Parametric detector as a tunable antenna for continuous gravitational radiation

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A way of using Weber-type antennas as tunable detectors of gravitational radiation from pulsars is described. Preliminary laboratory tests on the feasibility of such an instrument give encouraging results.

I. INTRODUCTION

It has been pointed out[1] that gravity-wave detectors can be divided in two classes, depending on whether the dispersion relation they exhibit is linear or quadratic. Consequently, for practical but highly plausible reasons, class-1 detectors are limited in operation to frequencies greater than a few hundred Hz, whereas those of class 2 are sensitive above a few tens of Hz.

This seems to exclude the possibility of using a class-1 detector, such as a Weber cylinder, to study the radiation produced by pulsars like Crab and Vela. The Tokyo group, in fact, which started a series of very elegant experiments aimed at the detection of the Crab pulsar emission, uses a class-2 antenna which is tuned by trial and error to the source frequency $2\nu_0 = 60.2$ Hz. One can represent the situation as a harmonic oscillator driven on resonance. However, this sets, apparently, in practice, severe limitations to the sensitivity (which, in terms of h, depends, among other factors, on $Q^{-1/2}$) because, for a nontunable oscillator, the Doppler shift and slowing down due to radiation losses do not allow a Q much greater than $\sim 10\,000$. To overcome this difficulty, recourse was made to the technique of cold damping,[2] which permits widening the bandwidth of the system without serious losses in sensitivity. Obviously, this method cannot be of any more help when the antenna is also cooled down to extremely low temperatures. A substantial gain in sensitivity will then only be obtained by making the antenna tunable over a wide range of frequencies. This is particularly desirable if one considers that unpredictable frequency variations, like microglitches, are to be added to the predictable ones which we mentioned above. In addition, differences in frequency between the electromagnetic and gravitational-radiation (GR) pulsation, as discussed by Fujimoto,[3] and more recently by Zimmerman and Szedenits,[4] cannot be excluded, which implies a systematic search in the neighborhood of $2\nu_0$ and, perhaps, of ν_0.

On the other hand, the recent discovery of a fast radio pulsar with a period of 1.7 msec, i.e., rotating near the maximum rate permitted before disruption, encourages the hope that other objects of this kind but more favorable from the point of view of GR emission may exist and possibly be found in the electromagnetic channel. However, strong energy loss (gravitational-wave luminosity depends on the frequency of rotation to the sixth power) means rapid variation of the period. It has been estimated,[5] for example, that a pulsar rotating at its maximum rate of about 2000 Hz may in one year by this mechanism change its rotation rate to about 100 Hz. This makes practically impossible the use of a tuned detector (one would not even have the time to tune it) so that we could be faced in the near future with the unpleasant situation that enough gravitational energy is radiated but it escapes detection as we are not fast enough to catch it.

We want to suggest here a "resonant high-Q wide-band" detector which could be obtained, for example, from a Weber-type antenna having a frequency of resonance ν_1 greater than that, ν_0, of the external force (pulsar radiation field). In this scheme the harmonic oscillator representing the antenna mode is made parametric by varying its elastic strength, which, for example, can be realized by means of piezoelectric ceramics whose coupling with the antenna is varied due to an applied tension at frequency ν_p. As we shall see in the following, if $\nu_1 = \nu_0 + \nu_p$, we have a response proportional to the Q of the oscillator, which is needed, as in the case of the harmonic oscillator, in order that the Brownian noise of the antenna dominates the wide-band noise of the electronic readout. Clearly, with this instrument, every variation of ν_0 can be compensated by a corresponding variation of ν_p, which is in the hands of the experimenter.

II. RESPONSE AND NOISE FIGURE

A Weber-type antenna, made up of an Al cylinder of mass M and length L, is equivalent, in its fundamental longitudinal mode, to a harmonic oscillator of mass $m = M/2$ and length $l = 4L/\pi^2$. The equation of motion of the corresponding parametric oscillator becomes

$$m[\ddot{x} + \gamma\dot{x} + (\omega_1{}^2 - \omega_c{}^2\cos\omega_p t)x] = f_G \sin(\omega_0 t + \phi) , \tag{1}$$

where $f_G = \frac{1}{2} m l \omega_0{}^2 h$ represents the force exerted by the gravitational wave.

Assuming $\omega_1 = \omega_0 + \omega_p$, the solution at frequency ω_1, for sufficiently high $Q = \omega_1/\gamma$, is [Eq. (A19)]

$$x = \frac{\omega_c{}^2 f_G}{2\omega_p \omega_1 m \gamma \omega_1} \cos(\omega_1 t + \phi) , \tag{2}$$

similar to that for the harmonic oscillator, except for the factor $\omega_c{}^2/2\omega_p\omega_1 < 1$, which can have a maximum value of the order of 0.2 compatible with the stability of the oscillator. It turns out, also, that the phases of the pump and of the force are irrelevant.

Similarly, by using a formal analogy with the harmonic oscillator, one can estimate the contribution due to thermal noise, which is represented by a force having a constant power spectral density $4kTm\gamma$. The Fourier components that mainly contribute in the range $\Delta\nu$ around ν_1 can be thought of as being represented by two forces f_0 and f_1 with the above spectral density and varying with the frequency ν_0 and ν_1, respectively. One obtains formally a displacement like in Eq. (2) for f_0, whereas for f_1 one gets [Eq. (A20)]

$$x=\frac{f_1}{m\gamma\omega_1}\cos(\omega_1 t+\phi) . \tag{3}$$

From these, by taking the modulus square, one easily obtains for the noise power

$$N=\left[1+\left[\frac{\omega_c^2}{2\omega_p\omega_1}\right]^2\right]\frac{4kTQ}{m\omega_1^3}\Delta\nu\simeq\frac{4kT}{m\omega_1^3}\frac{Q}{\Delta t} , \tag{4}$$

where Δt is the measuring time. For the signal power we have

$$S=\left[\frac{\omega_c^2}{2\omega_p\omega_1}\right]^2\frac{Q^2 f_G^2}{m^2\omega_1^4} . \tag{5}$$

For sufficiently high Q the wide-band noise of the electronics can be neglected, so the signal-to-noise ratio (SNR) becomes

$$\frac{S}{N}=\left[\frac{\omega_c^2}{2\omega_p\omega_1}\right]^2\frac{Qf_G^2}{4m\omega_1 kT}\Delta t \tag{6}$$

and the minimum detectable h with unit SNR turns out to be

$$h=\frac{8\omega_p\omega_1}{\omega_c^2 l\omega_0^2}\left[\frac{kT\omega_1}{Qm\Delta t}\right]^{1/2} , \tag{7}$$

which is worse by the factor $(2\omega_p\omega_1/\omega_c^2)\sqrt{\omega_1/\omega_0}$ with respect to the case of the harmonic oscillator driven at resonance.

III. DISCUSSION

As we have seen, the parametric oscillator has, in principle, a sensitivity lower than that of the tuned harmonic oscillator by a factor

$$(2\omega_p\omega_1/\omega_c^2)\sqrt{\omega_1/\omega_0}\simeq 5\sqrt{\omega_1/\omega_0}$$

compatible with stability and provided one is able to get a sufficiently strong periodic variation of the elastic coefficient. Preliminary tests on a 400-kg Al cylinder seems to indicate that such a possibility exists. We describe briefly the experimental set up we used and give an example of a possible practical configuration of a system having an equation of motion of the form of Eq. (1). The 400-kg antenna was suspended by a tuning fork. In front of one of its end faces a second Al cylinder of mass 16 kg was placed over an optical bench allowing micrometric displacement of its position parallel to the axes of both cylinders. These two masses were coupled together by means of four piezoelectric ceramics of dimensions $5\times5\times2$ cm^3 encapsulated inside an Al frame terminating on both sides with two steel half spheres which realize the contact with the ends of the two cylinders. This contact was made variable by applying to the ceramics a periodic voltage from a frequency synthesizer after amplification through a power amplifier plus transformer (Fig. 1). In this configuration, one mode of the antenna constitutes the basic harmonic oscillator, whereas the above variable contact results in a modulation of its equivalent elastic coefficient. The degree of variable coupling could be adjusted both by varying the voltage of the pump or the distance at the mechanical contacts through the micrometer screw.

Two small piezoelectric ceramics glued on the antenna served to force it at a given frequency and to detect the resulting motion, respectively.

After application of a forcing voltage outside resonance, the frequency of the pump was so adjusted to select one antenna mode in which the forcing frequency was translated. An example of the results we obtained is presented in Fig. 2, showing the power spectrum of the output of the ceramics used as displacement detector. Here a forced motion at 1031 Hz is brought by the pump at an antenna mode of 6324 Hz. The tests we made were mainly concerned with the degree of variable coupling one could obtain in this configuration, particularly with the force at 60 Hz and the fundamental longitudinal mode of the antenna at 1700 Hz. The measurement of the displacement at these two frequencies and the knowledge of the Q of the mode allowed evaluation of ω_c^2. We obtained a value of about 3×10^5 sec^{-2} to be compared with the maximum possible value of 1.6×10^6 sec^{-2}, as estimated on the basis of the elastic modulus, dimension, and geometry of the device used.

It is not clear how, with this method, the Q of the system is degraded. The above measurements were in fact not very significant in this respect as they were performed in air so that acoustic emission was the dominant loss.

It is very difficult to estimate *a priori* the dissipation this configuration will show when high-Q antennas are used. Critical points will certainly be the suspensions of the two masses and the variable contacts. One may hope that accurate working and choice of materials will keep these kinds of losses at very low levels so that the dominant mechanism would become the dissipation in the ceramics which, however, can be replaced by quartz. This would mean that the overall loss of sensitivity with respect to the tuned oscillator could be between a factor of 10 and a factor of 25. This loss, however, could be largely compensated by a greater mass and length of the antenna and by the possibility of keeping, in practice, a more favorable

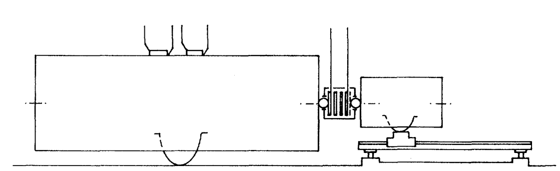

FIG. 1. Experimental setup used to test the feasibility of the proposed detector.

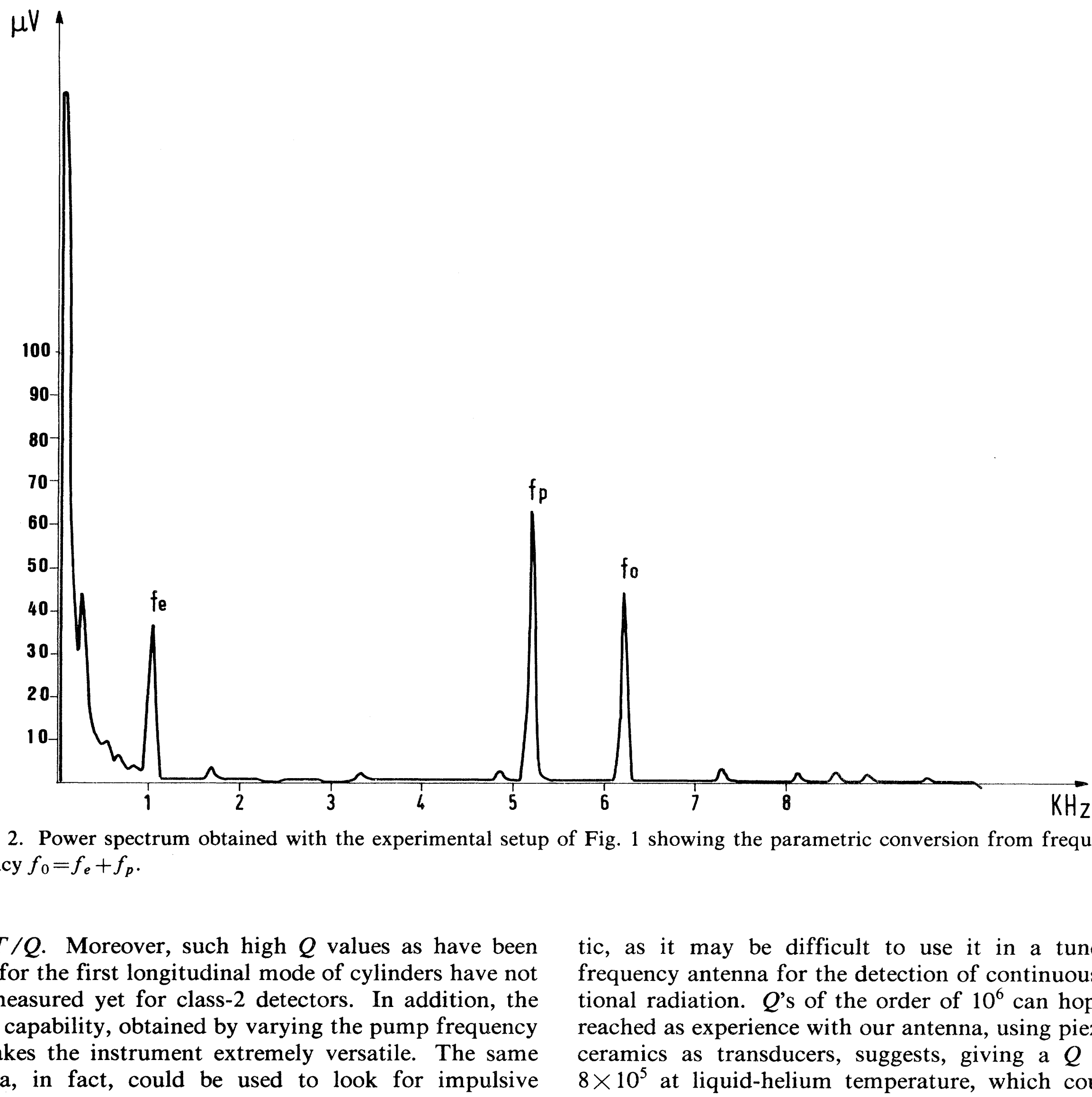

FIG. 2. Power spectrum obtained with the experimental setup of Fig. 1 showing the parametric conversion from frequency f_e to frequency $f_0 = f_e + f_p$.

ratio T/Q. Moreover, such high Q values as have been found for the first longitudinal mode of cylinders have not been measured yet for class-2 detectors. In addition, the tuning capability, obtained by varying the pump frequency ν_p, makes the instrument extremely versatile. The same antenna, in fact, could be used to look for impulsive events, or to study the continuous radiation emitted by the Crab and the Vela pulsars. In Table I are reported the sensitivities which, in principle, could be obtained with the cooled cylindrical antennas available to our group, one, in operation, of 400 kg and the second of 5000 kg almost completed. A hypothetical, but within the reach of the technology of the 1980's antenna is also considered. Figure 3 shows the results already obtained[6] as compared with the above sensitivities and the theoretical predictions.

Obtaining a Q value as high as 10^8 is probably unrealistic, as it may be difficult to use it in a tuned fixed-frequency antenna for the detection of continuous gravitational radiation. Q's of the order of 10^6 can hopefully be reached as experience with our antenna, using piezoelectric ceramics as transducers, suggests, giving a Q of about 8×10^5 at liquid-helium temperature, which could be at least sufficient to look at possible new rapidly decaying pulsars.

The calculations were performed by using approximations. To check the correctness of the results we performed a simulation with an analog computer. As discussed in Appendix B, the agreement is very good.

ACKNOWLEDGMENTS

We thank S. Ugazio for help in setting up the apparatus for laboratory tests. Simulations were performed at the

TABLE I. Sensitivity of the instruments mentioned in the text. Values of ω_c^2 are the greatest allowed by the stability of the oscillator.

m (kg)	l (m)	T (K)	Q	Δt (sec)	ω_1 (sec^{-1})	ω_c^2 (sec^{-2})	h_0 Crab	h_0 Vela
200	0.6	4.2	10^6	1.5×10^7	10^4	2.3×10^7	5.4×10^{-21}	4.1×10^{-20}
2500	1.2	4.2	10^6	1.5×10^7	5×10^3	8.5×10^6	3.5×10^{-22}	2.5×10^{-21}
5000	2.4	10^{-3}	10^8	1.5×10^7	2.5×10^3	3×10^6	9×10^{-26}	7.0×10^{-25}

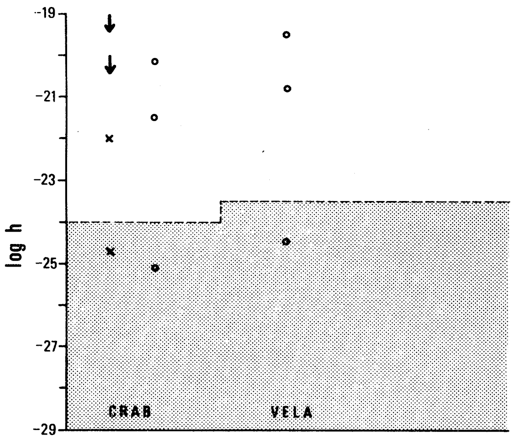

FIG. 3. Predicted sensitivities to gravitational radiation from Crab and Vela pulsars. Cross: previsions from Tokyo Group. Circles: our proposal, see Table I. Arrows show the upper limits to Crab radiation already set by the Tokyo group. Shadowed area gives the limits of theoretical predictions.

Analog Computer Facility of CNEN-CASACCIA. We are greatly indebted to the staff, and in particular to Dr. D. Taglienti, for making available to us computer facilities, and for their kind advice.

APPENDIX A

Letting $z=\omega_p t/2$, Eq. (1) of the text takes the standard form of the Mathieu equation

$$\frac{d^2x}{dz^2}+2\alpha\frac{dx}{dz}+(a-2q\cos 2z)x=\bar{f}\sin(pz+\phi)\ , \quad \text{(A1)}$$

where

$$\frac{\gamma}{\omega_p}=\alpha,\quad \frac{4\omega_1{}^2}{\omega_p{}^2}=a,\quad \frac{4f_G}{m\omega_p{}^2}=\bar{f},\quad \frac{2\omega_0}{\omega_p}=p,\quad \frac{2\omega_c{}^2}{\omega_p{}^2}=q\ . \quad \text{(A2)}$$

By further putting

$$x=e^{-\alpha z}y(z)\ , \quad \text{(A3)}$$

Eq. (A1) becomes, if $a \gg \alpha^2$,

$$y''+(a-2q\cos 2z)y=e^{\alpha z}\bar{f}\sin(pz+\phi) \quad \text{(A4)}$$

of which a particular integral is

$$y=(1/W)\left[y_1(z)\int^z y_2(u)f(u)du - y_2(z)\int^z y_1(u)f(u)du\right]\ , \quad \text{(A5)}$$

where $f(u)$ represents the right-hand side of Eq. (A4), $W=y_1y_2'-y_2y_1'$ is the Wronskian (constant for the Mathieu equation), and $y_1(z),y_2(z)$ are two independent solutions of the corresponding homogeneous equation. In our case, where a varies reasonably between 4 and 5, they may be written in the form

$$y_1=\sum_{-\infty}^{+\infty} c_j\cos(\sqrt{a}-2j)z\ , \quad \text{(A6)}$$

$$y_2=\sum_{-\infty}^{+\infty} c_j\sin(\sqrt{a}-2j)z\ . \quad \text{(A7)}$$

After a lengthy calculation we finally obtain

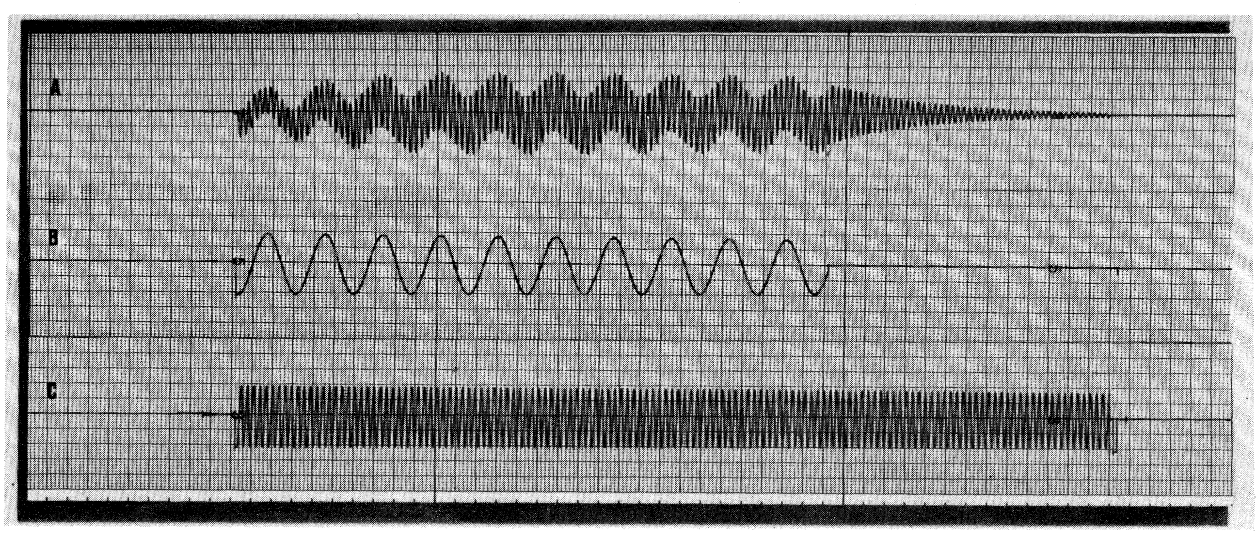

FIG. 4. Results from analog computer showing visually the qualitative behavior of the detector. A: Output from oscillator. B: External force. C: Pump. At turnoff of the force only the damped resonant oscillation is present.

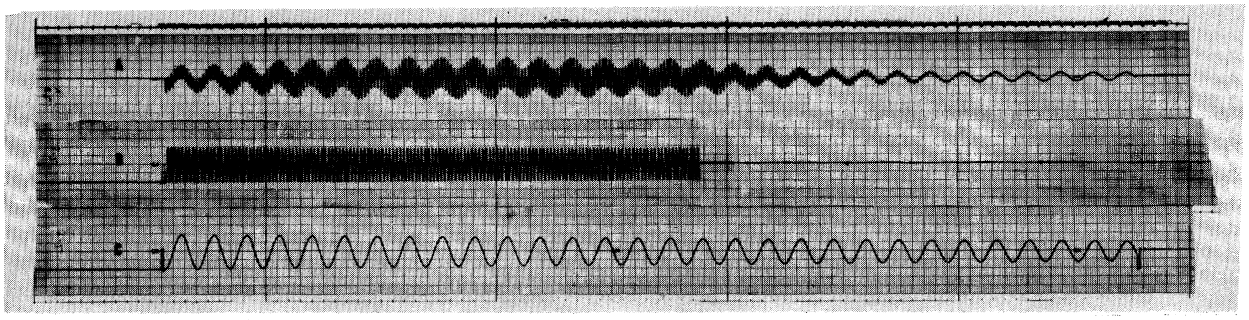

FIG. 5. Other results of simulation. A: Output from oscillator. B: Pump. C: External force. At turnoff of the pump only the forced motion remains whereas the resonant oscillation dies out with its characteristic time.

$$y=\frac{e^{\alpha z}\bar{f}}{2W}\left[\sum_{-\infty}^{+\infty} c_j \cos(\sqrt{a}-2j)z \sum_{-\infty}^{+\infty} c_j \left[\frac{\alpha\cos(A_j z+\phi)+A_j\sin(A_j z+\phi)}{\alpha^2+A_j^2}-\frac{\alpha\cos(B_j z+\phi)+B_j\sin(B_j z+\phi)}{\alpha^2+B_j^2}\right]\right.$$
$$\left.-\sum_{-\infty}^{+\infty} c_j \sin(\sqrt{a}-2j)z \sum_{-\infty}^{+\infty} c_j \left[\frac{\alpha\sin(A_j z+\phi)-A_j\cos(A_j z+\phi)}{\alpha^2+A_j^2}+\frac{\alpha\sin(B_j z+\phi)-B_j\cos(B_j z+\phi)}{\alpha^2+B_j^2}\right]\right], \quad \text{(A8)}$$

where

$$A_j=p-\sqrt{a}+2j,\quad B_j=p+\sqrt{a}+2j\ . \quad \text{(A9)}$$

This expression simplifies drastically if one considers that, in the cases of interest, we have high Q's, i.e., very small α's, and that the variable coupling is moderate, which implies a rapid decrease of the coefficients c_j. This means that only terms for which

$$A_j=0 \text{ or } B_j=0 \quad \text{(A10)}$$

will be important. Our choice

$$\omega_0+\omega_p=\omega_1 \quad \text{(A11)}$$

gives automatically $A_1=0$. Keeping only terms in A_1, one then gets in the first approximation,

$$y=\frac{e^{\alpha z}\bar{f}}{2W}\frac{c_1}{\alpha}\sum_{-\infty}^{+\infty} c_j \cos[(\sqrt{a}-2j)z+\phi]\ , \quad \text{(A12)}$$

and hence

$$x=(\bar{f}/2\alpha W)c_1 \sum c_i \cos[(\sqrt{a}-2i)z+\phi]\ . \quad \text{(A13)}$$

The component at the resonance x_R, obtained by putting $i=0$, is

$$x_R=(\bar{f}/2\alpha W)c_0 c_1 \cos(\sqrt{a}z+\phi)\ . \quad \text{(A14)}$$

To evaluate the noise contribution, as mentioned in the text, it is interesting to have a solution at the resonance for the case of a force f_1 having an angular frequency ω_1. It is easily seen that condition (A10) is now fulfilled by $j=0$, giving the solution

$$x=(\bar{f}_1/2\alpha W)c_0^2 \cos\sqrt{a}z\ . \quad \text{(A15)}$$

The Wronskian is immediately calculated

$$W=\sum_{-\infty}^{+\infty} c_i \sum_{-\infty}^{+\infty}(\sqrt{a}-2j)c_j \quad \text{(A16)}$$

and a rough approximation gives us for the coefficients

$$c_s=J_s(q/\sqrt{a})\ . \quad \text{(A17)}$$

Hence

$$W=\sqrt{a}-2q/\sqrt{a}\simeq\sqrt{a} \quad \text{(A18)}$$

and the solutions (A14) and (A15) finally become

$$x_R=\frac{\bar{f}}{2\alpha\sqrt{a}}J_0\left[\frac{q}{\sqrt{a}}\right]\cos(\sqrt{a}z+\phi)$$
$$\simeq\frac{\bar{f}}{4\alpha a}q\cos(\sqrt{a}z+\phi)\ , \quad \text{(A19)}$$

$$x=\frac{\bar{f}_1}{2\alpha W}J_0^2\left[\frac{q}{\sqrt{a}}\right]\cos\sqrt{a}z\simeq\frac{\bar{f}_1}{2\alpha\sqrt{a}}\cos\sqrt{a}z\ , \quad \text{(A20)}$$

respectively.

APPENDIX B

As the values (A17) for the coefficients come from an approximation which is not fully valid in our case, we checked the solution by simulation with an analog computer. Some results, showing visually the qualitative behavior of the oscillator are reported in Figs. 4 and 5.

A quantitative evaluation was performed by comparing the displacement at the resonance, x_R, with that, x_g, at the frequency of the driving force, whose ratio, on the basis of the previous calculations, should be

$$\frac{x_R}{x_g}=\frac{q}{4\alpha} \quad \text{(B1)}$$

for all cases we considered, where, for practical reasons, the α's used were not very small. The results indicate that the solution we adopted is fully justified within the range of the coupling coefficients we are interested in.

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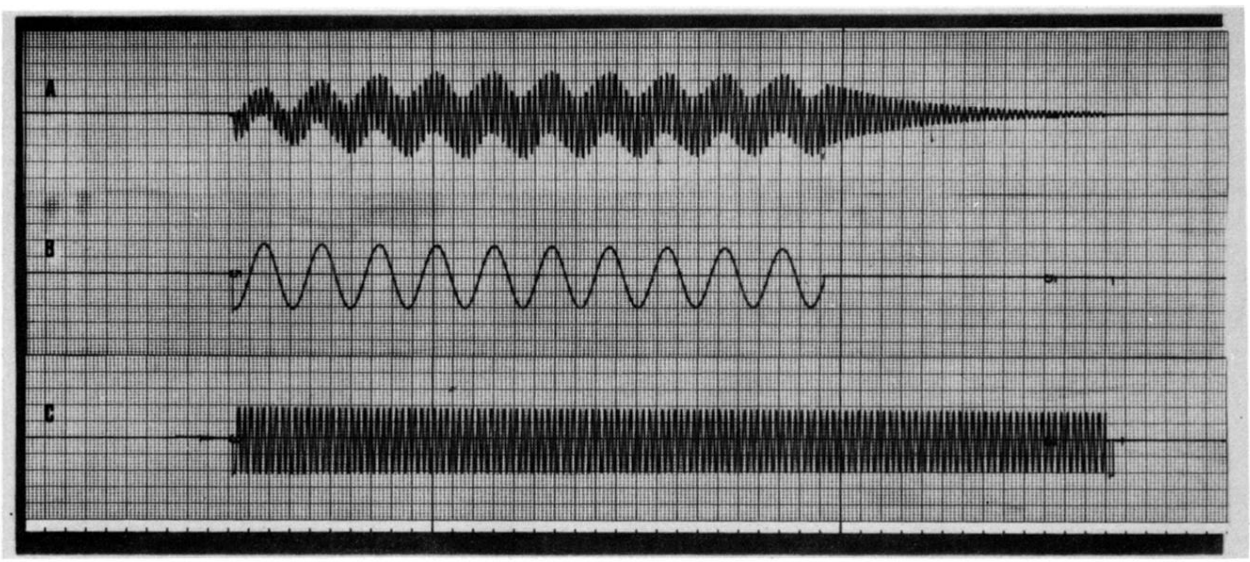

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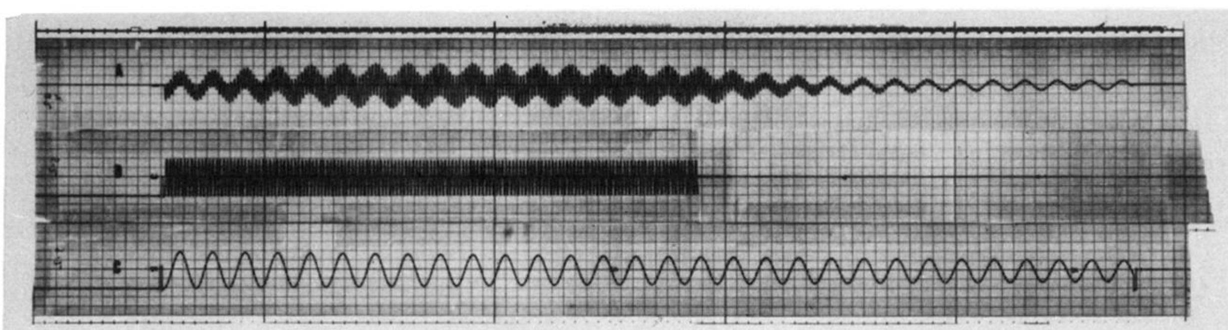

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