PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOLUME 28, NUMBER 6 15 SEPTEMBER 1983

Modeling in chaotic relativity

Andrew Zardecki
Theoretical Division, MS-B279, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 3 May 1983)

The chaotic behavior of solutions to Einstein's equations has recently been studied by Barrow within the framework of the dynamical systems theory. Barrow's program of gravitational turbulence is implemented in part by considering the solutions of type VII_0 and IX as well as some intermediate types. Quantitative measures of chaos, such as the power spectrum and Lyapunov characteristic exponent, are computed. By converting the equations of motion for the cosmic scale factors to stochastic Langevin's equations, the Mixmaster cosmology in the presence of driving noise terms is investigated. Possible sources of noise can be attributed to an imperfect cancellation of the effective vacuum energy density and the energy density associated with the Higgs field. An ensemble average over random trajectories leads to the suppression of chaotic behavior for type-IX cosmology.

I. INTRODUCTION

As is now well known, any isotropic homogeneous universe governed by Einstein's equations must have started with a singularity of infinite density.[1] This type of singularity, referred to as a Friedmann-Robertson-Walker (FRW) singularity, is characterized by the fact that the vanishing of spatial distances occurs according to the same law in all directions. The singularity of the oscillatory type, as exemplified by the Mixmaster[2] cosmology, is a general feature of solutions to the Einstein equations. This feature is independent of the assumption about the space homogeneity.[3] There exist numerous reasons for considering non-FRW models. The arguments of primordial chaos, as advanced by Misner[4] and Rees,[5] have been framed quantitatively in a general classification scheme of anisotropic models.[6] The generic singularity of the Belinskii, Khalatnikov, and Lifshitz (BKL) type[7] has recently been discussed by Barrow,[8,9] and Chernoff and Barrow,[8] in the context of dynamical systems theory. In Ref. 9, Barrow raised an intriguing question about the possibility of gravitational turbulence. As the curvature parameter is varied along the Bianchi sequence of cosmological models, the system would—through period doubling or a succession of three Hopf's bifurcations (Ruelle-Takens mechanism)—tend to an increasingly chaotic behavior.

The Mixmaster evolution is highly chaotic in the sense that the approach toward the singular point is made up of successive series of oscillations the lengths of which have the character of a random process. Since the model also exhibits a sensitive dependence on initial conditions, it implies stochasticity. For dissipative systems in three or more dimensions, there exist the structures which are characterized as having a fractional dimension—strange attractors.[10] The statistical self-similarity or the Cantor set behavior, which is a characteristic criterion of the strange attractor, is not contained, however, in the Mixmaster model.

Within the context of the chaotic dynamics, the interrelation between the inherent chaotic behavior and the noise-impressed fluctuations has repeatedly been investigated.[11] The purpose of this paper is to study the effect of white noise on Mixmaster oscillatory evolution. It has been stressed by BKL that in the perfect-fluid limit one may neglect the influence of the stress-energy terms in the Einstein equations. However, the oscillatory Mixmaster behavior need not persist in a stiff-matter era where the equation of state for density ρ and pressure p takes the form $p=\rho$. The initial state of the Universe was then isotropic and quiescent rather than chaotic.[12] When matter is described by an effective energy-momentum tensor containing the vacuum energy due to the cosmological term, the stress-energy terms become unimportant in the hard-thermal-state limit $p=\rho/3$. What remains is the effective vacuum energy density ρ_Λ and the energy density ρ_0 associated with spontaneous-symmetry-breaking terms.[13] The imperfect compensation between ρ_Λ and ρ_0 is a possible source of noise in the Einstein equations. Another source of noise can be due to the fluctuations in the Euclidean four-volume as considered by Hawking.[14]

In the remainder of this paper, we summarize, in Sec. II, the basic equations of the Mixmaster model. In Sec. III, the chaotic behavior of the Mixmaster dynamics is analyzed quantitatively by computing the Lyapunov exponent and the power spectrum of the system. The sources of noise are described in more detail in Sec. IV. Finally, by considering an ensemble of Mixmaster trajectories, starting all from the same initial point in the phase space, we show in Sec. V that the fluctuating effective cosmological term leads to suppression of the chaotic behavior.

II. EINSTEIN'S EQUATIONS IN ANISOTROPIC HOMOGENEOUS SPACE.

Following the Landau and Lifshitz[15] timelike convention, we write Einstein's field equations with a cosmological term in the form

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}-\Lambda g_{\mu\nu}=8\pi G T_{\mu\nu}\ , \tag{1}$$

where the metric has signature $(+---)$ and where we have set the velocity of light $c=1$. For a perfect relativistic fluid the energy-momentum tensor has the form

$$T_{\mu\nu}=-pg_{\mu\nu}+(p+\rho)U_\mu U_\nu\ , \tag{2}$$

where p is the pressure, ρ the energy density, and U the velocity of fluid in a comoving frame. By defining an effective (generalized) energy-momentum tensor

$$T^*_{\mu\nu}=(\Lambda/8\pi G)g_{\mu\nu}+T_{\mu\nu}\ , \tag{3}$$

we can write Eq. (1) as

$$R_{\mu\nu}=8\pi G S^*_{\mu\nu}\ , \tag{4}$$

where the source term $S^*_{\mu\nu}$ is

$$S^*_{\mu\nu}=T^*_{\mu\nu}-\tfrac{1}{2}T^*g_{\mu\nu}\ . \tag{5}$$

Here

$$T^*_{\mu\nu}=-p^*g_{\mu\nu}+(p^*+\rho^*)U_\mu U_\nu\ , \tag{6a}$$

$$p^*=p-\Lambda/8\pi G\ , \tag{6b}$$

$$\rho^*=\rho+\Lambda/8\pi G\ , \tag{6c}$$

$$T^*=g^{\mu\nu}T^*_{\mu\nu}\ . \tag{6d}$$

In a synchronous (Gaussian normal) system of coordinates the interval element is given as

$$ds^2=dt^2-dl^2\ , \tag{7}$$

where the spatial line element involves components γ_{ij} $(i,j=1,2,3)$ of the spatial metric tensor

$$dl^2=\gamma_{ij}dx^i dx^j\ . \tag{8}$$

For a homogeneous space of Bianchi type VIII or IX, γ_{ij} is

$$\gamma_{ij}(t)=\eta_{pq}(t)e_i^{(p)}e_j^{(q)}\ , \tag{9}$$

where

$$\eta_{pq}(t)=\mathrm{diag}(a^2(t),b^2(t),c^2(t))$$

and $e_i^{(p)}$ $(p=1,2,3)$ are the three frame vectors determined by the structure of constants of the group of motions.[15]

The matter filling the space cannot, in general, be at rest relative to the synchronous reference frame. In fact, the velocity four-vector U_μ satisfies the hydrodynamic equations[7,15]

$$abcU_0\rho^{3/4}=\text{const}\ , \tag{10a}$$

$$U_k\rho^{1/4}=\text{const}\quad (k=1,2,3)\ . \tag{10b}$$

However, if the synchronous frame is also comoving with respect to matter, we have $U_0=1$ and $U_k=0$ [Ref. 15(a)]. In this case the spatial components of Einstein's equations (4) yield

$$(\ln a^2)''=(\mu b^2-\nu c^2)^2-\lambda^2a^4+8\pi G(\rho^*-p^*)a^2b^2c^2\ , \tag{11a}$$

$$(\ln b^2)''=(\nu c^2-\lambda a^2)^2-\mu^2b^4+8\pi G(\rho^*-p^*)a^2b^2c^2\ , \tag{11b}$$

$$(\ln c^2)''=(\lambda a^2-\mu b^2)^2-\nu^2c^4+8\pi G(\rho^*-p^*)a^2b^2c^2\ , \tag{11c}$$

where the prime denotes $\partial_\tau\equiv abc\partial_t$, and λ, μ, and ν are the only nonvanishing structure constants. The time component R_{00} in Eq. (4) becomes

$$[\ln(abc)]''-2[(\ln a)'(\ln b)'+(\ln a)'(\ln c)'+(\ln b)'(\ln c)'] =-4\pi G(\rho^*+3p^*)a^2b^2c^2\ . \tag{12}$$

In the limit of the Kasner regime, $\tau\sim\ln t$, which causes us to name τ a logarithmic time scale. Equations (11) can obviously be written in a dimensionless form by choosing one of the structure constants, e.g., λ as an inverse length unit. In the numerical calculations that follow we set $\lambda^{-1}=1$ cm.

For a space of Bianchi type IX the structure constants λ, μ, and ν have the same sign and one usually sets $\lambda=\mu=\nu=1$. For a space of Bianchi type VIII, two constants have signs opposite to that of the third and we can put $\lambda=\mu=1$, $\nu=-1$. Bianchi type VII_0, characterized by $\lambda=\mu=1$, $\nu=0$, provides an example of nonchaotic behavior. In the particular case where the elements of the diagonal tensor η_{pq} are equal, $a=b=c\equiv R$, the equations of the FRW cosmology are recovered. As can readily be shown, by reverting to the time variable t, Eqs. (11) and (12) become identical with the Einstein equations of Ref. 1, provided one identifies $\lambda=\mu=\nu$ with the conventional curvature parameter $4k$.

Equations (11) and (12) can be combined to yield the first integral of the system of Eqs. (11),

$$(\ln a)'(\ln b)'+(\ln a)'(\ln c)'+(\ln b)'(\ln c)'=\tfrac{1}{4}(\lambda^2a^4+\mu^2b^4+\nu^2c^4-2\lambda\mu a^2b^2-2\lambda\nu a^2c^2-2\mu\nu b^2c^2)+8\pi G\rho^*a^2b^2c^2\ . \tag{13}$$

Equation (13) plays the role of a constraint imposed on the initial conditions of Eqs. (11). Although a detailed qualitative study of solutions to Eqs. (11) in the neighborhood of the singular point is contained in Refs. 7 and 15, a numerical analysis encompassing a wide range of initial conditions is desirable.[16] This will be the subject of the next section.

III. CHAOTIC BEHAVIOR

The first-integral condition expressed by Eq. (13) cannot be satisfied exactly. Experimentally or numerically the initial conditions are known only within the accuracy of a measuring device or the computer round-off error. For this reason, when solving Eqs. (11), we consider three types of initial conditions.

(a) Free—the relationship between the scale factors and their derivatives remains arbitrary.

(b) Kasner—the temporal evolution commences from the Kasner regime at $\tau=\tau_0$, i.e.,

$$a=\exp(\tau_0 p_1)\ ,\quad b=\exp(\tau_0 p_2)\ ,\quad c=\exp(\tau_0 p_3)\ ,\quad a'=p_1 a\ ,\quad b'=p_2 b\ ,\quad c'=p_3 c\ , \tag{14}$$

where the numbers p_1, p_2, and p_3 are given in parametric form through the formulas

$$p_1(u)=\frac{-u}{1+u+u^2}\ ,\quad p_2(u)=\frac{1+u}{1+u+u^2}\ ,\quad p_3(u)=\frac{u(1+u)}{1+u+u^2}\ . \tag{15}$$

(c) First integral—the first-integral error ϵ defined (in empty space) as

$$\epsilon=(\ln a)'(\ln b)'+(\ln a)'(\ln c)'+(\ln b)'(\ln c)'-\tfrac{1}{4}(\lambda^2a^4+\mu^2b^4+\nu^2c^4-2\lambda\mu a^2b^2-2\lambda\nu a^2c^2-2\mu\nu b^2c^2)\ , \tag{16}$$

is minimized.

Table I gives the values of scale factors and their derivatives together with the values of ϵ. Similarly to the work of BKL, we consider the evolution of the metric at $t\to 0$ ($\tau\to-\infty$), so that the initial conditions correspond to a later and not an earlier time.

We come now to the description of our numerical results. The computations were performed on a CDC 7600 computer, with double-precision arithmetic to minimize the round-off error. The integration algorithm was the multistep Hamming's predictor-corrector method,[17] having a small per-step truncation error. Starting values were determined by using a fourth-order Runge-Kutta method. Typically, the time step was $\Delta\tau=0.001-0.0001$ and the computations involved 10^5 steps.

The general behavior of the solution of Eqs. (11) is shown in Fig. 1 for initial conditions A. Figure 1(a) furnishes an illustration of chaotic trajectory of Bianchi type IX, while Fig. 1(b), shown for the sake of comparison, refers to Bianchi type VII_0. A more conventional representation is given in Fig. 2 which depicts the oscillatory evolution of the Mixmaster scale factors for three different initial conditions. In Fig. 2(b), the evolution starts with the initial nonzero energy density $\rho(0)=8.4163\times10^{12}$ g/cm^3. As the matter is dominated by relativistic particles, we see from Eq. (10a) that $(abc)^{4/3}=\text{const}$. The

TABLE I. Initial conditions for the equations of motion.

Type	τ_0	$a(\tau_0)$ / $a'(\tau_0)$	$b(\tau_0)$ / $b'(\tau_0)$	$c(\tau_0)$ / $c'(\tau_0)$	ϵ
(a) Free	0.0	1.2570	0.7882	0.2523	1.068×10^2
		-0.0480	0.0313	0.2519	
(b) Kasner	-2.6670	1.8541	0.4385	0.0854	3.378×10^1
		-0.4292	0.1355	0.0788	
(c) First integral imposed	0.0	1.8540	0.4385	0.0854	5.464×10^{-9}
		-0.4292	0.1355	2.9655	

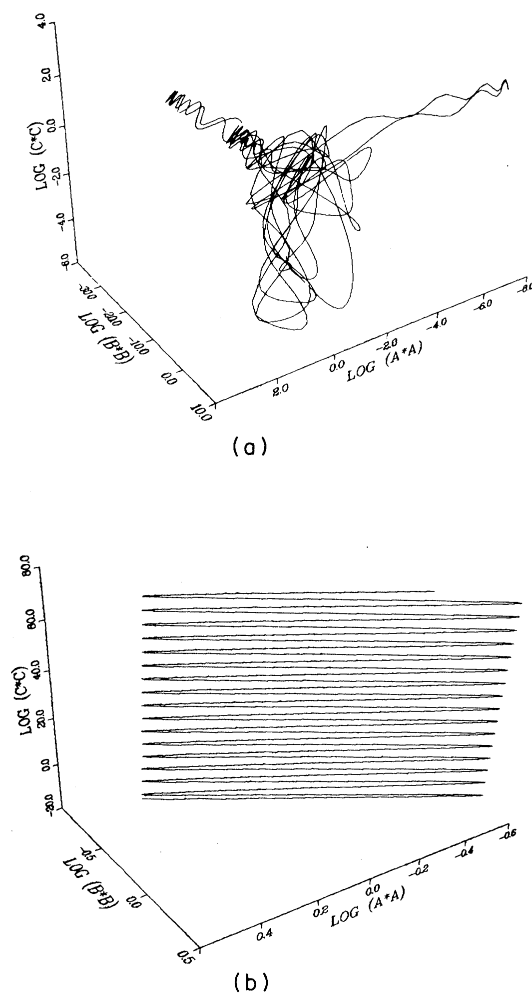

FIG. 1. Three-dimensional trajectory in the space $(a(\tau),b(\tau),c(\tau))$ corresponding to the free initial conditions. (a) Bianchi type IX, (b) Bianchi type VII_0.

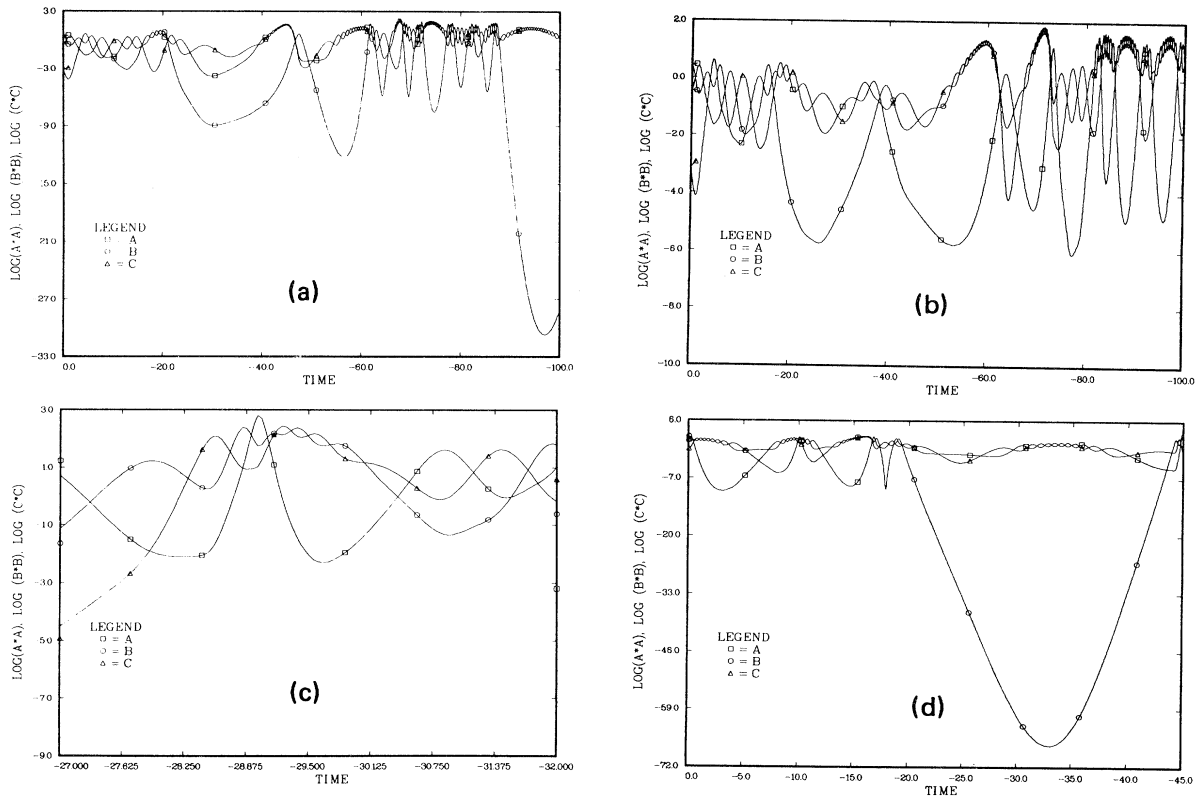

FIG. 2. Oscillatory evolution of the Mixmaster scale factors as a function of the logarithmic time τ. (a) Initial conditions A, empty space, (b) initial conditions A, matter density $\rho(0)=8.4163\times10^{12}$ g cm^{-3}, (c) initial conditions B, Kasner, (d) initial condition C, first integral imposed.

chosen value of $\rho(0)$ corresponds to the temperature of $10^{12}\,°K$, which is slightly above the temperature $1.3\times10^{11}\,°K$ marking the neutrino decoupling. The continuation of the temporal evolution towards large negative logarithmic times is based on a bold assumption that the quantum gravity era also possesses a Mixmaster description.[18] Note that the Planck length $t_p=1.616\times10^{-33}$ cm is translated as $\tau_p=-75.5$.

When the motion of a dynamical system is periodic, the power spectrum of an appropriate dynamical variable contains a discrete number of components localized at characteristic frequencies. In contrast, for a chaotic motion we would expect a broad-band spectrum. This simple criterion is illustrated in Fig. 3, where the power spectrum of the scale factor $a(\tau)$ is shown. The notion of chaos in a dynamical system can be formalized in terms of the Lyapunov characteristic exponent[19] whose positive algebraic sign indicates sensitive dependence of initial conditions.

If we write Eqs. (11) in autonomous form, then, symbolically, we have

$$\vec{x}\,'=\vec{F}(\vec{x},\lambda,\mu,\nu)\,, \tag{17}$$

where $\vec{x}=(a,b,c,a',b',c')$ and $\vec{F}$ is determined explicitly by Eqs. (11). Given a solution $\vec{x}(\tau)$ with $\vec{x}(0)=\vec{x}_0$, one defines the Lyapunov characteristic exponent by

$$\chi=\lim_{|\tau|\to\infty}\chi_\tau\,, \tag{18}$$

where χ_τ is an approximate Lyapunov exponent given as

$$\chi_\tau=\frac{1}{|\tau|}\ln\|\vec{\xi}(\tau)\|\,. \tag{19}$$

In Eq. (19), $\vec{\xi}(\tau)$ is the solution of the corresponding variational equation with initial condition $\vec{\xi}(0)=\vec{\xi}_0$ and $\|\cdots\|$ is the Euclidean norm.

As our phase space is six dimensional, χ takes actually six different values and the outlined procedure yields the largest characteristic exponent, independently of the initial value of $\vec{\xi}_0$. Figure 4 shows the approximate Lyapunov exponent as a function of time τ with the parameter ν taking the values of 1.0, 0.5, and 0.0. In the last case the Lyapunov exponent remains negative. This is in accord with the fact that Bianchi type-VII$_0$ solution exhibits order (nonchaos). A similar behavior of χ_τ is found for Bianchi type-VIII models, where $\nu<0$. As $\tau\to-\infty$, $\chi_\tau\to7.813\times10^{-2}$ and 1.037×10^{-1}, for $\nu=-0.5$ and $\nu=-1.0$, respectively.

IV. SOURCES OF NOISE

The arguments that we invoke to introduce driving noise terms into equations of motion are related to the

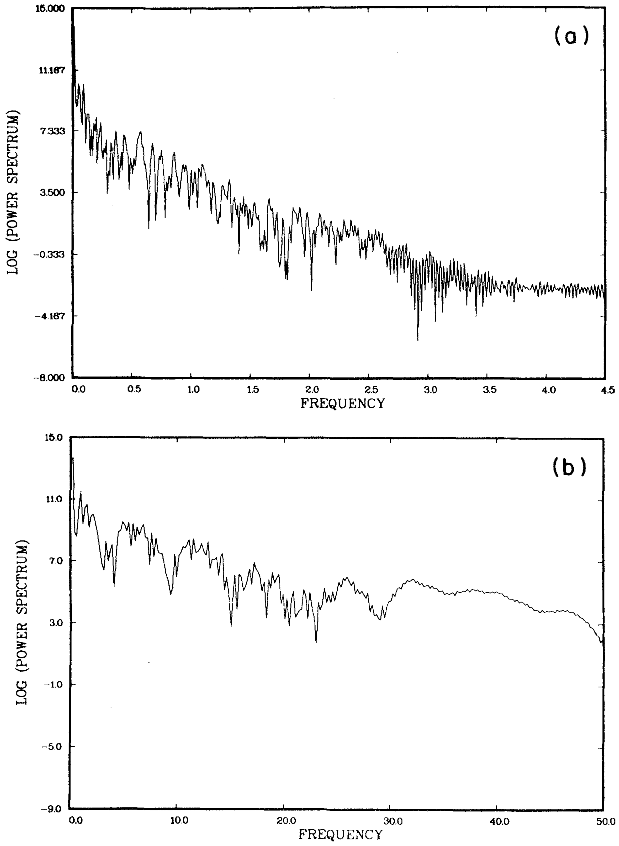

FIG. 3. Power spectrum, defined as the square of the absolute value of the temporal Fourier transform of $a(\tau)$, for (a) free initial conditions and (b) initial conditions satisfying the first integral.

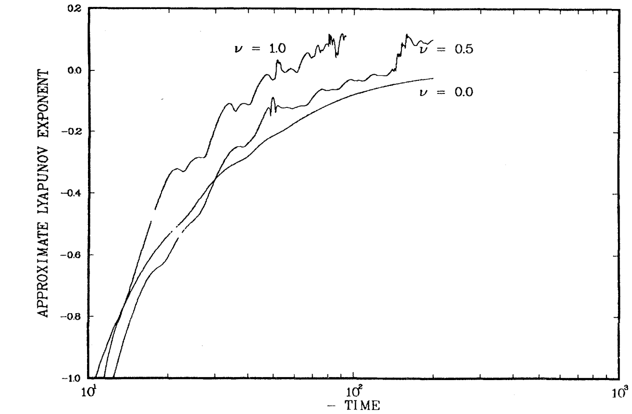

FIG. 4. Approximate Lyapunov exponent as a function of logarithmic time. The parameters $\lambda=\mu=1$, while ν takes on the values 1.0, 0.5, and 0.0.

smallness of the apparent cosmological constant. In terms of the Planck mass $m_P=G^{-1/2}$, one can place an upper limit[14] of about 10^{-60} on $|\Lambda/m_P|^{1/2}$.

When matter is described by a field theory with a scalar Higgs field ϕ responsible for the spontaneous symmetry breaking, the energy-momentum tensor will contain an additional term due to energy density of the vacuum. If $V(\phi)$ denotes the effective potential, the classical expectation value of the energy-momentum tensor is

$$\langle 0|T_{\mu\nu}|0\rangle=V(\phi_0)g_{\mu\nu}\equiv\rho_0 g_{\mu\nu} . \tag{20}$$

In Eq. (20), $|0\rangle$ denotes the true vacuum state and ϕ_0, the classical field, is the value of ϕ for which $V(\phi)$ attains an extremum.

The present small value of the apparent cosmological constant can be thought of as a result of cancellation of the vacuum induced energy ρ_0 and the effective vacuum energy $\rho_\Lambda=\Lambda/8\pi G$. In other words, to avoid contradiction with observation, one demands

$$\rho_{\text{tot}}=\rho_0+\rho_\Lambda\cong 10^{-29}\,\text{g cm}^{-3} . \tag{21}$$

Classically, the Higgs field will fluctuate about the value $\phi=\phi_0$ leading, by virtue of Eq. (20), to a fluctuating value of ρ_0. Assuming that in the absence of fluctuations ρ_{tot} exactly equals zero, we see that ρ_{tot} will become a fluctuating function of time $\delta\rho_{\text{tot}}(t)$. Equivalently, one can think in terms of the fluctuations of ρ_Λ due to the fluctuating cosmological constant $\delta\Lambda(t)$. Neglecting the contributions arising from the p and ρ terms in Eq. (6), we thus write the spatial components of Eq. (4) in the form

$$R_{kl}=-\delta\Lambda(t)g_{kl} . \tag{22}$$

Equation (22) is a Langevin stochastic equation, which can be converted into equations for three expansion factors. They read

$$(\ln a^2)''=(\mu b^2-\nu c^2)^2-\lambda^2a^4+a^2b^2c^2\delta\Lambda_a(\tau) , \tag{23a}$$

$$(\ln b^2)''=(\nu c^2-\lambda a^2)^2-\mu^2b^4+a^2b^2c^2\delta\Lambda_b(\tau) , \tag{23b}$$

$$(\ln c^2)''=(\lambda a^2-\mu b^2)^2-\nu^2c^4+a^2b^2c^2\delta\Lambda_c(\tau) . \tag{23c}$$

That the average value of ρ_{tot} now is small or zero does not imply that it was vanishing once the Universe was hotter than $T_c\sim 10^{14}$ GeV, when the cosmological phase transition took place. In fact, according to the inflationary scenario,[20,21] the era of $\rho_{\text{tot}}=0$ was preceded by two earlier eras. First, when $T\gg T_c$, the hot big-bang scenario was characterized by $\rho\gg\rho_{\text{tot}}=\rho_\Lambda$. At this stage $p=\rho/3$. Second, as the process of supercooling was initiated, $\rho_{\text{tot}}=\rho_\Lambda\gg\rho$ and the equation of state became $P=-\rho_\Lambda$. (Actually, these conditions are usually written by introducing the false vacuum energy in place of ρ_Λ.) Therefore, Eqs. (23) describing the fluctuations of the scale factors in the absence of matter can only apply once the de Sitter phase of uniform expansion is terminated.[22] We stress that the cosmological constant terms have, in general, a negligible effect on Mixmaster as $t\to 0$. If Mixmaster dynamics exists at temperatures when the grand-unified-gauge-theories phase transition occurs, the inflationary phase can be prevented.[23]

The balance of the symmetry breaking and cosmological contributions implied by Eq. (21) is unsatisfactory, chiefly because of the required accuracy of cancellations having such different origins. To resolve this puzzle, Hawking[14] assumed that the quantum state of the Universe is not chosen at random, but contains only states with a large Euclidean four-volume, V_0. On length scales, L, such that $\Lambda^{-1/2} \ll L \ll V_0{}^{1/4}$, there are the solutions of Einstein's equations that appear nearly flat. In the limit that $V_0 \to \infty$, the observed cosmological constant would be zero. One can thus consider large but finite fluctuating values of V_0 as a possible source of fluctuations of the cosmological constant.

V. NOISE-INDUCED SUPPRESSION OF CHAOS

In this section, we assume explicitly that the $\delta\Lambda$ terms in Eqs. (23) are δ-correlated Gaussian random processes with zero mean:

$$\langle \delta\Lambda_a \rangle = \langle \delta\Lambda_b \rangle = \langle \delta\Lambda_c \rangle = 0 , \tag{24}$$

$$\langle \delta\Lambda_a(\tau_1)\delta\Lambda_a(\tau_2) \rangle = Q_a \delta(\tau_1 - \tau_2) , \tag{25a}$$

$$\langle \delta\Lambda_b(\tau_1)\delta\Lambda_b(\tau_2) \rangle = Q_b \delta(\tau_1 - \tau_2) , \tag{25b}$$

$$\langle \delta\Lambda_c(\tau_1)\delta\Lambda_c(\tau_2) \rangle = Q_c \delta(\tau_1 - \tau_2) . \tag{25c}$$

The constants Q_a, Q_b, and Q_c in Eqs. (25) describe the strength of the random process. The assumption that the $\delta\Lambda$'s form a white-noise process, as expressed by Eqs. (25), implies a correct interpretation of Eqs. (23) in terms of the Ito differential equations.[24] To integrate Eqs. (23) numerically, one usually seeks a polygonal approximation to the random trajectories $a(\tau)$, $b(\tau)$, and $c(\tau)$ and a smooth representation of the noise terms. More specifically, if we go back to the generic equation (17), we write its stochastic equivalent as

$$\vec{x}\,' = \vec{F}(\vec{x},\lambda,\mu,\nu) + F_1(\vec{x})\vec{\xi}(\tau) , \tag{26}$$

where $\vec{\xi}(\tau)$ is a white-noise-type random vector. The approximate random trajectories at the points $\vec{x}_j = \vec{x}(\tau_j)$, $j = 1, \ldots, n$, can then be obtained according to the formula[25]

$$\vec{x}(\tau_{j+1}) = \vec{x}(\tau_j) + \Delta\tau \vec{F}(\vec{x}_j,\lambda,\mu,\nu) + F_1(\vec{x}_j)\sqrt{\Delta\tau}\,\vec{\xi}_j , \tag{27}$$

where $\Delta\tau$ is the time step and $\vec{\xi}_j$ are independent normal random vectors with zero mean and the variance equal to $\vec{Q} = (Q_a, Q_b, Q_c)$. In determining the strength of the fluctuations what counts then are the products $q_a = (Q_a \Delta\tau)^{1/2}$, $q_b = (Q_b \Delta\tau)^{1/2}$, and $q_c = (Q_c \Delta\tau)^{1/2}$. For the initial conditions (c) that satisfy the first integral we show in Fig. 5 the result of numerical integration of Eqs. (25) with $q_a = q_b = q_c = 10^{-5}$ ($Q_a = Q_b = Q_c = 10^{-6}$). Since the random nature of the process $(a(\tau), b(\tau), c(\tau))$ makes it more meaningful to ask questions about ensemble averages rather than sample values, we show in Fig. 6 the result of an ensemble averaging over 10 and 200 trajectories. The averaging is performed by starting with the same initial conditions for each sample trajectory. As can be seen by inspection of Fig. 6, the chaotic oscillations present in Figs. 2 and 5 tend to disappear as the logarithmic time τ becomes more and more negative. Similar behavior is shown in Fig. 7 for the initial conditions (a) and (b). We interpret the quenching of chaotic behavior as being due to the smearing of the scale-factor oscillations during each Mixmaster era. The driving-noise terms shift the length of the oscillations however slightly, averaging out the oscillations when the ensemble of random trajectories is constructed.

VI. CONCLUSIONS

The numerical analysis of the chaotic behavior of the Mixmaster universe is formulated within a sufficiently

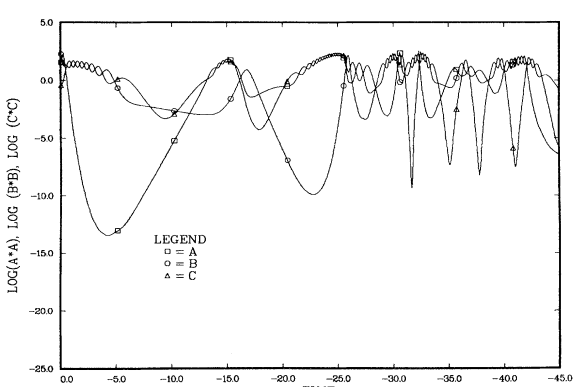

FIG. 5. Sample trajectory for fluctuation parameters $q_a = q_b = q_c = 10^{-5}$. Initial conditions C.

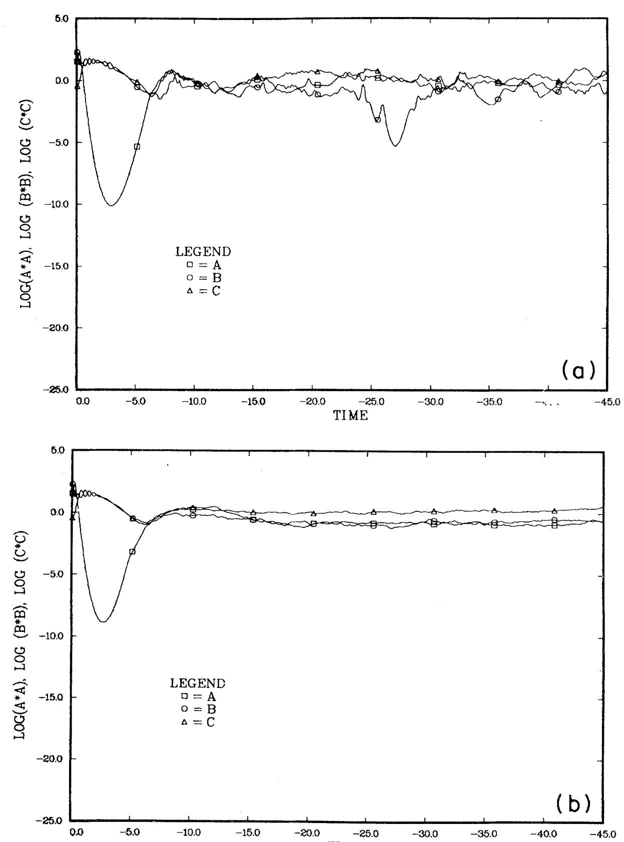

FIG. 6. Ensemble average over (a) 10 and (b) 200 random trajectories. First integral satisfied. $q_a = a_b = q_c = 10^{-5}$.

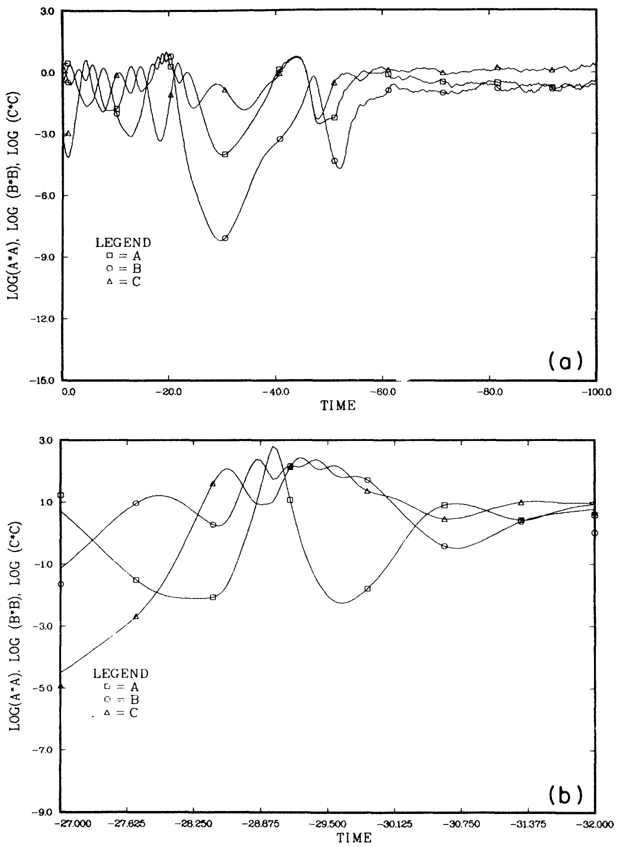

FIG. 7. Ensemble average over 200 random trajectories for (a) free and (b) Kasner initial conditions.

general framework to allow extensions for models with Bianchi types different from IX or VIII. We hope that this investigation constitutes the first step towards the program of gravitational turbulence as formulated by Barrow.[9] This consists in examining the behavior of Eqs. (11) as the parameters $\{\lambda,\mu,\nu\}$ vary in the range $(-\infty,\infty)$. A physical interpretation of such a procedure is in imagining that the cases $\{\lambda,\mu,\nu\}=(1,1,\pm 1)$ represent a spatial variation in the best-fit homogeneous model as one moves about the Universe. As the parameter ν decreases from the value $\nu=1$ to $\nu=0$, we have found the solutions to become less and less chaotic. The full implementation of the Barrow program remains yet to be accomplished. The main difficulty of such an undertaking lies in the runaway solutions, closely related to the open potential walls in the Hamiltonian formulation of the Einstein equations. What we have found in our numerical simulation is that the solutions depend strongly both on λ, μ, ν, and on the initial conditions. A sweep through the nine-dimensional parameter space to find out stable solutions of Eqs. (11) is a formidable task. By the same token, we are prevented from investigating the effect of the "noise" created by a whole ensemble of different Mixmasters on a single Mixmaster model.

The main result of this paper is the suppression of chaotic behavior for Bianchi type IX when an ensemble average over trajectories driven by random-noise terms is taken. As is evident from Figs. 5—7, the cosmic scale factors attain constant values for sufficiently large negative times. The problems of quantum gravity and cosmological phase transition aside, the time for which the scale-factor functional dependence is flat could conveniently be taken as the time t_0 defining the initial-value problem for the Einstein equations. Guth[20] has suggested to begin the hot big-bang scenario at some temperature $T_0 \sim 10^{17}$ GeV, which corresponds to $t_0 \sim 10^{-31}$ cm$=3.3\times 10^{-41}$ sec. Depending on the initial conditions, the origin τ_0 of the logarithmic time scale satisfies $-20 \lesssim \tau_0 \lesssim -10$. This would correspond to $10^{-31} \lesssim t_0 \lesssim 10^{-10}$ cm $(3.3\times 10^{-41} \lesssim t_0 \lesssim 3.3\times 10^{-20}$ sec).

ACKNOWLEDGMENTS

I am grateful for valuable conversations with J. D. Farmer, E. W. Kolb, A. S. Lapedes, and H. A. Rose.

[1] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[2] C. W. Misner, Phys. Rev. Lett. 22, 1071 (1969).
[3] V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 62, 1606 (1972) [Sov. Phys.—JETP 35, 838 (1972)].
[4] C. W. Misner, Astrophys. J. 151, 431 (1968).
[5] M. J. Rees, Phys. Rev. Lett. 25, 1669 (1972).
[6] M. McCallum, in *Physics of the Expanding Universe,* edited by M. Demianski (Springer, New York, 1979); also in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[7] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. 19, 525 (1970); V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk. 102, 463 (1970) [Sov. Phys.—Usp. 13, 745 (1971)]; J. D. Barrow and F. J. Tipler, Phys. Rep. 56, 371 (1979); Y. Elskens, Phys. Rev. D 28, 1033 (1983).
[8] J. D. Barrow, Phys. Rev. Lett. 46, 963 (1981); 46, 1436(E) (1981); D. F. Chernoff and J. D. Barrow, *ibid.* 50, 134 (1983).
[9] J. D. Barrow, Phys. Rep. 85, 1 (1982).
[10] R. Shaw, Z. Naturforsch. 36A, 80 (1981); J. P. Eckmann, Rev. Mod. Phys. 53, 643 (1981); E. Ott, *ibid.* 53, 655 (1981).
[11] J. P. Crutchfield and B. A. Huberman, Phys. Lett. 77A, 407 (1980); J. P. Crutchfield, M. Nauenberg, and J. Rudnick, Phys. Rev. Lett. 46, 933 (1981); G. Mayer—Kress and H. Haken, J. Stat. Phys. 26, 149 (1981); A. Zardecki, Phys. Lett. 90A, 274 (1982); J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, Phys. Rep. 92, 45 (1982).
[12] J. D. Barrow, Nature (London) 272, 211 (1978).
[13] E. W. Kolb and S. Wolfram, Astrophys. J. 239, 428 (1980).
[14] S. W. Hawking, in *Unified Theory of Elementary Particles,* edited by P. Breitenlohner and H. P. Dürr (Springer, New York, 1982); Nucl. Phys. B143, 349 (1978).
[15] L. D. Landau and E. M. Lifshitz, *The Classical Theory of*

Fields (Pergamon, New York, 1975).

15a When $U_k \neq 0$, the Bianchi type-VII_0 models also have Mixmaster oscillations [J. D. Barrow (unpublished)]. Also, in the presence of electromagnetic fields Bianchi type-VII_0 has oscillations.

16 For a special choice of initial conditions, the results of numerical integration are given by V. A. Belinskii and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 56, 4701 (1969) [Sov. Phys.—JETP 29, 911 (1969)]. Numerical solutions for symmetric Bianchi type-IX universes were found by A. R. Moser, R. A. Matzner, and M. P. Ryan, Jr., Ann. Phys. (N.Y.) 79, 558 (1973), utilizing the Hamiltonian approach.

17 B. Carnahan, H. A. Luther, and J. O. Wilkes, *Applied Numerical Methods* (Wiley, New York, 1969).

18 This viewpoint is advocated by Barrow in Ref. 9 and C. W. Misner, Phys. Rev. 186, 1328 (1969). Restrictions on the model of a Mixmaster universe are discussed by A. V. Doroshkevich and I. D. Novikov, Astron. Zh. 47, 948 (1970) [Sov. Astron. 14, 763 (1971)].

19 G. Benettin, L. Galgani, and J. M. Strelcyn, Phys. Rev. A 14, 2338 (1976); I. Shimada and T. Nagashima, Prog. Theor. Phys. 61, 1605 (1979). Our computational algorithm is based on the work of G. Benettin and L. Galgani, J. Stat. Phys. 27, 153 (1982).

20 A. Guth, Phys. Rev. D 23, 347 (1981).

21 A. Guth, in *Birth of the Universe,* Proceedings of the XVII Rencontre de Moriond, Les Arcs, France, 1982, edited by J. Audouze and J. Trân Thanh Vân (Éditions Frontières, Gif-sur-Yvette, 1982); A. Guth and E. J. Weinberg, Phys. Rev. D 23, 876 (1981).

22 A. Vilenkin and L. H. Ford, Phys. Rev. D 26, 1231 (1982).

23 J. D. Barrow and M. S. Turner, Nature (London) 298, 801 (1982).

24 L. Arnold, *Stochastic Differential Equations* (Wiley-Interscience, New York, 1974).

25 A. Zardecki, Phys. Rev. A 22, 1664 (1980). This procedure neglects the terms of order $\Delta\tau$ in the last term of Eq. (27). See N. J. Rao, J. D. Borwankar, and D. Ramkrishna, SIAM (Soc. Ind. Appl. Math.) J. Control. 12, 124 (1974).