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Rapid Communications

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Generalized Koba-Nielsen-Olesen scaling

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Implications for generalized Koba-Nielsen-Olesen (KNO) scaling, namely KNO scaling at a fixed pseudorapidity interval, are discussed. Predictions can be obtained for semi-inclusive pseudorapidity distributions and vice versa. These predictions agree very well with the CERN ISR and $\bar{p}p$ collider data and they should be tested against e^+e^-, lepton-nucleon, and other hadronic data. A theorem about the relative magnitudes of the moments for different generalized KNO curves is proven. Consequences and validity of a "strong scaling hypothesis" are discussed.

It has been known for some time that the multiplicity distributions for hadron-nucleon collisions[1] at fixed-target energies obey Koba-Nielsen-Olesen (KNO) scaling.[2] Moreover, they are universal in the sense that one KNO curve is sufficient to describe collisions with all available hadronic beams. It is even possible that scaling and universality are valid for all nuclear targets,[3] though there are uncertainties about the universality aspect.[4] KNO scaling is valid also for the decay products of diffractively excited clusters,[5] and for e^+e^- (Ref. 6) and lepton-nucleon[7] collisions. These KNO curves do not have the same shape as the one for hadronic collisions.

Recent experiments at the ISR (Ref. 8) and the $\bar{p}p$ collider[9,10] at CERN have demonstrated the validity of scaling and universality perhaps to c.m. energy $\sqrt{s}=540$ GeV. This has aroused renewed theoretical interest in the subject.[11] But at this time there seems to be no general consensus on the basic underlying dynamical mechanism, so more tests capable of discriminating various models must be devised. One such test is now available. The ISR and collider experiments[8-10] showed that KNO scaling is already valid in a fixed pseudorapidity range $|\eta| \leq \eta_0$, at least for $\eta_0=1.5$ and 3.5. We will refer to this new phenomenon as the generalized KNO (GKNO) scaling. The GKNO curve at $\eta_0=1.5$ is wider than that at $\eta_0=3.5$, in the sense that all the γ moments[12] of the former are larger than those of the latter (see Table I). At ISR energies, almost all the events fall inside $|\eta|<3.5$; the GKNO curve at $\eta_0=3.5$ agrees with the universal KNO curve at fixed target energies. At $\sqrt{s}=540$ GeV, the KNO curve for the complete pseudorapidity range is not known. It is the GKNO scaling that is actually verified.

This new phenomenon of GKNO scaling merits considerable attention and exploration. It relates multiplicity and rapidity distributions in a definite way and, it is hoped, this may provide new insights into the dynamical mechanism for multiparticle productions. Although we will not discuss detailed dynamical models in this Rapid Communication, we will explore the phenomenological implications of GKNO scaling. Since data are available only for pp and $\bar{p}p$ processes

TABLE I. Measured values of γ moments and ϕ_m moments [Eq. (8)]. They are related as follows: $\gamma_2=\phi_2-1$, $\gamma_3=\phi_3-3\phi_2+2$, $\gamma_4=\phi_4-4\phi_3+6\phi_2-3-3\gamma_2^2$.

	γ_2	γ_3	γ_4	ϕ_2	ϕ_3	ϕ_4
UA1, $\sqrt{s}=540$ GeV $\lvert\eta\rvert<1.5$	0.441 $\pm$0.017	0.308 $\pm$0.021	0.216 $\pm$0.050	1.441	2.631	5.677
ISR, $\sqrt{s}=63$ GeV $\lvert\eta\rvert<1.5$	0.46 $\pm$0.01	0.28 $\pm$0.02	0.29 $\pm$0.05	1.46	2.66	5.805
UA1, $\sqrt{s}=540$ GeV $\lvert\eta\rvert<3.5$	0.296 $\pm$0.011	0.122 $\pm$0.007	0.027 $\pm$0.008	1.296	2.01	3.554
ISR, $\sqrt{s}=63$ GeV all η	0.297 $\pm$0.010	0.125 $\pm$0.007	0.051 $\pm$0.006	1.297	2.016	3.598

at $|\eta| \lesssim 3.5$, we shall base our following comparisons on $\eta_0 = 3.5$, although discussions could have been carried out for general intervals and indeed for general processes.

Let N and n be the number of detected particles in $|\eta| \le 3.5$ and $|\eta| \le \eta_0$, respectively, and let $\bar{N}(s)$ and $\bar{n}(s,\eta_0)$ be their averages. Let us assume that for a given N, number fluctuations in the interval $|\eta| < \eta_0$ are unimportant for the following purposes. Then in terms of the KNO variables $z = N/\bar{N}$ and $z' = n/\bar{n}$, the multiplicity distribution is given by the probability density

$$dP = \psi(z)dz = \phi(z',\eta_0)dz' \ . \tag{1}$$

where ψ and ϕ are the GKNO functions for $|\eta| \le 3.5$ and $|\eta| \le \eta_0$, respectively. Since ψ is also the KNO function at fixed-target and ISR energies, we will for simplicity refer to it as the KNO function. By the GKNO function we will now mean ϕ.

GKNO scaling thus implies that there exists a function f such that

$$z' = f(z,\eta_0), \quad z = f^{-1}(z',\eta_0) \ , \tag{2}$$

and

$$\phi(z',\eta_0) = \psi(f^{-1}(z',\eta_0)) \frac{\partial f^{-1}(z',\eta_0)}{\partial z'} \ . \tag{3}$$

Let

$$\rho(z,\eta,s) \equiv \frac{1}{\sigma_N}\frac{d\sigma_N}{d\eta} \tag{4}$$

be the semi-inclusive pseudorapidity distribution function. Its integral over $|\eta| \le 3.5$ is N. Its integral over $|\eta| \le \eta_0$ is n. Thus,

$$I(z,\eta_0,s) \equiv \int_{-\eta_0}^{\eta_0} \rho(z,\eta,s)\,d\eta = n = \bar{n}z' = \bar{n}(s,\eta_0)f(z,\eta_0) \ .$$

Thus,

$$\begin{aligned}\int_{-\eta_0}^{\eta_0} \frac{1}{\sigma_I}\frac{d\sigma_I}{d\eta}d\eta &= \sum_N \frac{\sigma_N}{\sigma_I}\int_{-\eta_0}^{\eta_0}\frac{1}{\sigma_N}\frac{d\sigma_N}{d\eta} \\ &= \int_0^\infty dz\,\psi(z)f(z,\eta_0)\bar{n}(s,\eta_0) \\ &= \bar{n}(s,\eta_0)\int_0^\infty dz'\phi(z',\eta_0)z' \\ &= \bar{n}(s,\eta_0)\end{aligned} \tag{5}$$

as expected, and

$$\begin{aligned}\int_{-\eta_0}^{\eta_0}\frac{1}{\sigma_N}\frac{d\sigma_N}{d\eta}d\eta \Big/ \int_{-\eta_0}^{\eta_0}\frac{1}{\sigma_I}\frac{d\sigma_I}{d\eta}d\eta &= \frac{I(z,\eta_0,s)}{\bar{n}(s,\eta_0)} \\ &= f(z,\eta_0)\end{aligned} \tag{6}$$

scales with energy. Moreover, the resulting scaling function f is the same one that connects the KNO with the GKNO functions as given by Eq. (3).

To test Eq. (6), we estimate $I(z,\eta_0,s)$ for $\eta_0 = 1.5$ by performing graphical integrations[13] from the published curves of $\rho(z,\eta,s)$ in Refs. 8 and 10. After dividing by the values of $\bar{n}$,[8,10] we plot this value of z' against the z value at the middle of the z range for which data for $\rho(z,\eta,s)$ are shown. The resulting points are exhibited in Fig. 1. We see that points corresponding to different energies all seem to lie on a smooth curve, thus verifying the scaling property given by Eq. (6). Since the points in Fig. 1 are obtained by graphical integrations, errors are hard to estimate but they are likely to be larger than the dispersion between the various points from the smooth curve.

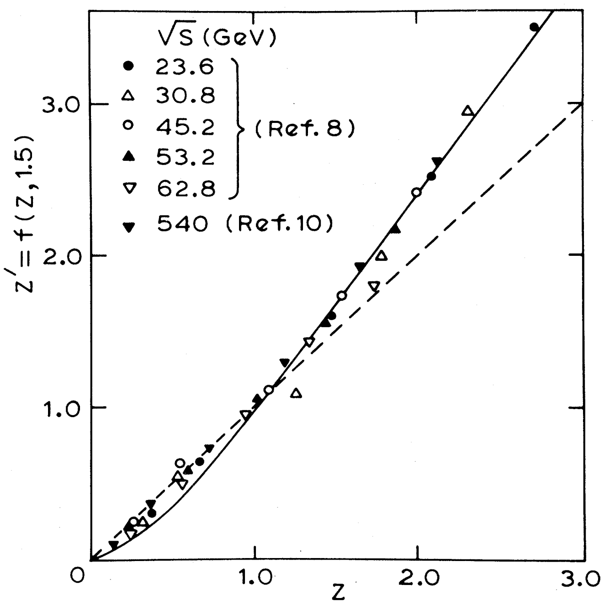

FIG. 1. Test of scaling with energy. The data points represent the left-hand side of Eq. (6), with the unintegrated data coming from Refs. 8 and 10. The solid curve is given by Eq. (7) and the dashed curve is $z' = z$.

The solid curve in Fig. 1 represents a simple parametrization of the data points. It is given by

$$f(z,1.5) = \frac{Az^2}{1+Bz} \ , \tag{7}$$

with $A = 2.88$ and $B = 1.90$. The small-z behavior of this parametrization is determined by the requirement that $\phi(z',1.5)$ is finite at $z' = 0$ and that $\psi(z) \simeq \beta z$ for small z. This requires $f(z,1.5) \simeq Az^2$, with A determined from Eq. (3) to be $A = \beta/2\phi(0,1.5)$. Using the Slattery parametrization[14] where $\beta = 3.79/2$, and the experimental value of $\phi(0,1.5)$, the aforementioned value of A was obtained. The denominator of (7) was put in by the desire (see below) that f should rise as a linear function of z at large z. The parameter B is then chosen to enable the solid curve to go near the data points.

Now we examine the other prediction of Eq. (6), that the KNO and the GKNO functions are related by the curve in Figs. 1. Using the Slattery parametrization[14] for $\psi(z)$, the parametric form (7) for f, and Eq. (3), we can calculate the GKNO curve for $\eta_0 = 1.5$. The result is shown as the solid curve in Fig. 2, together with the experimental data. Considering the simplicity of (7), the agreement between theory and experiment is quite astonishing.

It is possible to understand, even without detailed parametrization, why the GKNO curve is wider than the KNO curve. To this end we prove the following theorem.

Theorem. Let $f_1(z)$ and $f_2(z)$ be monotonically increasing, continuous, and non-negative functions of z. Suppose $f_1 \ge f_2$ for $z \ge z_0$ and $f_1 \le f_2$ for $z \le z_0$, for some $z_0 > 0$. Let $\phi_1(z')$ and $\phi_2(z')$ be two GKNO functions related to the KNO function $\psi(z)$ by $f_1(z)$ and $f_2(z)$, respectively, as in (3). Then $\phi_{1m} \geqslant \phi_{2m}$ for all positive integers m, where ϕ_{im} are the moments of $\phi_i(z')$:

$$\phi_{im} = \int_0^\infty \phi_i(z')(z')^m dz' \ . \tag{8}$$

Proof. From (1) and (2), we have

$$\phi_{1m} - \phi_{2m} = \int_0^\infty \psi(z)\, dz \{[f_1(z)]^m - [f_2(z)]^m\} \ . \tag{9}$$

Now

$$L(f_1(z), f_2(z)) \equiv \frac{f_1{}^m(z) - f_2{}^m(z)}{f_1(z) - f_2(z)} = \sum f_1{}^k(z) f_2{}^{m-1-k}(z)$$

is symmetric in f_1 and f_2, and it is a non-negative, continuous, and monotonically increasing function of z. Thus,

$$\begin{aligned}\phi_{1m} - \phi_{2m} &= \int_0^\infty \psi(z) L(f_1(z), f_2(z))[f_1(z) - f_2(z)]dz \\ &\geqslant L(f_1(z_0), f_2(z_0)) \left[- \int_0^{z_0} [f_2(z) - f_1(z)]\psi(z)dz + \int_{z_0}^\infty [f_1(z) - f_2(z)]\psi(z)dz \right] \\ &= L(f_1(z_0), f_2(z_0)) \int_0^\infty [\phi_1(z') - \phi_2(z')]z'dz' = 0 \ .\end{aligned}$$

Now we see from Fig. 1 that the solid curve $f(z, \eta_0)$ rises faster in z than the dashed curve z. This is expected because of the following general features of the semi-inclusive pseudorapidity distribution.[8,10] For small N, the distribution has a dip of $\eta = 0$. As N increases, this dip is gradually filled up to a plateau and later to a central peak. This means that n/N, the fraction of particles inside $|\eta| \leqslant \eta_0$, rises with N. It also means that $f(z, \eta_0)/z$ rises with z. So if we apply the theorem to $f_1 = f(z, 1.5)$ and $f_2 = z$, we will get the conclusion that

$$\phi_m(\eta_0 = 1.5) \geqslant \phi_m(\eta_0 = 3.5) \ .$$

This agrees with the experimental moments shown in Table I. If we now apply the theorem to two GKNO curves, at $\eta_0 = 3.5$ and $\eta_0 = \infty$, respectively, then it is hard to escape the conclusion that the genuine KNO curve ($\eta_0 = \infty$) at $\sqrt{s} = 540$ GeV is narrower than the GKNO curve at $\eta_0 = 3.5$, although precisely how much narrower cannot be estimated without having the knowledge of the full pseudorapidity distribution.

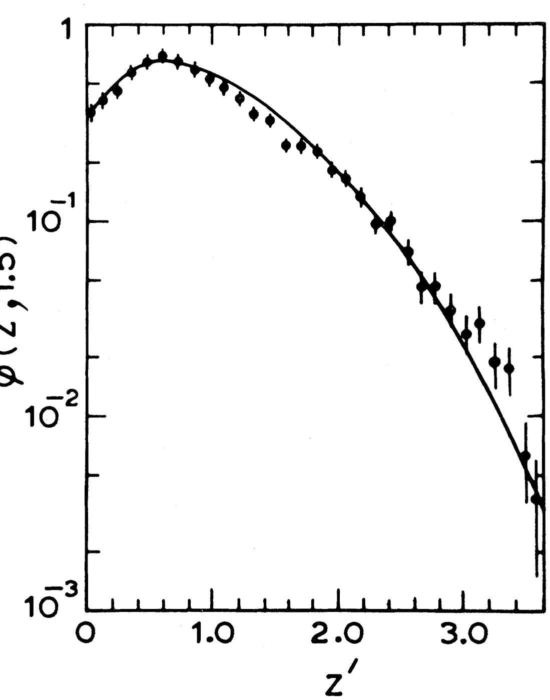

FIG. 2. Prediction of the GKNO scaling function for $\eta_0 = 1.5$ is given by the solid curve. The data points are from Ref. 10.

Let us now explore the consequences of the "strong scaling hypothesis" (SSH), namely, the assumption that GKNO scaling is exactly valid even for $z \gg 1$. Present tests of scaling extend out to $z = 3$ or 4, and not beyond. Thus, there is no compelling reason for the SSH to be valid. Nevertheless, it is interesting to see what can be obtained by making the hypothesis. The fraction of particles $x = n/N$ inside $|\eta| \leqslant \eta_0$ is expected to approach unity for very large N, because energy conservation forces almost all the particles in a large-multiplicity event to have small rapidities. In that case, $z' = xz(\bar{N}/\bar{n}) \to z(\bar{N}/\bar{n})$ for large z. With SSH, $z' = f(z, \eta_0)$ is independent of s; thus $(\bar{n}/\bar{N}) = R$ is a function of η_0 only. We thus arrive at the conclusion that

$$R(\eta_0) = \frac{1}{\bar{N}(s)} \int_{-\eta_0}^{\eta_0} \frac{1}{\sigma_I} \frac{d\sigma_I}{d\eta} d\eta \tag{10}$$

must be energy independent. This is phenomenologically incorrect because the UA5 data[9] clearly show that the pseudorapidity distribution at $\sqrt{s} = 540$ GeV is wider than that at $\sqrt{s} = 53$ GeV. So SSH is not exactly valid; the truth lies somewhere between Feynman-Yang scaling and that predicted by SSH. Nevertheless, SSH may not be too badly violated because $R'(0)$ is fairly energy independent. From the data of Refs. 8 and 10, we obtain $\frac{1}{2}R'(0) = 0.17$, 0.16, 0.15, 0.15, 0.15, and 0.16, respectively, for $\sqrt{s} = 23.6$, 30.8, 45.2, 53.8, 62.8, and 540 GeV. The errors in $R'(0)$ are roughly ± 0.01 for all of the values given. For comparison, note that throughout the same energy range, $\sigma_I{}^{-1} d\sigma_I/d\eta$ at $\eta = 0$ rises from 1.4 to 3.27.

Incidentally, whether SSH is valid or not, the arguments above Eq. (10) show that $f(z, \eta_0)$ grows asymptotically like z/R for large z. This jusitifies the parametrization of Eq. (7) for large z. The value $A/B = 1.5$ obtained there, however, is still smaller than $R^{-1} \simeq 2.0$, showing that at $z \sim 3$ we are still not yet in the asymptotic region for z.

In summary, we have discussed some of the predictions of GKNO scaling. It is important at this stage to test the range of validity of GKNO scaling for all processes. This is particularly so for e^+e^- and lepton-nucleon collisions because of the relative simplicity of their dynamics. Even for pp and $\bar{p}p$ collisions we still do not know whether GKNO scaling is valid only in the central region or for all η_0. For that matter we are not certain whether pseudorapidity, rapidity, or some other variable is the most appropriate one to use either. We hope that these and many other questions can be clarified in the near future.

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