$\widetilde{\lambda}^{(1)\alpha}=\widetilde{\varphi}^{(1)\alpha}I+\widetilde{\varphi}_\beta^{(1)\alpha}\gamma^\beta+\cdots+\widetilde{\varphi}_{\beta\lambda\mu\rho}\gamma^{\beta\lambda\mu\rho} . \tag{5.55}$$

From Eq. (5.17) we see that only $\widetilde{\varphi}_\beta^{(1)\alpha}$ is coupled to $\widetilde{\pi}_{1\alpha}^{(1)}$ and that the remaining quantities will have to satisfy the same equations when there is no interaction. Therefore we shall assume that

$$\widetilde{\varphi}^{(1)\alpha}=0,\quad \widetilde{\varphi}_{\beta_1\cdots\beta_r}^{(1)\alpha}=0,\ \text{for } r\ge 2 . \tag{5.56}$$

From Eq. (4.42) it follows that $\widetilde{\varphi}_\beta^{(1)\alpha}$ must be skew symmetric

$$\widetilde{\varphi}_{\alpha\beta}^{(1)}+\widetilde{\varphi}_{\beta\alpha}^{(1)}=0,\quad \widetilde{\varphi}_{\alpha\beta}^{(1)}\equiv\eta_{\alpha\rho}\widetilde{\varphi}_\beta^{(1)\rho} , \tag{5.57}$$

and from Eq. (4.24) it follows that $\widetilde{\varphi}_{\alpha\beta}^{(1)}$ must be a Poincaré-invariant tensor[21]

$$\mathscr{L}(\vec{\mathrm{P}}_a)\widetilde{\varphi}_{\alpha\beta}^{(1)}=0 ,$$
$$\mathscr{L}(\vec{\mathrm{J}}_{\lambda\mu})\widetilde{\varphi}_{\alpha\beta}^{(1)}=\eta_{\alpha\lambda}\widetilde{\varphi}_{\mu\beta}^{(1)}-\eta_{\alpha\mu}\widetilde{\varphi}_{\lambda\beta}^{(1)}+\eta_{\beta\lambda}\widetilde{\varphi}_{\alpha\mu}^{(1)}-\eta_{\beta\mu}\widetilde{\varphi}_{\alpha\lambda}^{(1)} . \tag{5.58}$$

Defining

$$\widetilde{\omega}_\alpha^{(1)}\equiv\widetilde{\varphi}_{\alpha\beta}^{(1)}k_1^\beta \tag{5.59}$$

and taking into account the simplifying assumptions (5.56), Eq. (5.17) becomes

$$D_2\widetilde{\omega}_\alpha^{(1)}-\frac{i\hbar}{2}\Box(\widetilde{\pi}_{1\alpha}^{(1)}+\widetilde{\omega}_\alpha^{(1)})=0 . \tag{5.60}$$

This equation shows that the function $\Phi^{(1)}$ in Eqs. (5.5) and (5.6) cannot be chosen arbitrarily. In fact, because of Eqs. (5.57) we get

$$\widetilde{\omega}_\alpha^{(1)}k_1^\alpha=0 \tag{5.61}$$

and therefore Eq. (5.60) tells us that necessarily we must have

$$\Box(\widetilde{\pi}_{1\alpha}^{(1)}k_1^\alpha)=0 . \tag{5.62}$$

Actually this condition is also a sufficient condition which guarantees the existence of $\widetilde{\omega}_\alpha^{(1)}$ satisfying the required properties. The general formula which enables us to calculate $\widetilde{\omega}_\alpha^{(1)}$ order by order in $\hbar$ is

$$\widetilde{\omega}_\alpha^{(1)}=\frac{i\hbar}{2}\int_{-\infty}^{0}d\lambda\, R_2(\lambda)\Box(\widetilde{\pi}_{1\alpha}^{(1)}+\widetilde{\omega}_\alpha^{(1)})+\widetilde{\omega}_\alpha^{*(1)} , \tag{5.63}$$

where $\widetilde{\omega}_\alpha^{*(1)}$ must be a solution of

$$D_2\widetilde{\omega}_\alpha^{*(1)}=0,\quad \widetilde{\omega}_\alpha^{*(1)}k_1^\alpha=0 , \tag{5.64}$$

satisfying order by order the asymptotic conditions

$$\lim_{\nu\to-\infty}R_1(\nu)\widetilde{\omega}_\alpha^{*(1)}=0 ,$$
$$\lim_{\nu\to+\infty}R_1(\nu)\widetilde{\omega}_\alpha^{*(1)}=-\frac{i\hbar}{2}\lim_{\nu\to\infty}R_1(\nu)\int_{-\infty}^{+\infty}d\lambda\, R_2(\lambda)\Box(\widetilde{\pi}_{1\alpha}^{(1)}+\widetilde{\omega}_\alpha^{(1)}) . \tag{5.65}$$

For each acceptable solution $\widetilde{\omega}_\alpha^{(1)}$, the general expression for $\widetilde{\varphi}_{\alpha\beta}^{(1)}$ will be

$$\widetilde{\varphi}_{\alpha\beta}^{(1)}=(k_1^\rho k_{1\rho})^{-1}(\widetilde{\omega}_\alpha^{(1)}k_{1\beta}-\widetilde{\omega}_\beta^{(1)}k_{1\alpha})+\widetilde{\varphi}_{\alpha\beta}^{*(1)} , \tag{5.66}$$

$\widetilde{\varphi}_{\alpha\beta}^{*(1)}$ being any skew-symmetric tensor satisfying the appropriate asymptotic conditions and the constraints:

$$\widetilde{\varphi}_{\alpha\beta}^{*(1)}k_1^\beta=0 . \tag{5.67}$$

Let us assume that a function $\Phi^{(1)}$, leading to the condition (5.62), and the solution $\tilde{\omega}_\alpha^{(1)}$ of Eq. (5.60) have been found. The first group of evolution equations (5.26) take now the simpler form

$$(-i\hbar\partial_1+m_1)\Psi^{(1)}=-(\tilde{\pi}_{1\alpha}^{(1)}+\tilde{\omega}_{1\alpha}^{(1)})\gamma^\alpha\Psi^{(0)} . \quad (5.68)$$

The operator $-(i\hbar\partial_1+m_1 I)$ acting on both terms of this equation yields

$$(\hbar^2\Box_1-m_1{}^2)\Psi^{(1)} = \left[2\tilde{\pi}_{1\alpha}^{(1)}k_1^\alpha+\frac{i\hbar}{2}\gamma^{\alpha\beta}\tilde{F}_{\alpha\beta}^{(1)}-i\hbar\tilde{Q}^{(1)}\right]\Psi^{(0)} , \quad (5.69)$$

where

$$\tilde{F}_{\alpha\beta}^{(1)}\equiv\frac{\partial}{\partial x^\alpha}\tilde{\mu}_\beta^{(1)}-\frac{\partial}{\partial x^\beta}\tilde{\mu}_\alpha^{(1)},\quad \tilde{Q}^{(1)}\equiv\frac{\partial}{\partial x_\alpha}\tilde{\mu}_\alpha^{(1)} ,$$
$$\tilde{\mu}_\alpha^{(1)}\equiv\tilde{\pi}_{1\alpha}^{(1)}+\tilde{\omega}_\alpha^{(1)} . \quad (5.70)$$

Let us assume now that Ψ is an eigenstate of $\hat{P}_\alpha$:

$$\hat{P}_\alpha\Psi=P_\alpha\Psi , \quad (5.71)$$

where here P_α are four real numbers. From Eqs. (4.30) and from (5.24) it follows that

$$P_\alpha=k_{1\alpha}+k_{2\alpha} , \quad (5.72)$$

$$\Psi^{(0)}=A\exp\left[\frac{i}{2\hbar}(k_1^\rho-k_2^\rho)x_\rho\right]\exp\left[\frac{i}{2\hbar}P_\alpha(x_1^\alpha+x_2^\alpha)\right] ,$$
$$\Psi^{(1)}=\psi^{(1)}(x^\rho)\exp\left[\frac{i}{2\hbar}P_\alpha(x_1^\alpha+x_2^\alpha)\right] . \quad (5.73)$$

Substituting these expressions for $\Psi^{(0,1)}$ in Eq. (5.68), after a left multiplication by γ^0, in the second equation (5.26) and in Eq. (5.69), we obtain in the center-of-mass frame of reference

$$P_0=k_{10}+k_{20}\equiv-P,\quad P_i=k_{1i}+k_{2i}=0 , \quad (5.74)$$

$$\left[-i\hbar\frac{\partial}{\partial x^0}-\tfrac{1}{2}P\right]\psi^{(1)} =-(\alpha^K\hat{p}_k+\beta m_1)\psi^{(1)}-(\tilde{\mu}_0^{(1)}+\alpha^K\tilde{\mu}_k^{(1)})\psi^{(0)} , \quad (5.75)$$

$$\left[\hbar^2\Box-i\hbar P\frac{\partial}{\partial x^0}+\tfrac{1}{4}P^2\right]\psi^{(1)} =m_2{}^2\psi^{(1)}+2\tilde{\pi}_{2\rho}^{(1)}k_1^\rho\psi^{(0)} , \quad (5.76)$$

$$\left[\hbar^2\Box+i\hbar P\frac{\partial}{\partial x^0}+\tfrac{1}{4}P^2\right]\psi^{(1)} =m_1{}^2\psi^{(1)}+\left[2\tilde{\pi}_{1\rho}^{(1)}k_1^\rho+\frac{i\hbar}{2}\gamma^{\alpha\beta}\tilde{F}_{\alpha\beta}^{(1)}-i\hbar\tilde{Q}^{(1)}\right]\psi^{(0)} , \quad (5.77)$$

where $\beta=\gamma^0$, $\alpha^i=\gamma^0\gamma^i$, $\hat{p}_k=-i\hbar\partial/\partial x^k$, and where, because of Eqs. (5.25) and (5.74) it is understood that each function of k_a^α is in fact a function of three variables $\vec{k}=(k^i)$ which is obtained from the original function by making the following substitutions[22]:

$$k_1^i=k^i,\quad k_2^i=-k^i,\quad k_1^0=+(m_1{}^2+\vec{k}^2)^{1/2} ,$$
$$k_2^0=+(m_2{}^2+\vec{k}^2)^{1/2} . \quad (5.78)$$

Subtracting Eq. (5.77) from Eq. (5.76) we get an expression for $(\partial/\partial x^0)\psi^{(1)}$ and this expression substituted in Eq. (5.75) gives the equation

$$(m_1\beta+\alpha^k\hat{p}_k)\psi^{(1)} =\frac{1}{2P}(P^2+m_1{}^2-m_2{}^2)\psi^{(1)}+\left[\frac{1}{P}(\tilde{\pi}_{1\rho}^{(1)}k_1^\rho-\tilde{\pi}_{2\rho}^{(1)}k_2^\rho)-\tilde{\mu}_0^{(1)}-\tilde{\mu}_k^{(1)}\alpha^k+\frac{i\hbar}{4P}\gamma^{\alpha\beta}\tilde{F}_{\alpha\beta}^{(1)}-\frac{i\hbar}{2P}\tilde{Q}^{(1)}\right]\psi^{(0)} . \quad (5.79)$$

From now on we shall assume that each quantity in this equation which depends on x^0 is restricted to $x^0=0$. This equation then becomes an initial condition for the evolution equation (5.68). The corresponding solutions are solutions also of the second equation (5.26) and therefore, in its restricted form, the wave equation (5.79) is equivalent to the system (5.26). Remember of course that $\psi^{(0)}$ is now

$$\psi^{(0)}=A\exp\left[\frac{i}{\hbar}k_jx^j\right] \quad (5.80)$$

and must be, as it is easy to see, a solution of

$$(m_1\beta+\alpha^k\hat{p}_k)\psi^{(0)}=\frac{1}{2P}(P^2+m_1{}^2-m_2{}^2)\psi^{(0)} . \quad (5.81)$$

VI. THE SCALAR AND VECTOR INTERACTIONS[23]

We consider here the problem of two interacting scalar or vector "charges" e_a. The linearized PIS's corresponding to these interactions are given by the functions[24,25]

$$\theta_a^{(1)\alpha} = r_a^{-3}(1+\mu r_a)\exp(-\mu r_a) \times(\eta_a A_a h^\alpha + L_{aa'} z_a t_{a'}^\alpha) , \tag{6.1}$$

where

$$\begin{aligned} z_a &= \eta_a \Lambda^{-2}[\pi_{a'}{}^2(x\pi_a) - k(x\pi_{a'})] , \\ \eta_a &= (-1)^{a+1}, \quad k = -\pi_1^\alpha \pi_{2\alpha} , \\ \pi_a{}^2 &= -\pi_a^\rho \pi_{a\rho}, \quad \Lambda^2 = k^2 - \pi_1{}^2\pi_2{}^2 , \\ (x\pi_a) &= x^\alpha \pi_{a\alpha}, \quad x^\alpha = x_1^\alpha - x_2^\alpha , \\ h^\alpha &= x^\alpha - z_1\pi_1^\alpha + z_2\pi_2^\alpha, \quad h^2 = h^\alpha h_\alpha , \\ t_a^\alpha &= \pi_{a'}^2\pi_a^\alpha - k\pi_{a'}^\alpha , \\ r_a &= (h^2 + \Lambda^2\pi_{a'}{}^{-2}z_a{}^2)^{1/2} = [x^2 + \pi_{a'}{}^{-2}(x\pi_{a'})^2]^{1/2}, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} A_a &= -\pi_a \text{ or } A_a = \pi_{a'}{}^{-1}k , \\ L_{aa'} &= \pi_a{}^{-1}\pi_{a'}{}^{-2}k \text{ or } L_{aa'} = -\pi_{a'}{}^{-1} , \end{aligned} \tag{6.3}$$

depending on whether we consider the scalar or vector interactions. $\mu \geq 0$ is the inverse of the range of the interaction.

The separability index s of these interactions is ∞ if $\mu > 0$ and 2 if $\mu = 0$. The PIS's with $s=2$ must be treated with special care[15] and for this reason, we have assumed up to now that s was greater than 2. But as we mentioned in Ref. 1 the complications that appear when $s=2$ can be dealt with very easily at the lowest order of approximation that we are considering here. We shall not exclude then the case $\mu=0$ and actually in the next section we shall concentrate our attention to the case of the vector interaction with $\mu=0$ (electromagnetic interaction). The PIS's defined by Eq. (6.1) are conservative. Moreover a straightforward calculation proves that

$$\Box\theta_a^{(1)\alpha} = \mu^2\theta_a^{(1)\alpha} . \tag{6.4}$$

In our preceding paper (Ref. 1) where we considered a system of two spinless particles we used the quantizer

$$\Phi^{(1)} = -2BR^{-1}\exp(-\mu R) , \tag{6.5}$$

where

$$\begin{aligned} R &= [x^2 + \Pi^{-2}(x\Pi)^2]^{1/2}, \quad \Pi^\alpha = \pi_1^\alpha + \pi_2^\alpha , \\ \Pi^2 &= -\Pi^\rho\Pi_\rho, \quad (x\Pi) = x^\alpha\Pi_\alpha , \end{aligned} \tag{6.6}$$

and where

$$B = -\pi_1\pi_2 \text{ or } B = k \tag{6.7}$$

depending on whether we consider the scalar or vector interactions. This function $\Phi^{(1)}$ is a solution of

$$\Box\Phi^{(1)} = \mu^2\Phi^{(1)} . \tag{6.8}$$

From Eqs. (5.5), (5.11), and (2.6) it follows that the functions $\tilde{\pi}_{a\alpha}^{(1)}$ associated with the quantizer (6.5) are such that

$$\tilde{\pi}_{a\alpha}^{(1)} k_a^\alpha = -\tfrac{1}{2}\tilde{\Phi}^{(1)} . \tag{6.9}$$

We see then using Eq. (6.8) that Eq. (5.62) will be satisfied if and only if $\mu=0$. Therefore function (6.1) leads to a quantization of a mixed-spin PIS if $\mu=0$ but does not if $\mu\neq 0$.

If $\mu=0$, from Eqs. (5.5), (5.11), (6.4), and (6.8) it follows that the functions $\tilde{\pi}_{a\alpha}^{(1)}$ are solutions of

$$\Box\tilde{\pi}_{a\alpha}^{(1)} = 0 \tag{6.10}$$

and therefore the simplest solution of Eq. (5.60) is

$$\tilde{\omega}_\alpha^{(1)} = 0 \tag{6.11}$$

for which, using also Eq. (6.9), the wave equation (5.79) becomes

$$\begin{aligned} (m_1\beta + \alpha^k\hat{p}_k)\psi^{(1)} &= \frac{1}{2P}(P^2 + m_1{}^2 - m_2{}^2)\psi^{(1)} \\ &\quad - \left[\tilde{\pi}_0^{(1)} + \tilde{\pi}_{1k}^{(1)}\alpha^k - \frac{i\hbar}{4P}\gamma^\alpha\gamma^\beta\tilde{F}_{\alpha\beta}^{(1)} + \frac{i\hbar}{2P}\tilde{Q}^{(1)} \right]\psi^{(0)} , \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} \tilde{F}_{\alpha\beta}^{(1)} &= \frac{\partial}{\partial x^\alpha}\tilde{\pi}_{1\beta}^{(1)} - \frac{\partial}{\partial x^\beta}\tilde{\pi}_{1\alpha}^{(1)}, \\ \tilde{Q}^{(1)} &= \frac{\partial}{\partial x_\alpha}\tilde{\pi}_{1\alpha}^{(1)} . \end{aligned} \tag{6.13}$$

If $\mu > 0$ a possible choice of the function $\Phi^{(1)}$ would be

$$\Phi^{(1)} = -B\pi_1{}^{-1}r_2{}^{-1}\exp(-\mu r_2) . \tag{6.14}$$

But for this choice, or for any other, Eq. (5.60), will never have the solution (6.11). This makes the quantization of short-range scalar or vector interactions more difficult than the quantization of long-range ones. We want to stress though that this is just a technical complication and that nothing would prevent us from developing the theory for $\mu > 0$ as we shall do for the case of the electromagnetic interaction.

VII. THE ELECTROMAGNETIC INTERACTION

We shall consider in this section three approximations of Eqs. (5.81) and (6.12) for the case of the electromagnetic interaction. In this case the functions $\theta_a^{(1)\alpha}$ are ($g = e_1e_2$)

$$\theta_a^{(1)\alpha} = \pi_{a'}{}^{-1}r_a{}^{-3}(\eta_a kh^\alpha - z_a t_{a'}^\alpha) . \tag{7.1}$$

A straightforward calculation shows that

$$\frac{\partial}{\partial x^\alpha}\theta_a^{(1)\alpha}=0 \qquad (7.2)$$

and therefore, from Eqs. (5.5), (5.11), and (6.8) for $\mu=0$, we see that the functions $\tilde{\pi}_{a\alpha}^{(1)}$ associated with the quantizer

$$\Phi^{(1)}=-2k\Pi^{-1}R^{-1} \qquad (7.3)$$

satisfy the equation

$$\tilde{Q}^{(1)}=\frac{\partial}{\partial x_\alpha}\tilde{\pi}_{1\alpha}^{(1)}=0\ . \qquad (7.4)$$

Using Eqs. (5.5), (7.1), and (7.3) we can calculate $p_a^{(1)\alpha}$ and the result is

$$p_a^{(1)\alpha}=-\frac{1}{c}\eta_a kh^{-2}(\pi_{a'}{}^{-1}z_a r_a{}^{-1}-\Pi^{-1}ZR^{-1})h_\alpha$$
$$-\frac{1}{c}\Lambda^{-2}k\Pi^{-1}R^{-1}(t_{a\alpha}-t_{a'\alpha})$$
$$-\frac{1}{c}\Lambda^{-2}\pi_{a'}r_a{}^{-1}t_{a'a}\ , \qquad (7.5)$$

where we have restored the speed of light in vacuum c assuming that π_a^α have dimensions of linear momentum and x^α have dimensions of a length. From (7.5) we obtain the functions $\tilde{\pi}_{a\alpha}^{(1)}$ by changing the sign and substituting everywhere π_a^α by k_a^α.

The corresponding expressions for $\tilde{F}_{\alpha\beta}^{(1)}$ are

$$\tilde{F}_{\alpha\beta}^{(1)}=-\frac{1}{c}\Lambda^{-2}\pi_2{}^{-1}r_1{}^{-3}[k(h_\alpha t_{1\beta}-h_\beta t_{1\alpha})$$
$$+\pi_2{}^2(h_\alpha t_{2\beta}-h_\beta t_{2\alpha})$$
$$+z_1(t_{1\alpha}t_{2\beta}-t_{1\beta}t_{2\alpha})]\ . \qquad (7.6)$$

Let us introduce two-component spinors $\psi_1^{(0,1)}$, and $\psi_2^{(0,1)}$ such that

$$\psi^{(0,1)}=\begin{bmatrix}\psi_1^{(0,1)}\\ \psi_2^{(0,1)}\end{bmatrix}\ . \qquad (7.7)$$

Using the usual representation of the Dirac matrices the zeroth-order wave equation (5.81) can be written as

$$m_1c\psi_1^{(0)}+\sigma^k\hat{p}_k\psi_2^{(0)}=\frac{1}{2P}(P^2+m_1{}^2c^2-m_2{}^2c^2)\psi_1^{(0)}\ , \qquad (7.8)$$

$$-m_1c\psi_2^{(0)}+\sigma^k\hat{p}_k\psi_1^{(0)}=\frac{1}{2P}(P^2+m_1{}^2c^2-m_2{}^2c^2)\psi_2^{(0)}\ , \qquad (7.9)$$

σ^k being the Pauli matrices. And taking into account Eq. (7.4), the first-order wave equation (6.12) can be written as

$$m_1c\psi_1^{(1)}+\sigma^k\hat{p}_k\psi_2^{(1)}=\frac{1}{2P}(P^2+m_1{}^2c^2-m_2{}^2c^2)\psi_1^{(1)}-\left[\tilde{\pi}_{10}^{(1)}+\frac{i\hbar}{4P}\sigma^i\sigma^k\tilde{F}_{jk}^{(1)}\right]\psi_1^{(0)}-\sigma^k\left[\tilde{\pi}_{1k}^{(1)}+\frac{i\hbar}{2P}\tilde{F}_{k0}^{(1)}\right]\psi_2^{(0)}\ , \qquad (7.10)$$

$$-m_1c\psi_2^{(1)}+\sigma^k\hat{p}_k\psi_1^{(1)}=\frac{1}{2P}(P^2+m_1{}^2c^2-m_2{}^2c^2)\psi_2^{(1)}-\left[\tilde{\pi}_{10}^{(1)}+\frac{i\hbar}{4P}\sigma^i\sigma^k\tilde{F}_{jk}^{(1)}\right]\psi_2^{(0)}-\sigma^k\left[\tilde{\pi}_{1k}^{(1)}+\frac{i\hbar}{2P}\tilde{F}_{k0}^{(1)}\right]\psi_1^{(0)}\ . \qquad (7.11)$$

We are reminded that in Eqs. (7.8)–(7.11) $x^0=0$, and that the substitutions (5.78) have to be made.

A. The "slow-motion approximation method" to deal with Eqs. (7.8)–(7.11) will consist in using formal series expansions in $1/c$ and neglecting in the final result all terms of order higher than $1/c^2$. Let us write for any function f which has a limit for $c\to\infty$:

$$f={}_0f+\frac{1}{c}{}_1f+\cdots+\frac{1}{c^n}{}_nf+\cdots\ . \qquad (7.12)$$

Using this notation, a rather long but straightforward calculation gives

$${}_1\tilde{\pi}_{10}^{(1)}=\frac{m_2}{Mx},\quad {}_3\tilde{\pi}_{10}^{(1)}=-\frac{m_2}{2Mx}\left[\frac{(\vec{x}\cdot\vec{k})^2}{m_2{}^2x^2}+\frac{k^2}{m_1m_2}\right],$$
$${}_0\tilde{\pi}_{10}^{(1)}={}_2\tilde{\pi}_{10}^{(1)}=0$$
$${}_2\tilde{\pi}_{1i}^{(1)}=\frac{1}{2Mm_2x}\left[m_1\frac{\vec{x}\cdot\vec{k}}{x^2}x_i+(m_1+2m_2)k_i\right],$$
$${}_0\tilde{\pi}_{1i}^{(1)}={}_1\tilde{\pi}_{1i}^{(1)}=0$$
$${}_1\tilde{F}_{i0}^{(1)}=-\frac{1}{x^3}x^i,{}_0\tilde{F}_{ij}^{(1)}=0\ , \qquad (7.13)$$
$${}_2\tilde{F}_{ij}^{(1)}=-\frac{1}{m_2x^3}(x_ik_j-x_jk_i),\quad {}_0\tilde{F}_{ij}^{(1)}={}_1\tilde{F}_{ij}^{(1)}=0\ ,$$

where

$$M=m_1+m_2,\ \vec{x}=(x^i),\ x=|\vec{x}|\ ,\quad \vec{k}=(k^i),\ k=|\vec{k}|\ . \tag{7.14}$$

Let us now write

$$P\equiv Mc+\frac{W}{c}\ . \tag{7.15}$$

The expressions (7.13) and this definition of W imply the following form of the formal solutions of Eqs. (7.8)—(7.11):

$$\psi_1^{(0,1)}={}_0\psi_1^{(0,1)}+\frac{1}{c^2}{}_2\psi_1^{(0,1)}+O(1/c^4)\ ,\quad \psi_2^{(0,1)}=\frac{1}{c}{}_1\psi_2^{(0,1)}+\frac{1}{c^3}{}_3\psi_2^{(0,1)}+O(1/c^5)\ . \tag{7.16}$$

At the first order in $1/c$ Eqs. (7.8) and (7.10) give

$$\sigma^k\hat{p}_k{}_1\psi_2^{(0)}=\frac{m_2}{M}W_0\psi_1^{(0)}\ , \tag{7.17}$$

$$\sigma^k\hat{p}_k{}_1\psi_2^{(1)}=\frac{m_2}{M}W_0\psi_1^{(0)}-{}_1\tilde{\pi}_{10}^{(1)}{}_0\psi_1^{(0)}\ , \tag{7.18}$$

and at the zeroth order Eqs. (7.9) and (7.11) give

$$-2m_1{}_1\psi_2^{(0,1)}+\sigma^k\hat{p}_k{}_0\psi_1^{(0,1)}=0\ . \tag{7.19}$$

The left-hand terms of Eqs. (7.17) and (7.18) can be calculated using Eq. (7.19) and then the former can be written

$$W_0\psi_1^{(0)}=\frac{1}{2\mu}\hat{\vec{p}}^2{}_0\psi_1^{(0)}\ ,\quad W_0\psi_1^{(1)}=\frac{1}{2\mu}\hat{\vec{p}}^2{}_0\psi_1^{(1)}+\frac{1}{x}{}_0\psi_1^{(0)}\ , \tag{7.20}$$

μ being here the reduced mass of the system: $\mu=m_1m_2/M$.

A similar (but much longer) calculation to the next order yields the equations

$$W_2\psi_1^{(0)}=\frac{1}{2\mu}\hat{\vec{p}}^2{}_2\psi_1^{(0)}-\frac{1}{8}\left[\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right]\hat{\vec{p}}^4{}_0\psi_1^{(0)}\ , \tag{7.21}$$

$$\begin{aligned}W_2\psi_1^{(1)}=&\frac{1}{2\mu}\hat{\vec{p}}^2{}_2\psi_1^{(1)}-\frac{1}{8}\left[\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right]\hat{\vec{p}}^4{}_0\psi_1^{(1)}+\frac{1}{x}{}_2\psi_1^{(0)}\\&+\left[\frac{k^2}{m_1m_2x}-\hbar\left[\frac{1}{4m_1{}^2}+\frac{1}{2m_1m_2}\right]\frac{1}{x^3}\vec{\sigma}\cdot(\vec{x}\times\vec{k})+\frac{\hbar^2}{4}\left[\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right]\Delta\frac{1}{x}\right.\\&\left.-i\hbar\frac{2m_1{}^2+m_2{}^2-2m_1m_2}{4m_1{}^2m_2{}^2}\frac{\vec{x}\cdot\vec{k}}{x^3}\right]{}_0\psi_1^{(0)}\ ,\end{aligned} \tag{7.22}$$

where $\vec{\sigma}=(\sigma^i)$.

We reconsider now in a restricted form the equivalence problem that we mentioned in Secs. IV and V. Let E be the linear vector space generated by the formal solutions of Eqs. (7.20), (7.21), and (7.22) which satisfy the appropriate regularity conditions implied by the conditions (4.34) and let E^t be vector space generated by the states obtained from E by the following transformations:

$${}_{0,2}\psi_1^{t(0)}={}_{0,2}\psi_1^{(0)},\quad {}_0\psi_1^{t(1)}={}_0\psi_1^{(1)},\quad {}_2\psi_1^{t(1)}={}_2\psi_1^{(1)}+\tilde{T}^{(1)}{}_0\psi_1^{(0)}\ , \tag{7.23}$$

where $\tilde{T}^{(1)}$ is a function of (x^i,k^j), which is invariant under rotations and such that the states ψ_1^t satisfy the appropriate regularity conditions. E^t is then the vector space generated by the regular solutions of Eqs. (7.20), (7.21), and

$$\begin{aligned}W_2\psi_1^{t(1)}=&\frac{1}{2\mu}\hat{\vec{p}}^2{}_2\psi_1^{t(1)}+\frac{1}{x}{}_2\psi_1^{(0)}-\frac{1}{8}\left[\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right]\hat{\vec{p}}^4{}_0\psi_1^{(1)}\\&+\left[\frac{k^2}{m_1m_2x}-\hbar\left[\frac{1}{4m_1{}^2}+\frac{1}{2m_1m_2}\right]\frac{1}{x^3}\vec{\sigma}\cdot(\vec{x}\times\vec{k})+\frac{\hbar^2}{4}\left[\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right]\Delta\frac{1}{x}\right.\\&\left.-i\hbar\frac{2m_1{}^2+m_2{}^2-2m_1m_2}{4m_1{}^2m_2{}^2}\frac{\vec{x}\cdot\vec{k}}{x^3}+\left\{\tilde{T}^{(1)},\frac{k^2}{2\mu}\right\}\right]{}_0\psi_1^{(0)}\ .\end{aligned} \tag{7.24}$$

The system of Eqs. (7.20)—(7.22) can be considered equivalent to the system of Eqs. (7.20), (7.21), and (7.24) in so far as their regular solutions can be put in a one-to-one correspondence by the transformation (7.23). Let us take

$$\tilde{T}^{(1)}=\frac{i}{2\hbar M}\frac{\vec{x}\cdot\vec{k}}{x}-\frac{2m_1^2+m_2^2}{4Mm_1m_2}\frac{1}{x}\ . \tag{7.25}$$

Equation (7.24) becomes

$$W_2\psi_1^{t(1)}=\frac{1}{2\mu}\hat{\vec{p}}^2\,{}_2\psi_1^{t(1)}+\frac{1}{x}\,{}_2\psi_1^{(0)}-\frac{1}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]\hat{\vec{p}}^4\,{}_0\psi_1^{(1)}$$
$$+\left\{\frac{1}{2m_1m_2x}\left[k^2+\frac{(\vec{x}\cdot\vec{k})^2}{x^2}\right]-\hbar\left[\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right]\frac{1}{x^3}\vec{\sigma}\cdot(\vec{x}\times\vec{k})-\frac{\hbar^2\pi}{2m_1^2}\delta(\vec{x})\right\}{}_0\psi_1^{(0)}\ , \tag{7.26}$$

where we have used the equation

$$\Delta\frac{1}{x}=-4\pi\delta(\vec{x})\ . \tag{7.27}$$

We are reminded that in Eqs. (7.20), (7.21), and (7.26), ${}_{0,2}\psi_1^{(0)}$ are plane-wave two-component spinors

$${}_{0,2}\psi_1^{(0)}={}_{0,2}A_1\exp\left[\frac{i}{\hbar}\vec{k}\cdot\vec{x}\right] \tag{7.28}$$

and therefore these equations will be useful for scattering problems only. Let us assume now that a general equation exists,

$$\hat{H}\psi_1=W\psi_1\ , \tag{7.29}$$

where the Hamiltonian $\hat{H}$ has the expression

$$\hat{H}={}_0\hat{H}^{(0)}+g\,{}_0\hat{H}^{(1)}+\frac{1}{c^2}\,{}_2\hat{H}^{(0)}+\frac{g}{c^2}\,{}_2\hat{H}^{(1)} \tag{7.30}$$

and being such that its formal solutions

$$\psi_1={}_0\psi_1^{(0)}+\frac{1}{c^2}\,{}_2\psi_1^{(0)}+g\,{}_0\psi_1^{(1)}+\frac{g}{c^2}\,{}_2\psi_1^{(1)}+\cdots \tag{7.31}$$

with ${}_{0,2}\psi_1^{(0)}$ as in Eq. (7.28) coincide with the solutions of Eqs. (7.20), (7.21), and (7.26). Substituting these expressions and (7.30) in Eq. (7.29) and identifying term by term with the latter we see that $\hat{H}$ must necessarily be the operator

$$H=\frac{1}{2\mu}\hat{\vec{p}}^2+\frac{g}{x}-\frac{1}{8c^2}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]\hat{\vec{p}}^4+\frac{g}{2m_1m_2c^2x}\left[\hat{\vec{p}}^2+\frac{x^ix^j}{x^2}\hat{p}_i\hat{p}_j\right]$$
$$-\frac{g\hbar}{c^2}\left[\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right]\frac{1}{x^3}\vec{\sigma}\cdot(\vec{x}\times\vec{p})-\frac{g\hbar^2\pi}{2c^2m_1^2}\delta(\vec{x})\ , \tag{7.32}$$

where it is important to notice that the operators $\hat{p}_i$ are always to the right of any function of the operators x^j. We shall call $\hat{H}$ the induced Hamiltonian. This Hamiltonian coincides, except for the Darwin-type last term, with Breit's[2] Hamiltonian if we drop in it all terms which contain the matrices $\vec{\sigma}_2$ which refer to particle 2.

The introduction of the induced Hamiltonian and the consideration of the wave equation (7.29) enlarges the scope of the theory insofar as we are now in a position to deal with general problems including, in the attractive case, the bound energy problems. It must be emphasized nevertheless that Eq. (7.29) remains an approximate equation and that any "exact" results which we could get from it should be discussed with special care. Actually since the first two terms in (7.32) coincide with the exact nonrelativistic Hamiltonian $\hat{H}_{NR}$, the safest position that we can take in interpreting the induced Hamiltonian is to write it as

$$\hat{H}=\hat{H}_{NR}+\hat{H}_{RC} \tag{7.33}$$

and to consider $\hat{H}_{RC}$ as a small relativistic correction to $\hat{H}_{NR}$. *From this point of view ordinary first-order perturbation theory must be used to take into account* $\hat{H}_{RC}$. This does not present any difficulty because $\hat{H}_{RC}$ is Hermitian with respect to the usual nonrelativistic scalar product. If we had defined $\hat{H}$ using Eq. (7.22) instead of Eq. (7.26) the corresponding $\hat{H}_{RC}$ would not have been Hermitian and the safest interpretation of $\hat{H}$ would have been more difficult. This is the reason why we used the equivalence transformation (7.23) with the function (7.25).

B. The "heavy-spin approximation method" to deal with Eqs. (7.8)–(7.11) will consist in using formal series expansions in $1/m_1$ and neglecting in the final result all terms of order higher than $1/m_1$. Let us write for any function f which has a limit when $m_1 \to \infty$:

$$f = {}_0f + \frac{1}{m_1}\,{}_1f \cdots \,. \tag{7.34}$$

Using this notation, a straightforward calculation gives[26]

$$\begin{aligned}
&{}_1\tilde{\pi}_{10}^{(1)} = m_2\tilde{r}_1^{\,-1}, \quad {}_2\tilde{\pi}_{10}^{(1)} = -m_2\tilde{r}_1^{\,-1}\tilde{\pi}_2^{\,0}, \quad {}_0\tilde{\pi}_0^{(1)} = 0 \,,\\
&{}_0\tilde{\pi}_{1i}^{(1)} = \tilde{\pi}_2^{\,0}\tilde{h}^{-2}k^{-2}(\vec{x}\cdot\vec{k})(m_2^{\,-1}\tilde{\pi}_2^{\,0}\tilde{r}_1^{\,-1} - x^{-1})\tilde{h}_i\\
&\qquad + k^{-2}(\tilde{\pi}_2^{\,0}x^{-1} - m_2\tilde{r}_1^{\,-1})k_i \,,\\
&{}_1\tilde{\pi}_{1i}^{(1)} = -m_2^{\,2}k^{-2}\tilde{h}^{-2}(\vec{x}\cdot\vec{k})(m_2^{\,-1}\tilde{\pi}_2^{\,0}\tilde{r}_1^{\,-1} - x^{-1})\tilde{h}_i\\
&\qquad + k^{-2}(-m_2^{\,2}x^{-1} + m_2\tilde{\pi}_2^{\,0}\tilde{r}_1^{\,-1})k_i \,,\\
&{}_0\tilde{F}_{i0}^{(1)} = -m_2^{\,-1}\tilde{\pi}_2^{\,0}\tilde{r}_1^{\,-3}x_i, \quad {}_1\tilde{F}_{i0}^{(1)} = 0 \,,\\
&{}_0\tilde{F}_{ij}^{(1)} = -m_2^{\,-1}\tilde{r}_1^{\,-3}(x_ik_j - x_jk_i) \,,
\end{aligned} \tag{7.35}$$

where

$$\begin{aligned}
&\tilde{h}_i = x_i - k^{-2}(\vec{x}\cdot\vec{k})^2k_i, \quad \tilde{h}^2 = \tilde{h}_i\tilde{h}^i \,,\\
&\tilde{r}_1 = [x^2 + m_2^{\,-2}(\vec{x}\cdot\vec{k})^2]^{1/2}, \tilde{\pi}_2^{\,0} = (m_2^{\,2} + \vec{k}^2)^{1/2} \,.
\end{aligned} \tag{7.36}$$

Let us now define W by the formula

$$P \equiv m_1 + W \,. \tag{7.37}$$

The expressions (7.35) and this definition of W imply the following form of the formal solutions of Eqs. (7.8)–(7.11):

$$\begin{aligned}
&\psi_1^{(0,1)} = {}_0\psi_1^{(0,1)} + \frac{1}{m_1}\,{}_1\psi_1^{(0,1)} + O(1/m_1^{\,2}) \,,\\
&\psi_2^{(0,1)} = \frac{1}{m_1}\,{}_1\psi_2^{(0,1)} + O(1/m_1^{\,2}) \,.
\end{aligned} \tag{7.38}$$

At the first order in $1/m_1$ Eqs. (7.8) and (7.10) give

$$\begin{aligned}
&\sigma^k\hat{p}_k\,{}_1\psi_2^{(0)} = \tfrac{1}{2}(W^2 - m_2^{\,2})\,{}_0\psi_1^{(0)} \,,\\
&\sigma^k\hat{p}_k\,{}_1\psi_2^{(1)} = \tfrac{1}{2}(W^2 - m_2^{\,2})\,{}_0\psi_1^{(1)}\\
&\qquad - \left[{}_1\tilde{\pi}_{10}^{(1)} + \frac{i\hbar}{4}\sigma^i\sigma^i\,{}_0\tilde{F}_{ij}^{(1)}\right]{}_0\psi_1^{(0)}\\
&\qquad - \sigma^k\,{}_0\tilde{\pi}_{1k}^{(1)}\,{}_1\psi_2^{(0)} \,,
\end{aligned} \tag{7.39}$$

and the zeroth-order equations (7.9) and (7.11) give

$$\begin{aligned}
&-2\,{}_1\psi_2^{(0)} + \sigma^k\hat{p}_k\,{}_0\psi_1^{(0)} = 0 \,,\\
&-2\,{}_1\psi_2^{(1)} + \sigma^k\hat{p}_k\,{}_0\psi_1^{(1)} = -\sigma^k\,{}_0\tilde{\pi}_{1k}^{(1)}\,{}_0\psi_1^{(0)} \,.
\end{aligned} \tag{7.40}$$

Using these equations as definitions of ${}_1\psi_2^{(0,1)}$ and substituting the corresponding expressions into Eqs. (7.39) we obtain

$$\begin{aligned}
&(W^2 - \hat{\vec{p}}^{\,2} - m_2^{\,2})\,{}_0\psi_1^{(0)} = 0 \,,\\
&(W^2 - \hat{\vec{p}}^{\,2} - m_2^{\,2})\,{}_0\psi_1^{(1)} - 2\frac{W}{x}\,{}_0\psi_1^{(0)} = 0 \,.
\end{aligned} \tag{7.41}$$

A similar (but longer) calculation to next order yields the following equations:

$$\begin{aligned}
&(W^2 - \hat{\vec{p}}^{\,2} - m_2^{\,2})\,{}_1\psi_1^{(0)} = W\hat{\vec{p}}^{\,2}\,{}_0\psi_1^{(0)} \,,\\
&(W^2 - \hat{\vec{p}}^{\,2} - m_2^{\,2})\,{}_1\psi_1^{(1)} - 2\frac{W}{x}\,{}_1\psi_1^{(0)}\\
&\qquad = W\hat{\vec{p}}^{\,2}\,{}_0\psi_1^{(1)} + \left[2\frac{k^2}{x} + \frac{i\hbar}{2m_2}W\sigma^i\sigma^i\tilde{f}_{ij}\right]{}_0\psi_1^{(0)} \,,
\end{aligned} \tag{7.42}$$

where

$$\tilde{f}_{ij} = \left[\tilde{r}_1^{\,-3} + \frac{3i\hbar}{2m_2}\tilde{r}_1^{\,-5}(\vec{x}\cdot\vec{k})\right](x_ik_j - x_jk_i) \,. \tag{7.43}$$

An induction procedure similar to the procedure we used in subsection A to obtain the induced Hamiltonian $\hat{H}$ leads here to the general induced wave equation:

$$\begin{aligned}
&\left[\left[W - \frac{g}{x}\right]^2 - \hat{\vec{p}}^{\,2} - m_2^{\,2}\right]\psi_1\\
&\qquad = \frac{1}{m_1}\left[W\hat{\vec{p}}^{\,2} + \frac{2g}{x}\vec{p}^{\,2} + \frac{i\hbar g}{2m_2}W\sigma^j\sigma^k\hat{f}_{jk}\right]\psi_1 \,,
\end{aligned} \tag{7.44}$$

where $\hat{f}_{jk}$ is the nonlocal operator with indicial function given by $\tilde{f}_{ij}$ [Eq. (7.43)]. Strictly speaking the g^2 term of the left-hand term should be disregarded because the whole calculation has been a first-order calculation in g. But we notice that in the limiting case $m_1 \to \infty$, Eq. (7.44) reduces to two uncoupled Klein-Gordon equations modified by a Coulomb field. This result can be derived directly by a standard quantization method in combination with the information which says that a spin-$\frac{1}{2}$ particle behaves as a monopole charge in the limit $m_1 \to \infty$. Therefore we can maintain the g^2 term in Eq. (7.44) because we know that this term would be the leading term (zeroth order) of the $1/m_1$ series at the second order in g.

Here again it must be emphasized that Eq. (7.44) remains an approximate equation and probably the best use we can make of it is to consider the operator in the right-hand term as small compared to the operator in the left-hand term.

C. Let us consider finally the "light-spin approximation method." This method is simpler than the preceding ones because it is not necessary to decompose ψ as a direct sum of two-component spinors, and will consist in using formal series expansions in $1/m_2$ and neglecting all terms of order higher than $1/m_2$. Let us write for any function which has a limit when $m_2 \to \infty$:

$$f = f_0 + \frac{1}{m_2}\,{}_1f + \cdots \ . \tag{7.45}$$

Using this notation, a straightforward calculation gives

$$
\begin{aligned}
&{}_0\tilde{\pi}^{(1)}_{10} = \frac{1}{x}, \quad {}_1\tilde{\pi}^{(1)}_{10} = -\frac{\tilde{\pi}^0_1}{x}, \\
&\qquad\qquad \text{with } \tilde{\pi}^0_1 = (m_1{}^2 + \vec{k}^2)^{1/2}, \\
&{}_0\tilde{\pi}^{(1)}_{1i} = 0, \quad {}_1\tilde{\pi}^{(1)}_{1i} = \frac{k_i}{x}, \\
&{}_0\tilde{F}^{(1)}_{i0} = -\frac{x_i}{x^3}, \quad {}_0\tilde{F}^{(1)}_{ij} = 0 \ .
\end{aligned} \tag{7.46}
$$

Let us write

$$\psi^{(0,1)} = {}_0\psi^{(0,1)} + \frac{1}{m_2}\,{}_1\psi^{(0,1)} \tag{7.47}$$

and define W by the formula

$$P \equiv m_2 + W \ . \tag{7.48}$$

Taking into account Eqs. (7.4) and (7.46) the zeroth-order equations (5.81) and (6.12) are

$$
\begin{aligned}
&(m_1\beta + \alpha^i k_i)_0\psi^{(0)} = W_0\psi^{(0)} \ , \\
&(m_1\beta + \alpha^i \hat{p}_i)_0\psi^{(1)} = W_0\psi^{(1)} - \frac{1}{x}\,{}_0\psi^{(0)} \ ,
\end{aligned} \tag{7.49}
$$

and the first-order equations are

$$(m_1\beta + \alpha^i k_i)_1\psi^{(0)} = W_1\psi^{(0)} + \tfrac{1}{2}(m_1{}^2 - W^2)_0\psi^{(0)} \ , \tag{7.50}$$

$$
\begin{aligned}
(m_1\beta + \alpha^k \hat{p}_k)_1\psi^{(1)} = {}& W_1\psi^{(1)} - \frac{1}{x}\,{}_1\psi^{(0)} \\
&+ \tfrac{1}{2}(m_1{}^2 - W^2)_0\psi^{(1)} \\
&+ \left[\frac{\tilde{\pi}_1{}^0}{x} - \frac{k_i}{x}\alpha^i \right. \\
&\qquad \left. + \frac{i\hbar}{2}\alpha^i \frac{x_i}{x^3} \right] {}_0\psi^{(0)} \ .
\end{aligned} \tag{7.51}
$$

Using the first equation (7.49) this last equation can also be written as

$$
\begin{aligned}
(m_1\beta + \alpha^k \hat{p}_k)_1\psi^{(1)} = {}& W_1\psi^{(1)} - \frac{1}{x}\,{}_1\psi^{(0)} \\
&+ \tfrac{1}{2}(m_1{}^2 - W^2)_0\psi^{(1)} \\
&+ \left[\frac{m_1}{x}\beta + \frac{i\hbar}{2}\alpha^i \frac{x_i}{x^3} \right] {}_0\psi^{(0)} \ .
\end{aligned} \tag{7.52}
$$

The general induced wave equation which can be derived from Eqs. (7.49), (7.50), and (7.52) is

$$
\begin{aligned}
(m_1\beta + \alpha^k \hat{p}_k)\psi = {}& \left[W - \frac{g}{x} \right] \psi \\
&+ \frac{1}{m_2} \left[\tfrac{1}{2}(m_1{}^2 - W^2) \right. \\
&\qquad \left. + g \left[\frac{m_1}{x}\beta + \frac{i\hbar}{2}\alpha^i \frac{x_i}{x^3} \right] \right] \psi \ .
\end{aligned} \tag{7.53}
$$

This equation which reduces to the exact Dirac equation for a spin-$\frac{1}{2}$ particle in a Coulomb field should be treated with the same care as that of Eqs. (7.32) and (7.44).

In most textbooks on relativistic quantum mechanics the Dirac equation is used to derive the energy-level patterns for bound states of hydrogen-like atoms. For hydrogen itself this equation is not accurate enough even if the magnetic moment of the proton which is responsible for the hyperfine structure and the Lamb-type effects are neglected. The reason is that the Dirac equation neglects the recoil of the proton. To deal with this difficulty it has sometimes been suggested that the mass of the electron should be replaced by the reduced mass of the electron-proton system. This recipe has never been justified theoretically and anyway it does not give with sufficient quantitative precision the fine-structure pattern. Instead the Breit[2] equation takes care of both the magnetic moment (not the anomalous one) of the proton and its recoil.

For ionized helium, H_e^+, the nucleus has spin zero and therefore one of the limitations of the Dirac equation, the magnetic moment of the nucleus, is not present. The ratio m_1/m_2 is four times smaller than for hydrogen, but the charge of the nucleus is twice that of the proton and this compensates the smallness of this ratio. The net result is that the Dirac equation is again an insufficient approximation to discuss the fine-structure pattern of H_e^+. This problem can, *a priori,* be consistently handled using either Eqs. (7.32) or (7.53).

The ratio m_1/m_2 is not always small for interesting cases. For instance if particle 1 is a proton and particle 2 is a π^- meson, this ratio is $\simeq 7$. It would be nonsense to use the Dirac equation and the reduced-mass recipe to deal with the electromagnetic contribution to this problem. On the other hand the Klein-Gordon equation modified by a Coulomb potential would be an insufficient approximation. Depending on the range of energies being considered our proposition is to use either Eqs. (7.32) or (7.44).

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[1]L. Bel, in *Contribution to Differential Geometry and Relativity,* edited by M. Cahen and M. Flato (Reidel, Dordrecht, Holland, 1976).

[2]See, for instance, L. D. Landau and E. M. Lifshitz, *Théorie Quantique Relativiste* (Editions de Moscou, 1974), p. 391; H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Systems,* Contribution to Handbuch der Physik, Vol. XXXV (Springer, Berlin, 1957).

[3]A. Messiah, *Mécanique Quantique* (Dunod, Paris, 1959), footnote p. 801.

[4]C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Mécanique Quantique* (Hermann, Paris, 1973), footnote p. 1204.

[5]This paper is intended to be self-contained as far as the formalism is concerned. But it has been written mainly for those readers who are acquainted with the late developments in predictive relativistic mechanics. To those who are not familiar with the philosophy behind this approach, we recommend the reading of the first three paragraphs of Sec. I in L. Bel, Phys. Rev. D 18, 4780 (1978).

[6]The subject of this paper has been considered in connection with the quasipotential approach in V. A. Rizov, I. T. Todorov, and B. L. Aneva, Nucl. Phys. B98, 447 (1975). We ignore whether an *a priori* connection between this approach and ours can be established.

[7]$\alpha,\beta,\gamma,\ldots=0,1,2,3$. We use the signature $+2$ of M_4.

[8]$a,b,c,\ldots=1,2$. We use these indices indistinctly in an up or down position.

[9]$\epsilon^a=1$. We use the summation convention both for the greek and latin indices.

[10]By convention $a'\neq a$.

[11]We take the speed of light in vacuum $c=1$.

[12]L. Bel and X. Fustero, Ann. Inst. Henri Poincaré 24, 411 (1976).

[13]In Ref. 1, we used a milder definition.

[14]In Ref. 1, we used a stronger definition.

[15]L. Bel and J. Martin, Ann. Inst. Henri Poincaré 22, 173 (1975).

[16]The definition that we gave in Ref. 1 included the symmetry under the permutation group S_2. The relaxation of this condition here will be needed in some of the problems that we shall consider in Sec. VI.

[17]This limit must be understood as in Eq. (2.17) by substituting π_a^α by k_a^α.

[18]In general there will exist an infinity of quantizers corresponding to any given system, and any one of them will define an associated quantum-mechanical system. As we shall see later on, the conditions that we have demanded to the concept of a quantizer are sufficient to guarantee the equivalence (at least at a formal level) of all these quantum systems.

[19]Any operator acting on a function of (x_a^α,π_b^β) acts on a function of (x_a^α,k_b^β) by substitution of π_a^α by k_a^α.

[20]Notice that the k_a^α's are labels of asymptotically free states. Their numerical value will coincide with the numerical value of the kinematical momenta Π_b^β of scattering particles and with the numerical value of the canonical momenta (which are functions of x_a^α and Π_b^β) in the incoming asymptotic free region, i.e., in the infinite past.

[21]In fact we already used this first equation when we wrote Eq. (5.17).

[22]For noninteracting particles the choice of the $+$ sign of the square roots corresponds to the choice of positive energy solutions.

[23]As we know (Ref. 12 and references therein) standard classical field theories of interacting particles can be cast into a predictive-relativistic-mechanics formalism without abandoning any of its field-theoretical aspects.

[24]L. Bel, A. Salas, and J. M. Sanchez, Phys. Rev. D 7, 1099 (1973).

[25]L. Bel and J. Martin, Phys. Rev. D 9, 2760 (1974).

[26]We take again $c=1$.