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Feynman propagator in a linearly expanding universe

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It is demonstrated that three reasonable physical criteria yield a unique Feynman propagator in a linearly expanding Robertson-Walker universe.

I. INTRODUCTION

The generalization of quantum-field-theory (QFT) propagators from flat to curved space-time is not yet solved (cf. Refs. 1—3), and there are different criteria to define these Green's functions in arbitrary geometries. On the other hand, to find the right criterion, i.e., to define a good particle model in curved space-time, is a very important problem not only for QFT in a curved background but also for quantum gravity.

This work is intended to be a concise mathematical study of four of these criteria, focused on the definition of the Feynman propagator in a Robertson-Walker universe with linear evolution $a(t)=t$. All these criteria have an "exact" statement in common, i.e., they can be formulated independently of a power development (for criteria stated in a power development, see DeWitt,[4] Parker,[2] and Castagnino *et al.*[5]).

First, we shall see how they work in flat space-time, and afterwards it will be shown that in a linear evolution three of these criteria single out a unique propagator, introduced for the first time by Chitre and Hartle.[6] Charach and Parker[7] made a deeper study of this original work and showed that the conditions stated by Chitre and Hartle were insufficient to define the propagator uniquely. They added a new condition that is studied and compared with the other criteria below (condition 2). Thus this work can be considered as a continuation of this line of thought.

Section II is devoted to establishing the basic properties of the Feynman propagator and its development in the "proper time" formalism. In Sec. III, we introduce the four criteria and use them in flat space-time. Later, in Sec. IV, we do the same in a linearly expanding universe. Finally, we discuss the results in Sec. V.

II. PRELIMINARIES

In this section, we review the principal equations and results, referring the reader to Refs. 3 and 8 for a complete treatment of the subject.

We shall study the problem in a Robertson-Walker universe. Its metric is

$$ds^2=-dt^2+a^2(t)(dx^2+dy^2+dz^2)\ , \tag{2.1}$$

where $a(t)$ is the radius of the universe, that we shall choose with a singularity at $t=0$, and with $R\to 0$ when $t\to\infty$. The Klein-Gordon equation

$$(-\nabla_\mu\nabla^\mu+m^2+\xi R)\phi=0 \tag{2.2}$$

can be solved by a function $\phi_{\vec{k}}(x)=e^{i\vec{k}\cdot\vec{x}}\psi_k(t)$ if $\psi_k(t)$ satisfies

$$\frac{1}{a^3}\frac{d}{dt}a^3\frac{d}{dt}\psi_k+\left[\frac{k^2}{a^2}+m^2+\xi R\right]\psi_k=0\ . \tag{2.3}$$

Let $\{P^-_{\vec{k}}\}\cup\{P^+_{\vec{k}}\}$ be the "out" basis, i.e., $P^+_{\vec{k}}$ ($P^-_{\vec{k}}$) is the particle (antiparticle) model wave function in the far future (when $R\ll 1$), and let

$$P^+_{\vec{k}}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\psi_k(t)\ , \tag{2.4a}$$

$$P^-_{\vec{k}}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\psi^*_k(t)\ . \tag{2.4b}$$

The $\psi_k(t)$ function is a solution of Eq. (2.3) that satisfies the normalization condition

$$\psi^*_k\frac{d}{dt}\psi_k-\psi_k\frac{d}{dt}\psi^*_k=\frac{-i}{a^3}\ . \tag{2.5}$$

This condition yields

$$(P^+_{\vec{k}},P^+_{\vec{k}'})=-(P^-_{\vec{k}},P^-_{\vec{k}'})=\delta(\vec{k}-\vec{k}')\ , \tag{2.6a}$$

$$(P^+_{\vec{k}},P^-_{\vec{k}'})=0\ , \tag{2.6b}$$

where (,) is the Klein-Gordon product

$$(g,h)=i\int_{\Sigma} d\Sigma_{\mu}(g^{*}\partial^{\mu}h-h\partial^{\mu}g^{*}) . \quad (2.7)$$

If functions g and h are solutions of Eq. (2.2), this product is independent of the spatial integration surface Σ.

Let now $\{F^{-}_{\vec{k}}\}\cup\{F^{+}_{\vec{k}}\}$ be the "in" basis. Both in and out bases are arbitrary, but we shall suppose that the in basis fixes the particle model at the singularity, i.e., at the time origin. The in basis can be obtained from the out basis by a Bogoliubov transformation,

$$F^{+}_{\vec{k}}=B_kP^{+}_{\vec{k}}+C_kP^{-}_{\vec{k}} , \quad (2.8a)$$

$$F^{-}_{\vec{k}}=B^{*}_kP^{-}_{\vec{k}}+C^{*}_kP^{+}_{\vec{k}} . \quad (2.8b)$$

As a result of the spatial homogeneity of the metric, the B_k and C_k can only depend on the length k, of $\vec{k}$, and not on its orientation. $F^{+}_{\vec{k}}$ and $F^{-}_{\vec{k}}$ satisfy a set of equations similar to (2.4), (2.5), and (2.6) with the normalization

$$|B_k|^2-|C_k|^2=1 . \quad (2.9)$$

The field $\phi(x)$ can be developed either in the out or the in basis as

$$\phi(x)=\int d^3k[a_{\vec{k}}P^{+}_{\vec{k}}(x)+a^{\dagger}_{-\vec{k}}P^{-}_{\vec{k}}(x)]$$
$$=\int d^3k[A_{\vec{k}}F^{+}_{\vec{k}}(x)+A^{\dagger}_{-\vec{k}}F^{-}_{\vec{k}}(x)] . \quad (2.10)$$

The creation and annihilation operators satisfy the usual canonical commutation relations, and they are related by

$$a_{\vec{k}}=B_kA_{\vec{k}}+C^{*}_kA^{\dagger}_{-\vec{k}} , \quad (2.11a)$$

$$a^{\dagger}_{-\vec{k}}=C_kA_{\vec{k}}+B^{*}_kA^{\dagger}_{-\vec{k}} . \quad (2.11b)$$

We define the Feynman propagator as

$$G_F(x,x')=i\langle 0_{\rm out}|0_{\rm in}\rangle^{-1}$$
$$\times\langle 0_{\rm out}|T(\phi(x)\phi(x'))|0_{\rm in}\rangle , \quad (2.12)$$

where $|0_{\rm in}\rangle$ and $|0_{\rm out}\rangle$ are the in and out vacua, respectively, and T means the time-ordered product. Taking into account the symmetries of the problem, we can write G_F as

$$G_F(x,x')=i\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left[\psi_k(t_>)\psi^{*}_k(t_<)+\left[\frac{C_k}{B_k}\right]^{*}\psi_k(t)\psi_k(t')\right] . \quad (2.13)$$

Here $t_>=\max(t,t')$, $t_<=\min(t,t')$. If we consider $G_F(x,x')$ as a function of x only, and $t\neq t'$, it is clear that it is a solution of Eq. (2.2). At $t=t'$, ∂_tG_F is not continuous. In fact, taking Eq. (2.5) into account,

$$\frac{\partial}{\partial t}G_F(t'^{+})-\frac{\partial}{\partial t}G_F(t'^{-})=\frac{1}{a^3}\delta(\vec{x}-\vec{x}') . \quad (2.14)$$

Therefore G_F is a solution of

$$(-\nabla_{\mu}\nabla^{\mu}+m^2+\xi R)G_F(x,x')$$
$$=\frac{1}{[a(t)]^3}\delta(\vec{x}-\vec{x}\,')\delta(t-t') . \quad (2.15)$$

We can try to write $G_F(x,x')$ using an integral representation (cf. Schwinger[9] and Parker[2]),

$$G_F(x,x')=i\int_0^{\infty}ds\,e^{-im^2s}\langle x,s\,|\,x',0\rangle , \quad (2.16)$$

where the kernel $\langle x,s\,|\,x',0\rangle$ is the solution of

$$i\frac{\partial}{\partial s}\langle x,s\,|\,x',0\rangle=(-\nabla_{\mu}\nabla^{\mu}+\xi R)\ \langle x,s\,|\,x',0\rangle , \quad (2.17a)$$

$$\langle x,0\,|\,x',0\rangle=\frac{1}{a^3(t)}\delta(x-x') . \quad (2.17b)$$

This integral representation is a natural generalization of the Schwinger representation of the Feynman propagator in flat space-time. Thus, we could state that only the real physical propagator can be developed in this way. In order to study this conjecture, that later on will prove to be wrong, it is interesting to find the inversion formula of Eq. (2.16), i.e., given G_F, the way to find $\langle x,s\,|\,x',0\rangle$. A natural idea is to consider Eq. (2.16) as a Laplace or Fourier transform and thus to take into account the set of all $G_F(x,x',m^2)$ as functions of the parameter m^2.

Since m^2 is real and positive, at first sight we can take $\langle x,s\,|\,x',0\rangle$ as the Laplace transform of $G_F(m^2)$. Nevertheless, we shall not find Eq. (2.16) in this way, but a similar one with positive sign in the exponent and integration bounds $\sigma-i\infty$ and $\sigma+i\infty$ in the complex variable $\sigma+is$.

If we consider Eq. (2.16) as a Fourier transform of $\langle x,s\,|\,x',0\rangle$, this problem disappears, but now both m^2 and s are to be considered as real variables, also taking negative values, and some ambiguities show up.

Anyhow, let us suppose that it is possible to find a function $\Gamma_F(x,x',\alpha)$ that satisfies

(a) $\Gamma_F(x,x',\alpha)=G_F(x,x',m^2)$ if $m^2=\alpha\geq 0$.

(b) $(-\nabla_\mu\nabla^\mu+\alpha+\xi R)\Gamma_F(x,x',\alpha)=\frac{1}{a^3}\delta(x-x')$ for all $\alpha\in\mathsf{R}$,

(c) For all x,x', Γ_F is a tempered distribution for the variable α .

(d) The Fourier transform of Γ_F ,

$$F[\Gamma_F]\equiv\gamma(x,x',s)=\int_{-\infty}^{\infty}d\alpha\, e^{i\alpha s}\Gamma_F(x,x',\alpha) \tag{2.18}$$

vanishes when $s<0$.

Then, putting

$$\langle x,s\,|\,x',0\rangle=\frac{1}{2\pi i}\gamma(x,x',s) ,$$

Eq. (2.16) is nothing else than the Fourier inversion formula. We can also find Eq. (2.17) from (2.18); for it we transform (2.18b) side by side, and we obtain

$$(-\nabla_\mu\nabla^\mu+\xi R)\gamma-i\frac{\partial\gamma}{\partial s}=\frac{2\pi}{a^3}\delta(x-x')\delta(s) . \tag{2.19}$$

Here $\partial\gamma/\partial s$ is the derivative of γ as a distribution. Taking into account the discontinuity of γ at $s=0$,

$$\frac{\partial\gamma}{\partial s}=\frac{\partial\gamma}{\partial s}+\gamma(0)\delta(s) . \tag{2.20}$$

On the right-hand side, $\partial\gamma/\partial s$ is the ordinary derivative. Putting (2.20) back into (2.19), we get (2.17a) for $s\neq 0$ and (2.17b) for $s=0$.

We can put (2.18d) in a slightly different manner. If $R(\alpha)$ and $I(\alpha)$ are the real and imaginary parts of $\Gamma(\alpha)$, (2.18d) proves to be equivalent to

$$I(\alpha_0)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\alpha\frac{R(\alpha)}{\alpha-\alpha_0} . \tag{2.18d'}$$

Later on we shall see that not all G_F of Eq. (2.13) have a Γ_F that satisfies (2.18). But in all cases we shall have

$$\begin{aligned}G_F(x,x',m^2)&=i\int_{-\infty}^{\infty}ds\, e^{-im^2 s}\langle x,s\,|\,x',0\rangle\\&=i\int_0^{\infty}ds[e^{-im^2 s}\langle x,s\,|\,x',0\rangle\\&\qquad+e^{im^2 s}\langle x,(-s)\,|\,x',0\rangle] .\end{aligned} \tag{2.21}$$

We shall also call this integral representation a "proper-time formalism" to avoid the invention of an additional name.

III. THE FEYNMAN PROPAGATOR IN FLAT SPACE-TIME

As an example, let us see what happens in flat space-time. The most general solution of (2.3) is

$$\psi_k(t)=\alpha_k\frac{e^{-i\omega_k t}}{(2\omega_k)^{1/2}}+\beta_k\frac{e^{i\omega_k t}}{(2\omega_k)^{1/2}} , \tag{3.1}$$

where $\omega_k=(k^2+m^2)^{1/2}$. Condition (2.5) gives

$$|\alpha_k|^2-|\beta_k|^2=1 , \tag{3.2}$$

and the most general Feynman propagator is

$$\begin{aligned}G_F(x,x')=i\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2\omega_k}&\times\Bigg[|\alpha_k|^2 e^{-i\omega_k(t_>-t_<)}+|\beta_k|^2 e^{i\omega_k(t_>-t_<)}+\alpha_k\beta_k^* e^{-i\omega_k(t+t')}+\beta_k\alpha_k^* e^{i\omega_k(t+t')}\\&+\left[\frac{C_k}{B_k}\right]^*[\alpha_k{}^2 e^{-i\omega_k(t+t')}+\beta_k{}^2 e^{i\omega_k(t+t')}+\alpha_k\beta_k(e^{-i\omega_k(t-t')}+e^{i\omega_k(t-t')})]\Bigg] .\end{aligned} \tag{3.3}$$

In flat space-time it is natural to demand that the real physical propagator must be invariant under space-time translations. This requirement yields

$$\alpha_k\left[\beta_k^*+\left[\frac{C_k}{B_k}\right]^*\alpha_k\right]=0 , \tag{3.4a}$$

$$\beta_k^* \left[\left(\frac{C_k}{B_k} \right) \beta_k^* + \alpha_k \right] = 0 \,. \tag{3.4b}$$

From Eq. (3.2), $\alpha_k \neq 0$ as

$$\det \begin{bmatrix} 1 & (C_k/B_k)^* \\ (C_k/B_k) & 1 \end{bmatrix} = \frac{1}{|B_k|^2} \neq 0 \,.$$

Using Eq. (2.9), we have $\beta_k = 0$, and now (3.4a) gives $C_k = 0$. We can choose α_k and B_k real [the G_F of Eq. (2.13) is invariant under phase transformation of the basis], and thus α_k and B_k become fixed by Eqs. (2.9) and (3.2). Finally, the Feynman propagator turns out to be

$$\Delta_F(x,x') = i \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}}{2\omega_k} e^{-i\omega_k(t_> - t_<)} \,. \tag{3.5}$$

Let us define now

$$\omega_k(\alpha) = \begin{cases} (k^2+\alpha)^{1/2} \text{ if } k^2+\alpha \geq 0 \,, \\ -i(\,|k^2+\alpha|\,)^{1/2} \text{ if } k^2+\alpha < 0 \,, \end{cases} \tag{3.6}$$

and

$$\Gamma_F(\alpha) = i \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}}{2\omega_k(\alpha)} e^{-i\omega_k(\alpha)(t_> - t_<)} \,. \tag{3.7}$$

This definition provides a generalization of Δ_F to negative values of m^2. It can be shown that Γ_F satisfies conditions (2.18) and its Fourier transform leads us to the Schwinger formula

$$\Delta_F(x,x') = \int_0^\infty i\,ds\, e^{-im^2 s} \left[i \frac{1}{(4\pi i s)^2} \exp\left[-i \frac{(t-t')^2 - (\vec{x}-\vec{x}\,')^2}{4s} \right] \right] \,. \tag{3.8}$$

Let us now give some alternative definitions of the physical propagator.

The invariance requirement under translation, that defines the physical Δ_F among all the possible propagators in flat space-time, has no analog in the Robertson-Walker universe, because the metric itself is not invariant under time translations. Therefore, it is interesting to find another criterion that can be generalized from flat to curved space-time. We shall study the most usual criteria that can be found in the literature on the subject.

Criterion 1 [see Schwinger (Ref. 9) and DeWitt (Ref. 4)]. The physical G_F can be developed as in Eq. (2.16), i.e., it is analytic in the lower semiplane of the variable m^2.

We recall the development[10]

$$\left[\frac{1}{2\omega_k} \right] e^{-i\omega_k |t-t'|} = \int_0^\infty i\,ds\, e^{-i\omega_k^2 s} \frac{e^{-i\pi/4}}{2\sqrt{\pi s}} e^{-i(|t-t'|^2/4s)} \,, \tag{3.9}$$

which comes from the spatial Fourier transform of Eq. (3.8). Using both Eq. (3.9) and its complex conjugate, we can write the most general G_F as

$$G_F(x,x')=\int_0^\infty i\,ds\frac{e^{-i\pi/4}}{2\sqrt{\pi s}}\times\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\times\left[e^{-im^2s}e^{-ik^2s}\left[\left[|\alpha_k|^2+\left[\frac{C_k}{B_k}\right]^*\alpha_k\beta_k\right]e^{-i(t-t')^2/4s}+\left\{\left[\alpha_k\beta_k^*+\left[\frac{C_k}{B_k}\right]^*\alpha_k{}^2\right]H(t+t')+\left[\beta_k\alpha_k^*+\left[\frac{C_k}{B_k}\right]^*\beta_k{}^2\right]H(-(t+t'))\right\}e^{-i(t+t')^2/4s}\right]-e^{im^2s}e^{ik^2s}\left[\left[|\beta_k|^2+\left[\frac{C_k}{B_k}\right]^*\alpha_k\beta_k\right]e^{i(t-t')^2/4s}+\left\{\left[\beta_k\alpha_k^*+\left[\frac{C_k}{B_k}\right]^*\beta_k{}^2\right]H(t+t')+\left[\alpha_k\beta_k^*+\left[\frac{C_k}{B_k}\right]^*\alpha_k{}^2\right]H(-(t+t'))\right\}e^{i(t+t')^2/4s}\right]\right],\quad(3.10)$$

where H is the Heaviside function. Therefore criterion 1 yields

$$\beta_k\left[\beta_k^*+\left[\frac{C_k}{B_k}\right]^*\alpha_k\right]=0\,,\quad(3.11a)$$

$$\beta_k^*\left[\left[\frac{C_k}{B_k}\right]\beta_k^*+\alpha_k\right]=0\,,\quad(3.11b)$$

$$\alpha_k\left[\beta_k^*+\left[\frac{C_k}{B_k}\right]^*\alpha_k\right]=0\,.\quad(3.11c)$$

This system is equivalent to Eqs. (3.4), so it has the same unique solution $\beta_k=C_k=0$ that yields Δ_F.

Criterion 2 [see Charach and Parker (Ref. 7)]. The physical G_F can be developed as in Eq. (2.21) with the square-integrable kernel in Euclidean space.

We can go to Euclidean space-time by writing $t=\pm is$. On the other hand, it is necessary to change s into $\sigma=is$, in order that function $e^{ix^2/4s}$ becomes $e^{-x^2/4\sigma}$ and turns out to be integrable. Then the function $e^{-it^2/4s}$ becomes $e^{-t^2/4\sigma}$, which is integrable, and the function $e^{+it^2/4s}$ becomes $e^{+t^2/4\sigma}$, which is divergent. Therefore, criterion 2 also yields system (3.11).

If we choose $\sigma=-is$ instead of $\sigma=+is$, we will find, besides the divergence in $e^{+x^2/4\sigma}$, the equations

$$\alpha_k^*\left[\alpha_k+\left[\frac{C_k}{B_k}\right]\beta_k^*\right]=0\,,\quad(3.12a)$$

$$\alpha_k\left[\left[\frac{C_k}{B_k}\right]^*\alpha_k+\beta_k^*\right]=0\,,\quad(3.12b)$$

$$\beta_k^*\left[\alpha_k+\left[\frac{C_k}{B_k}\right]\beta_k^*\right]=0\,.\quad(3.12c)$$

These equations yield $\alpha_k=0$ and so they are incompatible with Eq. (3.2).

Criterion 3 [see DeWitt (Ref. 4)]. The physical G_F is the analytic continuation of the unique Green's function of the operator $(-\Delta_4+m^2)$ in the four-dimensional Euclidean space.

Here Δ_n is the Laplace operator in n dimensions. Let us call τ the Euclidean time. The unique solution of

$$\left[-\frac{\partial^2}{\partial\tau^2}-\Delta_3+m^2\right]J(x,x')=\delta(\vec{x}-\vec{x}\,')\delta(\tau-\tau')\,,\quad(3.13)$$

that is a tempered distribution, is

$$J(x,x')=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{e^{-\omega_k(\tau_>-\tau_<)}}{2\omega_k}\,,\quad(3.14)$$

where $\omega_k=(k^2+m^2)^{1/2}$. Going back to Minkowski space-time, we can change τ for $t=-i\tau$ or $t=i\tau$. The first choice yields $G_F(x,x')=\Delta_F(x,x')$, up to an irrelevant phase factor. The second choice yields again system (3.12) and, as we already know, an incompatible result with Eq. (3.2). Therefore, the method works if we define the Euclidean time as $\tau = it$, $t = -i\tau$.

Criterion 4 [see Schwinger (Ref. 9), DeWitt (Ref. 4) and Rumpf and Urbantke (Ref. 11)]. The physical G_F is the limit, when $z \to m^2 - i0^+$, of the unique Green's function of the operator $-\Box + z$ in Minkowski space-time.

Let $z = m^2 \pm i\epsilon$, $\epsilon > 0$. The quoted Green's function is

$$Q(x,x') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega_k{}^2 \pm i\epsilon - \omega^2} . \quad (3.15)$$

We use the formula

$$\lim_{\epsilon \to 0} \frac{1}{x \pm i\epsilon} = \frac{1}{x} \mp i\pi\delta(x) . \quad (3.16)$$

After an elementary integration we can see that we find the correct result with the minus sign, i.e., $z = m^2 - i0^+$, while the plus sign yields again (3.12). Therefore, the limit we must take is $z \to m^2 - i0^+$.

Other criteria. A natural condition, that we can suppose the physical Feynman propagator has in the case of arbitrary metrics, is that G_F "resembles" Δ_F of flat space-time when $x \to x'$ (see Castagnino and Weder,[12] and Ceccato *et al.*[13] An attempt to make this statement precise is to ask that both G_F and Δ_F have the same singular structure when $x \to x'$ (see Castagnino *et al.* Ref. 5). When we have a universe such that $R \to 0$ when $t \to \infty$, we can also choose as the out solution a (WKB) approximation.

But, as we shall see immediately, these "inexact" criteria do not lead to a unique G_F even in flat space-time. If we put $\beta_k = 0$, $\alpha_k = 1$ in Eq. (3.3), we have

$$G_F(x,x') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2\omega_k} \left[e^{-i\omega_k(t_> - t_<)} + \left[\frac{C_k}{B_k} \right]^* e^{-i\omega_k(t+t')} \right] , \quad (3.17)$$

and we can choose $(C_k/B_k)^*$ in infinite ways such that the second term in the parentheses would be regular at the origin. In this way, G_F and Δ_F will have the same singular structure.

Let use remark that all these choices have a common limit $(C_k/B_k) \to 0$ when $k \to \infty$, because the Fourier transform interchanges regularity with vanishing at infinity.

Where these criteria are used, normally it is stated that $(C_k/B_k) = 0$ implicitly by choosing a power development in k or x that automatically eliminates terms in (C_k/B_k). Therefore, the asymptotic particle model for $k \to \infty$ is only a local value around the point where the development is made. Although they are useful, and well proved by their success in renormalization methods, they do not fit to solve our problem.

IV. ROBERTSON-WALKER UNIVERSE WITH LINEAR EXPANSION

(See Nariai and Azuma,[14] Chitre and Hartle,[6] and Charach and Parker.[7]) Let us consider Eq. (2.3) with $a(t) = t$, $R = 6t^{-2}$, and $\xi = \frac{1}{6}$ (conformal coupling). The general solution is

$$\psi_k(t) = \alpha_k \frac{H_{ik}^{(2)}(mt)}{t} + \beta_k \frac{H_{ik}^{(1)}(mt)}{t} , \quad (4.1)$$

where $H_{ik}^{(1)(2)}$ are the Hankel functions that satisfy

$$[H_\nu^{(2)}(z)]^* = H_{\nu^*}^{(1)}(z^*) , \quad (4.2a)$$

$$H_{-\nu}^{(1)}(z) = e^{i\nu\pi} H_\nu^{(1)}(z) ,$$

$$H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z) , \quad (4.2b)$$

$$H_\nu^{(1)}(z) \dot{H}_\nu^{(2)}(z) - \dot{H}_\nu^{(1)}(z) H_\nu^{(2)}(z) = \frac{4}{i\pi z} . \quad (4.2c)$$

The normalization equation (2.5) turns out to be

$$e^{-k\pi} |\alpha_k|^2 - e^{k\pi} |\beta_k|^2 = \frac{\pi}{4} . \quad (4.3)$$

Now we can introduce the Bessel functions

$$J_\lambda(z) = \tfrac{1}{2}[H_\lambda^{(1)}(z) + H_\lambda^{(2)}(z)] , \quad (4.4a)$$

$$J_{-\lambda}(z) = \tfrac{1}{2}[e^{i\lambda\pi} H_\lambda^{(1)}(z) + e^{-i\lambda\pi} H_\lambda^{(2)}(z)] . \quad (4.4b)$$

The most general G_F becomes

$$G_F(x,x')=\frac{2i}{tt'}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(1-e^{-2k\pi})^{-1}$$
$$\times\Bigg\{H^{(2)}_{ik}(mt_>)J_{ik}(mt_<)\left[e^{-k\pi}|\alpha_k|^2-e^{-k\pi}\alpha_k\beta^*_k+\left[\frac{C_k}{B_k}\right]^*(\alpha_k\beta_k-e^{-2k\pi}\alpha_k{}^2)\right]$$
$$-H^{(2)}_{-ik}(mt_>)J_{-ik}(mt_<)\left[e^{-3k\pi}|\alpha_k|^2-e^{-k\pi}\alpha_k\beta^*_k+\left[\frac{C_k}{B_k}\right]^*(e^{-2k\pi}\alpha_k\beta_k-e^{-2k\pi}\alpha_k{}^2)\right]$$
$$+H^{(1)}_{ik}(mt_>)J_{ik}(mt_<)\left[-e^{-k\pi}|\beta_k|^2+e^{-k\pi}\alpha^*_k\beta_k+\left[\frac{C_k}{B_k}\right]^*(\beta_k{}^2-e^{-2k\pi}\alpha_k\beta_k)\right]$$
$$-H^{(1)}_{-ik}(mt_>)J_{-ik}(mt_<)\left[-e^{k\pi}|\beta_k|^2+e^{-k\pi}\alpha^*_k\beta_k+\left[\frac{C_k}{B_k}\right]^*(\beta_k{}^2-\alpha_k\beta_k)\right]\Bigg\}. \quad (4.5)$$

We proceed now to test criteria 1—4.

Criterion 1′. We shall use an integral representation of the product of Hankel and Bessel functions (see Watson,[15] p. 439),

$$H^{(1)}_\nu(Z)J_\nu(z)=\frac{1}{i\pi}\int_0^\infty\frac{ds}{s}I_0\left[\frac{Zz}{is}\right]\exp\left\{(i/2)\left[s+\frac{1}{s}(Z^2+z^2)\right]\right\}, \quad (4.6a)$$

$$H^{(2)}_\nu(Z)J_\nu(z)=\frac{1}{i\pi}\int_0^\infty\frac{ds}{s}I_\nu\left[\frac{iZz}{s}\right]\exp\left\{-(i/2)\left[s+\frac{1}{s}(Z^2+z^2)\right]\right\}, \quad (4.6b)$$

where $\mathrm{Re}(\nu)>-1$, $|Z|>|z|$, and $I_\nu(z)$ is the modified Bessel function $I_\nu(z)=i^{-\nu}J_\nu(iz)$. If we change the variable s to $2m^2s$, with $m^2\geq 0$, it is clear that the terms rejected by criterion 1 are those with $H^{(1)}_{ik}$ or $H^{(1)}_{-ik}$, i.e., the last two terms in the curly brackets in (4.5).[16] Therefore we have

$$\beta_k\left[-e^{-k\pi}\beta^*_k+e^{-k\pi}\alpha^*_k+\left[\frac{C_k}{B_k}\right]^*(\beta_k-e^{-2k\pi}\alpha_k)\right]=0, \quad (4.7a)$$

$$\beta_k\left[-e^{k\pi}\beta^*_k+e^{-k\pi}\alpha^*_k+\left[\frac{C_k}{B_k}\right]^*(\beta_k-\alpha_k)\right]=0. \quad (4.7b)$$

Taking (4.3) into account, we find

$$\det\begin{bmatrix}-e^{-k\pi}\beta^*_k+e^{-k\pi}\alpha^*_k & \beta_k-e^{-2k\pi}\alpha_k\\ -e^{k\pi}\beta^*_k+e^{-k\pi}\alpha^*_k & \beta_k-\alpha_k\end{bmatrix}=\frac{-\pi}{4}(1-e^{-2k\pi})\neq 0. \quad (4.8)$$

The system (4.7) implies that $\beta_k=0$, which coincides with the WKB solution for the out basis (see Chitre and Hartle[6] and Charach and Parker[7]), but G_F is not fully determined because C_k/B_k remains undefined.

Criterion 2′. We must use the Euclidean metric

$$ds^2=d\tau^2+\tau^2(d\xi_1{}^2+d\xi_2{}^2+d\xi_3{}^2).$$

From our previous experience in flat space-time, we make the changes $t=-i\tau$ (see criterion 3) and $S=-i\sigma$ (see criterion 2). But these transformations alone are not enough, and we must also change the spatial coordinates. We have now two possibilities, $\vec{x}=\pm i\vec{\xi}$ and the correlative change $\vec{k}=\mp i\vec{\eta}$, in order that $\vec{\eta}$ remains the canonical conjugated coordinates of $\vec{\xi}$. Nevertheless, the magnitudes of physical relevance have spherical symmetries and both possibilities lead us to the same physical results. Furthermore, we must treat $\eta=(\vec{\eta},\vec{\eta})^{1/2}$ as a positive number, but, as $(\vec{\eta},\vec{\eta})=-(\vec{k},\vec{k})=-k^2$, thus $\eta=\pm ik$.

Let us consider again Eq. (4.6) and the integral representation (see Watson,[17] p. 181)

$$I_\nu(z)=\frac{1}{2\pi i}(\tfrac{1}{2}z)^\nu\int_{-\infty}^{0^+}dr\,r^{-\nu-1}e^{r+z^2/4r}. \quad (4.9)$$

If we take $\eta=ik$ and we go back to (4.5), we find (a) the second term has a

$$I_{-ik}\left[\frac{tt'}{2is}\right]=I_{-\eta}\left[\frac{-\tau\tau'}{2\sigma}\right]$$

in its proper-time development, and by (4.9) a

$(-\tau\tau'/4\sigma)^{-\eta}$ term, divergent when $\tau,\tau'\to 0$. (b) The third and fourth terms have the exponential

$$e^{i(t^2+t'^2)/4s}=e^{(\tau^2+\tau'^2)/4\sigma}$$

in their proper-time development, and so diverge when $\tau,\tau'\to\infty$. From (4.7) we know that this leads us to $\beta_k=0$, and therefore from (a) we conclude

$$e^{-3k\pi}|\alpha_k|^2-\left[\frac{C_k}{B_k}\right]^* e^{-2k\pi}\alpha_k{}^2=0\ . \qquad (4.10)$$

Taking α_k, B_k real, from (2.9) and (4.3) it follows that

$$\alpha_k=\frac{\sqrt{\pi}}{2}e^{1/2k\pi},\ \ \beta_k=0\ , \qquad (4.11a)$$

$$B_k=(1-e^{-2k\pi})^{1/2}\ ,$$

$$C_k=e^{-k\pi}(1-e^{-2k\pi})^{-1/2}\ . \qquad (4.11b)$$

Therefore the out basis turns out to be

$$P_{\vec{k}}^{+}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\frac{\sqrt{\pi}}{2t}e^{1/2k\pi}H_{ik}^{(2)}(mt)\ ,$$
$$P_{\vec{k}}^{-}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\frac{\sqrt{\pi}}{2t}e^{-1/2k\pi}H_{ik}^{(1)}(mt)\ , \qquad (4.12)$$

and the in basis is

$$F_{\vec{k}}^{+}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\frac{\sqrt{\pi}}{2t}(1-e^{-2k\pi})^{-1/2}[e^{k\pi/2}H_{ik}^{(2)}(mt)+e^{-3k\pi/2}H_{ik}^{(1)}(mt)]\ ,$$
$$F_{\vec{k}}^{-}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\frac{\sqrt{\pi}}{2t}(1-e^{-2k\pi})^{-1/2}[e^{-k\pi/2}H_{ik}^{(1)}(mt)+e^{ik\pi/2}H_{ik}^{(2)}(mt)]\ . \qquad (4.13)$$

More concisely,

$$F_{\vec{k}}^{\pm}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\frac{\sqrt{\pi}}{2t}(1-e^{-2k\pi})^{1/2}e^{-k\pi/2}J_{\mp ik}(mt)\ .$$

These bases appear for the first time in the work of Chitre and Hartle[6] as a consequence of a generalization of the Feynman path integral from flat to curved space-time. Charach and Parker[7] found again these bases by requiring that the kernel $\langle x,s\,|\,x'0\rangle$ would be square integrable, and the out base be the WKB basis. As we can see, the last condition is redundant.

If we take $\eta=-ik$, the divergence in the origin appears in the first term of (4.5), so we are led to

$$e^{-k\pi}\alpha_k{}^2\left[1-\left[\frac{C_k}{B_k}\right]^* e^{-k\pi}\right]=0\ . \qquad (4.14)$$

This is incompatible with (2.9) and (4.3).

Criterion 3' [see Mensky and Karmakov (Ref. 18), Wald (Ref. 19), and Candelas and Raine (Ref. 20)]. The Euclidean metric is

$$ds^2=d\tau^2+\tau^2(d\xi_1{}^2+d\xi_2{}^2+d\xi_3{}^2) \qquad (4.15)$$

with $R=-6\tau^{-2}$. In this metric the Euclidean version of Eq. (2.15) is

$$-\frac{\partial^2}{\partial\tau^2}Q_F-\frac{3}{\tau}\frac{\partial}{\partial\tau}Q_F-\frac{1}{\tau}\Delta_3 Q_F+m^2Q_F-\frac{1}{\tau^2}Q_F=\frac{1}{\tau^2}\delta(\tau-\tau')\delta(\vec{\xi}-\xi')\ . \qquad (4.16)$$

Let us write $Q_F(\vec{\xi},\xi')$ as

$$Q_F(\xi,\xi')=\int\frac{d^3\eta}{(2\pi)^3}\frac{e^{i\vec{\eta}\cdot\vec{\xi}}}{\tau\tau'}h(\tau,\tau',\vec{\xi}',\vec{\eta})\ . \qquad (4.17)$$

Thus h must satisfy

$$\frac{\partial^2}{\partial\tau^2}h+\frac{1}{\tau}\frac{\partial}{\partial\tau}h-\left(m^2+\frac{\eta^2}{\tau^2}\right)h=\frac{-1}{\tau'}\delta(\tau-\tau')e^{-i\vec{\eta}\cdot\vec{\xi}'}\ . \qquad (4.18)$$

The solution of this Bessel equation that is regular for all τ is (see Courant and Hilbert[17], Vol. I, Chap. VII, pp. 470 and 501)

$$h(\tau)=A(\vec{\xi}',\tau',\vec{\eta})J_\eta(im\tau)\ \text{ if } 0\le\tau\le\tau'\ , \qquad (4.19a)$$

$$h(\tau)=B(\vec{\xi}',\tau',\vec{\eta})H_\eta^{(1)}(im\tau)\ \text{ if } \tau'<\tau<\infty\ . \qquad (4.19b)$$

At $\tau=\tau'$, h must be a continuous function, but its first derivative $\partial h/\partial\tau$ must be discontinuous, with a jump of $-(1/\tau')e^{-i\vec{\eta}\cdot\vec{\xi}'}$. Therefore

$$B(\vec{\xi}',\tau',\vec{\eta})H_\eta^{(1)}(im\tau')-A(\vec{\xi}',\tau',\vec{\eta})J_\eta(im\tau')=0\ , \qquad (4.20a)$$

$$B(\vec{\xi}',\tau',\vec{\eta})H_\eta'^{(1)}(im\tau')-A(\vec{\xi}',\tau',\vec{\eta})J_\eta'(im\tau')=\frac{i}{m\tau'}e^{-i\vec{\eta}\cdot\vec{\xi}'}\ . \qquad (4.20b)$$

This system has a determinant $2(\pi m\tau')^{-1}$ and its solution is

$$A(\vec{\xi}',\tau',\vec{\eta})=\frac{i\pi}{2}H_{\eta}^{(1)}(im\tau')e^{-i\vec{\eta}\cdot\vec{\xi}'}\,, \qquad (4.21a)$$

$$B(\vec{\xi}',\tau',\vec{\eta})=\frac{i\pi}{2}J_{\eta}(im\tau')e^{-i\vec{\eta}\cdot\vec{\xi}'}\,. \qquad (4.21b)$$

The final result is

$$Q_F(\xi,\xi')=\frac{i\pi}{2\tau\tau'}\int\frac{d^3\eta}{(2\pi)^3}e^{i\vec{\eta}(\vec{\xi}-\vec{\xi}')}\times H_{\eta}^{(1)}(im\tau_>)J_{\eta}(im\tau_<)\,. \qquad (4.22)$$

We note that $t=-i\tau$, and

$$H_{\eta}^{(1)}(im\tau_>)=H_{\eta}^{(1)}(-mt_>)=-H_{-\eta}^{(2)}(mt_>)\,, \qquad (4.23a)$$

$$J_{\eta}(im\tau_<)=J_{\eta}(-mt_<)=-J_{-\eta}(mt_<)\,. \qquad (4.23b)$$

So if we try to return to Euclidean space by taking $\eta=ik$ as in criterion 2′, we find the incompatible Eq. (4.14), but if we put $\eta=-ik$, we lead again to Chitre and Hartle's solution, up to a numerical factor.

Criterion 4′. Now we shall try to solve Eq. (2.15) by translating, to the linear expanding universe, the Feynman prescription for displacing the poles of the propagator in the ω complex plane. In flat spacetime, the prescription is to turn $(\omega^2-k^2-m^2)$ to $(\omega^2-k^2-m^2-i\epsilon)$. This can be achieved by adding a negative imaginary part to m^2 (as is usually stated), but also by adding it to k^2, or both.

Although a divergence may appear when the $e^{i\vec{k}\cdot\vec{x}}$ term is integrated, it is not essential, because it disappears if this integration is postponed to the end.

If we put

$$G_F(x,x')=\int\frac{d^3k}{(2\pi)^3}\frac{e^{\vec{k}\cdot(\vec{x}\cdot\vec{x}')}}{tt'}h_k(t,t',\vec{x},\vec{x}')\,,$$

then h must satisfy

$$\frac{\partial^2 h}{\partial t^2}+\frac{1}{t}\frac{\partial h}{\partial t}+\left[m^2+\frac{k^2}{t^2}\right]h=\frac{1}{t'}\delta(\vec{x}-\vec{x}')\delta(t-t')\,. \qquad (4.24)$$

Now we "displace the poles" by putting $k^2-i\eta$ instead of k^2, and $m^2-i\epsilon$ instead of m^2. If we define

$$\lambda=a-ib,\quad \mu=c-id,\quad a,b,c,d>0\,;$$

$$\lambda^2=k^2-i\eta,\quad \mu^2=m^2-i\epsilon\,.$$

Then the solution of (4.24) that is regular for all t is

$$h_k(t,t')=\begin{cases}A(t')J_{i\lambda}(\mu t) \text{ if } 0\le t\le t'\,,\\ B(t')H_{i\lambda}^{(2)}(\mu t) \text{ if } t'\le t<\infty\,.\end{cases} \qquad (4.25)$$

Repeating the calculations of criterion 3′, and then going to the limit $\epsilon,\eta\to 0$, we shall find the Chitre and Hartle propagator.

DISCUSSION AND CONCLUSIONS

The remarkable coincidence of criteria 2′, 3′, and 4′ to single out the Chitre and Hartle propagator must be based on any of the following causes.

(a) The studied criteria are basically equivalent.

(b) The Chitre and Hartle propagator is the real physical propagator for the linear expansion.

(c) The Robertson-Walker universe with linear expansion and conformal coupling is such a special case that all criteria give the same reasonable result.

Obviously, proposition (b) can neither be accepted nor rejected by now, until we have at our disposal more soundly based physical criteria working in every kind of geometry. As regards proposition (c), we must also say that the problem with linear expansion and conformal coupling is really exceptional. In fact, it is the only metric plus coupling where the definition of the particle model via Hamiltonian diagonalization works, giving a finite creation of particles [see Fulling (Ref. 21)]. The propagator of this particle model is again the one of Chitre and Hartle.

Finally, on the possible equivalence of the different criteria, proposition (a), we would like to make the following remarks.

(i) Criterion 2 is basically a refinement of criterion 1. The conclusion we get in criterion 1′ is that in a general metric the Cauchy data [Eq. (2.17b)] is not enough to single out a unique solution for Eq. (2.17a), but it should be obvious from the structure of this equation that, if we add boundary conditions over the kernel $\langle x,s\,|\,x',0\rangle$, when x and x' go to ∞ or to the singularity, we get uniqueness. The most natural choice of that boundary value is the homogeneous one. So we can think of Parker and Charach's ansatz as being a particular way of imposing null boundary conditions and regularity at zero over Schwinger's kernel. Nevertheless, we stress the fact that formally criteria 2 and 2′ are independent of criteria 1′ because the terms with the wrong dependence upon m^2s are also divergent in ∞.

(ii) Both criteria 1 and 4 use the same idea, i.e., to make the analytic continuation of G_F to complex values of m^2. Nevertheless, criterion 4 gives a more definite result because it has stronger restrictions on the behavior of G_F at the singularity and at infinity.

These restrictions are completely natural since one tries to Fourier analyze G_F (in criterion 4′ this transform is only spatial, but it remains physically implausible that G_F diverges when time goes to zero or moreover to ∞ when the metric is "almost" flat). On the contrary, it does not seem natural to impose arbitrary limits to the growth of kernel $\langle x,s \mid x'0\rangle$ at infinity, since neither this kernel nor the parameter s have a clear physical meaning.

On the other hand, criterion 4 is only a formalized version of the commonly used shift of real poles to the complex plane, before integrating a function using the residue theorem, while the proper-time formalism appears to bear different motivation according to different authors (see Schwinger,[9] Chitre and Hartle,[6] and Parker[2]). It is the authors' hope that their treatment of the proper-time formalism as a Fourier transform could give a better insight into this subject in the future.[22]

(iii) Criteria 2 and 3 are based on a very common idea in the current literature: i.e., to shift problems of the pseudo-Euclidean metrics to Euclidean ones.

Again, we can see that criterion 3 has regularity requirements on the propagator that are not explicitly stated, but implicitly in the fact that the propagator has a Fourier transform, and in the physical meaning of both times involved, the Euclidean τ and the physical time t. On the other hand, in criteria 2 and 2′, we demand that the kernel be L^2, when really we only want to eliminate some divergences, that can equally be excluded by weaker conditions, e.g., that the kernel would be a tempered distribution.[23]

Nevertheless, there are some ambiguities in the concept of "Euclidean techniques," e.g., the subtle role played by the η sign in criteria 2′ and 3′, that can be a serious drawback for these methods in more general metrics.

We conclude that the Chitre and Hartle propagator, in a linearly expanding universe, has so many properties belonging to the Feynman propagator of flat space-time that it is impossible to deny that it is a good candidate to play that role.

Nevertheless, it is premature to draw conclusions until other geometries have been studied and we have a deeper knowledge of the physical foundations of our methods. These are the lines that we shall follow in our future research.

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[22]It should be noted that for a square-integrable function f, the vanishing of its Fourier transform on the negative real axis is a necessary and sufficient condition for f to be the limit as $y \to 0$ of an analytic function $f(x+iy)$ such that $\int_{-\infty}^{\infty} |f(x+iy)|^2 dx < K$ for all $y<0$ (see Titchmarsh, Ref. 24, Chap. V, theorem 95).

[23]We stress the fact that the kernels $\langle xs \mid x'0\rangle$ from (4.6) are locally L^2, because they are continuous functions. The reason why only one kernel is globally L^2 is that only one kernel has a regular behavior at the boundary of space-time. So, imposing square integrability over the kernel $\langle xs \mid x'0\rangle$ weakens the original claim of Chitre and Hartle (Ref. 6) that their proposal avoids imposing definite boundary conditions on the singularity.

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