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Vector-meson decays in the SLAC lattice quantum-chromodynamic theory

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The large-distance-dominated hadronic decays of the K^*, ϕ, and ω mesons of the lowest-lying vector-meson nonet of ordinary flavor SU(3) symmetry are discussed from the point of view of the SLAC lattice Hamiltonian QCD theory. The agreement between our theoretical results and observation is reasonable. This agreement we take as further evidence for the applicability of the two key ingredients in our calculational procedure to large-distance hadron dynamics. Here, we have reference to (a) the vacuum-insertion technique of Lee, Primack, and Treiman for the evaluation of light-hadron matrix elements of effective low-energy interacting densities and (b) the identification of the SLAC lattice currents with the physical hadron currents to leading order in the SLAC order-$1/g^2$ effective Hamiltonian for the fluxless light-hadron sector—in the spirit of Gell-Mann's work in current algebra. Indeed, when the results of the present paper are combined with our result for the decay $\rho\rightarrow\pi\pi$ in the same theoretical framework, we may say that the SLAC lattice QCD theory, taken together with these two techniques, represents a viable scheme for understanding the large-distance aspects of the hadronic decays of the entire vector-meson nonet of the ordinary SU(3) flavor symmetry.

I. INTRODUCTION

This paper represents a continuation of a previous paper,[1] hereafter referred to as I, in which we used the SLAC lattice Hamiltonian QCD theory[2] to discuss the decay

$$\rho\rightarrow\pi\pi \ . \tag{1}$$

Our result in I clearly indicated that the SLAC QCD theory, when combined with appropriate theoretical techniques already well tested in other areas of theoretical particle physics, may very well provide a viable scheme for calculating large-distance light-hadron dynamics. Here, the theoretical techniques which we have in mind are (a) the vacuum-insertion technique of Lee, Primack, and Treiman,[3] and (b) the abstraction to the physical hadron currents of the properties of the currents of quasirealistic model field theories (such as the SLAC theory)—after the fashion of Gell-Mann[4] in his work in current algebra. What we want to do in the present paper is to provide further evidence for the applicability of the calculational scheme under discussion to light-hadron decay systematics. This we want to do by applying the scheme to the large-distance aspects of the (Zweig-rule-allowed) hadronic decays of the remaining mesons in the lowest-lying vector-meson nonet of the ordinary SU(3) flavor symmetry. In this way, we may hope to determine the extent to which the success of our calculation in I was fortuitous. We shall begin by briefly reviewing the result which we obtained in I, in the interest of continuity and completeness.

What we accomplished in I was the following. Using the SLAC order-$1/g^2$ effective Hamiltonian for the fluxless light-hadron sector, together with the vacuum-insertion technique and the identification (in the spirit of Gell-Mann) of the SLAC lattice currents with the physical hadron currents to leading order in this effective Hamiltonian, we were able to express the decay width for the process in (1) in terms of the QCD coupling constant $g(m_\rho^2)$, the lattice constant a, the pion form factor $F_\pi(m_\rho^2)$, the ρ-meson decay constant f_ρ, the value of the quadratic Casimir operator of the fundamental representation of the color SU(3) group, and, of course, the masses of the ρ and π mesons. Thus, our expression in I for the decay width $\Gamma(\rho\rightarrow\pi\pi)$, which we shall effectively rederive in the ensuing sections, required for its evaluation a value of the lattice spacing a, for all of the remaining parameters in our result are known either theoretically or phenomenologically, although there is considerable uncertainty in the value of $g(m_\rho^2)$ because of the uncertainty[5] in the QCD Λ parameter Λ_{QCD}. We determined the value of the lattice constant a from a latticed PCAC (partial conservation of axial-vector current) condition,[6] as we shall again illustrate in the sections which follow. The result for a was

$$a \cong 5.74 \text{ GeV}^{-1} . \tag{2}$$

The corresponding result for $\Gamma(\rho^+ \to \pi^+\pi^0)$, for

$$g^2(m_\rho{}^2)/4\pi = 12\pi/23 \ln[m_\rho{}^2/(0.34 \text{ GeV})^2] , \tag{3}$$

is[1]

$$\Gamma(\rho^+ \to \pi^+\pi^0) \cong 162 \text{ MeV} , \tag{4}$$

to be compared with the experimental value of 158 MeV. Obviously, this is an encouraging state of affairs.

The question naturally arises as to why this SLAC lattice QCD-based approach to light-hadron large-distance dynamics should work as well as implied by (4). We feel that, *a posteriori,* the fact that the approach works reasonably well for $\rho \to \pi\pi$ suggests that, whenever a light-hadron decay process is dominated by large-distance effects, the approach should be applicable. Indeed, if one looks at (3) one sees that the momentum transfers in the process should be large enough that g^2 is reasonably well approximated by perturbation theory and yet these momentum transfers should be small enough that the lattice cutoff π/a does not significantly affect the dominant physical effects.

The value of g^2 can be obtained from perturbation theory whenever

$$\alpha_s/\pi$$

is significantly less than 1, where

$$\alpha_s \equiv g^2/4\pi . \tag{5}$$

From (3) we have, at $m_\rho{}^2$,

$$\alpha_s/\pi \cong 0.317 , \tag{6}$$

so that $g^2(m_\rho{}^2)$ can be reliably computed in perturbation theory. Also,

$$2\pi/a = 2\pi/(5.74 \text{ GeV}^{-1}) = 2(0.547 \text{ GeV}) \tag{7}$$

exceeds $m_\rho = 0.776$ GeV. Hence, it is not surprising that our approach works for $\rho \to \pi\pi$.

Clearly, these arguments suggest that the large-distance strong-interaction decays of the K^* (which has mass $m_{K^*} = 0.8918$ GeV), of the ϕ (which has mass 1.0196 GeV), and of the ω (which has mass 0.7824 GeV) should all be amenable to our approach. We have in mind the (Zweig-rule-allowed) large-distance strong-interaction decays $K^* \to K\pi$, $\phi \to \bar{K}K$, and $\omega \to \pi\pi\pi$. It will be our sole purpose in the following discussion to determine precisely to what extent the latter decays are indeed amenable to our approach.

Our work is organized as follows. In Sec. II we present the basic aspects of our approach to large-distance hadron dynamics. In this section we recapitulate the relevant aspects of the SLAC lattice QCD theory and of our use of this theory. In Sec. III, we apply the SLAC theory to the process $K^{*+} \to K\pi$. In Sec. IV, we apply the SLAC theory to the Zweig-rule-allowed process $\phi \to \bar{K}K$. In Sec. V, we consider the decay $\omega \to \pi\pi\pi$. Section VI contains some concluding remarks. The Appendix contains the evaluation of the lattice constant appropriate for purely kaonic transitions.

II. THE SLAC LATTICE QCD THEORY

In this section we wish to review the relevant aspects of the SLAC lattice QCD theory insofar as our calculation of large-distance hadron dynamics is concerned, for we feel that the present discussion should be self-contained.

The SLAC lattice Hamiltonian QCD theory consists of the Hamiltonian

$$H(g,a) = \frac{1}{a}\left\{ \sum_{\text{links}} \tfrac{1}{2} g^2 E^{\alpha\,2}_{\vec{j},\hat{\mu}} - \sum_{\text{loops}} \frac{1}{g^2} \operatorname{tr}\left[\prod_{\substack{\text{around}\\ \text{loop}}} U_{\vec{j},\hat{\mu}} \right] - \left[i \sum_{\substack{\vec{j},\hat{\mu}\\ n>0}} \delta'(n) \psi^{\dagger\alpha f}_{\vec{j}} \alpha_\mu \psi^{\beta f}_{\vec{j}+n\hat{\mu}} \times \left[\prod_{m=0}^{n-1} U_{\vec{j}+m\hat{\mu},\hat{\mu}} \right]^{\alpha\beta} + \text{H.c.} \right] \right\} , \tag{8}$$

where we have used the notation of Ref. 2. In this notation, the spinor field $\psi^{\alpha f}_{\vec{j}}$ at site $\vec{j}$ carries color index α and flavor index f. The operators $\vec{E}_{\vec{j},\hat{\mu}}$ measure the units of color flux created by the operator $U_{\vec{j},\hat{\mu}}$ on the link joining site $\vec{j}$ to site $\vec{j}+\hat{\mu}$. The α_μ are Dirac's matrices, which we always represent in the convention of Bjorken and Drell.[7] To repeat, the parameter a is the lattice spacing and the parameter g is the QCD gauge coupling constant. And, finally we note that the quantity $\delta'(n)$, which is the truly defining characteristic of the SLAC theory, is constructed so that

$$\partial_\mu \psi_{\vec{j}} = \frac{1}{a} \sum_n \delta'(n) \psi_{\vec{j}+n\hat{\mu}} \tag{9}$$

is the SLAC derivative on a lattice. In the infinite-volume limit in which we will always be interested, we have

$$\delta'(n) \to (-1)^{n+1}/n . \tag{10}$$

This completes the definition of the SLAC theory to the extent that we shall use it.

For the purpose of our calculations, we need to note the following key result of Ref. 2. Namely, to order $1/g^2$, the SLAC group has derived from (8) the effective second-order Hamiltonian

$$H^{(2)}_{\mathrm{eff}}=\frac{1}{a}\sum_{\vec{j},n,\mu}\frac{\delta'(n)\delta'(-n)}{\frac{1}{2}g^2\,|\,n\,|\,C_F}\times\psi^{\dagger\alpha f}_{\vec{j}}\alpha_\mu\psi^{\beta f}_{\vec{j}+n\hat{\mu}}\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}}\alpha_\mu\psi^{\alpha f'}_{\vec{j}} \tag{11}$$

using the standard degenerate-state perturbative techniques. The parameter C_F is the value of the quadratic Casimir operator in the fundamental representation of the color group $SU(N_c)$ so that

$$C_F=\frac{N_c{}^2-1}{2N_c}\ . \tag{12}$$

We will always take $N_c=3$. For completeness, we note that the matrix α_μ in (11) is, in the notation of Ref. 7, given by

$$\alpha_\mu=\gamma^0\gamma^\mu\ . \tag{13}$$

The interaction (11) forms the basis of our approach to large-distance strong-interaction decay processes for light hadrons. We will now apply our approach to the decays of interest, $K^{*+}\to K\pi$, $\phi\to K^+K^-$, and $\omega\to\pi^+\pi^-\pi^0$, in turn in the next three sections.

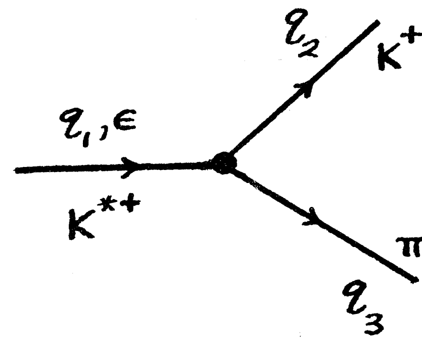

FIG. 1. The decay $K^{*+}\to K^+\pi^0$.

III. THE DECAY $K^{*+}\to K\pi$

In this section, we will compute the decay process $K^{*+}\to K^+\pi^0$; isospin considerations will then be used to obtain the rate for $K^{*+}\to K\pi$. We may proceed to do this in complete analogy with our calculation of $\rho\to\pi\pi$ in I. We shall so proceed.

More precisely, the amplitude of interest is

$$\mathscr{A}(K^{*+}\to K^+\pi^0)=\langle K^+\pi^0\,|\,-i\int_{-\infty}^{\infty}dt\,H^{(2)}_{\mathrm{eff}}\,|\,K^{*+}\rangle\ . \tag{14}$$

The relevant kinematics is summarized in Fig. 1. As in I, we work in the vector-meson rest frame so that $q_1=(m_{K^*},\vec{0})$. The polarization of this meson is described by ϵ_μ. The vacuum-insertion technique then leads us to consider the following expressions:

$$\langle K^+\pi^0\,|\,\psi^{\dagger\alpha f}_{\vec{j}}\alpha_\mu\psi^{\beta f}_{\vec{j}+n\hat{\mu}}\,|\,0\rangle\langle 0\,|\,\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}}\alpha_\mu\psi^{\alpha f'}_{\vec{j}}\,|\,K^{*+}\rangle\ , \tag{15}$$

$$-\langle K^+\pi^0\,|\,\psi^{\dagger\alpha f}_{\vec{j}\nu_1}(\alpha_\mu)_{\nu_1\nu_2}\psi^{\alpha f'}_{\vec{j}\sigma_2}\,|\,0\rangle\langle 0\,|\,\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}\sigma_1}(\alpha_\mu)_{\sigma_1\sigma_2}\psi^{\beta f}_{\vec{j}+n\hat{\mu}\nu_2}\,|\,K^{*+}\rangle\ , \tag{16}$$

$$\langle K^+\pi^0\,|\,\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}}\alpha_\mu\psi^{\alpha f'}_{\vec{j}}\,|\,0\rangle\langle 0\,|\,\psi^{\dagger\alpha f}_{\vec{j}}\alpha_\mu\psi^{\beta f}_{\vec{j}+n\hat{\mu}}\,|\,K^{*+}\rangle\ , \tag{17}$$

$$-\langle K^+\pi^0\,|\,\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}\sigma_1}(\alpha_\mu)_{\sigma_1\sigma_2}\psi^{\beta f}_{\vec{j}+n\hat{\mu}\nu_2}\,|\,0\rangle\langle 0\,|\,\psi^{\dagger\alpha f}_{\vec{j}\nu_1}(\alpha_\mu)_{\nu_1\nu_2}\psi^{\alpha f'}_{\vec{j}\sigma_2}\,|\,K^{*+}\rangle\ . \tag{18}$$

As in I, we note that matrix elements such as (15) and (17) vanish by the Wigner-Eckart theorem in the flavor space of f,f'. Thus, we only have to consider (16) and (18).

For the evaluation of (16) and (18), we may repeat the key steps used in the evaluation of the corresponding matrix elements in our computation of $\rho\to\pi\pi$ in I. In this way, we have (see I) the result that the sum of (16) and (18) is

$$-\tfrac{1}{4}\{\exp[i(q_2+q_3-q_1)\cdot x_{\vec{j}}]+\exp[i((q_2+q_3)\cdot x_{\vec{j}+n\hat{\mu}}-q_1\cdot x_{\vec{j}})]\}(-1)^{\delta_{i\mu}-1}\times\langle K^+\pi^0\,|\,\bar{\psi}^{\alpha f}(0)\gamma^i\psi^{\alpha f'}(0)\,|\,0\rangle\langle 0\,|\,\bar{\psi}^{\beta f'}(0)\gamma^i\psi^{\beta f}(0)\,|\,K^{*+}\rangle\ . \tag{19}$$

In writing (19), we have defined $x_{\vec{j}}=(t,\vec{j}a)$ and have reimplemented the Lorentz group in the spirit of Gell-Mann[4] as we explained in I, so that $\psi^{\alpha f'}(0)$ is now the fully interacting Heisenberg QCD quark quantum field at the origin of Minkowski space. Further, $\delta_{i\mu}$ is the Kronecker δ function, and we are summing on i in (19).

The result (19) may thus be expressed in terms of the corresponding matrix elements for $\rho\to\pi\pi$ using the flavor SU(3) symmetry: In (19) the K^{*+} selects, in f,f' space, the V-spin lowering operator

$$V^{-}=\lambda_4-i\lambda_5 \tag{20}$$

in the notation of Gell-Mann, so that we may write

$$\begin{aligned}\langle K^+\pi^0|\bar\psi^{\alpha f}(0)\gamma^i\psi^{\alpha f'}(0)|0\rangle\langle 0|\bar\psi^{\beta f'}(0)\gamma^i\psi^{\beta f}(0)|K^{*+}\rangle \\ =\langle K^+\pi^0|\bar\psi^{\alpha}(0)\gamma^iV^+\psi^{\alpha}(0)|0\rangle\langle 0|\bar\psi^{\beta}(0)\gamma^iV^-\psi^{\beta}(0)|K^{*+}\rangle \\ \underset{\mathrm{SU(3)}}{=}\frac{(1/\sqrt{2})F_\pi(m_{K^*}{}^2)(q_3-q_2)^i\sqrt{2}f_8\epsilon^i m_{K^*}}{(2q_2^02q_3^02q_1^0)^{1/2}}\ ,\end{aligned} \tag{21}$$

where in the last line of (21) we have used SU(3) together with the result of Ref. 8 to identify the SU(3) vector-meson decay constant f_8 with due account for the breaking of SU(3) in the value of this particular decay constant for the K^*. Note also that the pion form factor is evaluated at $m_{K^*}{}^2$. Clearly, (21) allows us to write (19) as

$$\begin{aligned}-\tfrac{1}{4}\{\exp[i(q_2+q_3-q_1)\cdot x_{\vec{j}}]+\exp[i((q_2+q_3)\cdot x_{\vec{j}+n\hat\mu}-q_1\cdot x_{\vec{j}})]\}(-1)^{\delta_{i\mu}-1} \\ \times F_\pi(m_{K^*}{}^2)(q_3-q_2)^if_8\epsilon^im_{K^*}/(2q_2^02q_3^02q_1^0)^{1/2}\ .\end{aligned} \tag{22}$$

On introducing (22) into (14), we have

$$\begin{aligned}\mathscr{A}(K^{*+}\to K^+\pi^0)&=-i\int_{-\infty}^{\infty}dt\,a^3\sum_{\vec{j},n,\mu}\frac{\delta'(n)\delta'(-n)}{a(\frac{1}{2})g^2|n|C_F}F_\pi(m_{K^*}{}^2)(q_3-q_2)^if_8\epsilon^im_{K^*} \\ &\quad\times(-\tfrac{1}{4})\{\exp[i(q_2+q_3-q_1)\cdot x_{\vec{j}}]+\exp[i((q_2+q_3)\cdot x_{\vec{j}+n\hat\mu}-q_1\cdot x_{\vec{j}})]\} \\ &\quad\times(-1)^{\delta_{i\mu}-1}a^3/(2q_1^02q_2^02q_3^0)^{1/2} \\ &=\frac{-i}{2}\int_{-\infty}^{\infty}dt\,a^3\sum_{\vec{j},n,\mu}\frac{(-1)^{n+1}(-1)^{-n+1}}{n(-n)|n|ag^2C_F}F_\pi(m_{K^*}{}^2)f_8a^3\epsilon^i(q_3-q_2)^im_{K^*} \\ &\quad\times\{\exp[i(q_2+q_3-q_1)\cdot x_{\vec{j}}]+\exp[i((q_2+q_3)\cdot x_{\vec{j}+n\hat\mu}-q_1\cdot x_{\vec{j}})]\} \\ &\quad\times(-1)^{\delta_{i\mu}}/(2q_1^02q_2^02q_3^0)^{1/2} \\ &=\frac{(2\pi)^4\delta^4(q_1-q_2-q_3)i[2\zeta(3)]F_\pi(m_{K^*}{}^2)f_8m_{K^*}\vec\epsilon\cdot(\vec{q}_3-\vec{q}_2)a^2}{g^2C_F(2q_1^02q_2^02q_3^0)^{1/2}}\ .\end{aligned} \tag{23}$$

In arriving at the last line of (23), we have passed to the limit of the infinite volume and identified

$$a^3\sum_{\vec{j}}\exp[-i(\vec{q}_2+\vec{q}_3-\vec{q}_1)\cdot\vec{j}a]=(2\pi)^3\delta^3(\vec{q}_2+\vec{q}_3-\vec{q}_1)\ . \tag{24}$$

The neglect of umklapps is justified by conservation of energy. Further, we have identified in (23) the Riemann function ζ of argument 3:

$$2\zeta(3)=\sum_{n\neq 0}\frac{1}{|n|^3}\ . \tag{25}$$

Its value is

$$\zeta(3) \cong 1.202 \ . \tag{26}$$

Finally, let us note that in writing (23) we have restored the powers of a which had been scaled[2] out of the fields in (11).

The amplitude (23), by the standard methods, corresponds to the decay width

$$\Gamma(K^{*+} \to K^+\pi^0) = \frac{1}{6\pi}\left\{\frac{[2\zeta(3)]^2 |F_\pi(m_{K^*}{}^2)|^2 |f_8|^2 a^4}{g^4 C_F{}^2}\right\}\left[\frac{m_{K^*}{}^4 - 2m_{K^*}{}^2(m_{K^+}{}^2 + m_{\pi^0}{}^2) + (m_{K^+}{}^2 - m_{\pi^0}{}^2)^2}{4m_{K^*}{}^2}\right]^{3/2} \tag{27}$$

Using[1,9–13]

$$\begin{aligned} m_{K^*} &= 0.8918 \text{ GeV} \ , \\ m_{K^+} &= 0.493\,669 \text{ GeV} \ , \\ m_{\pi^0} &= 0.134\,962\,6 \text{ GeV} \ , \\ |F_\pi(m_{K^*}{}^2)| &= 0.611\,|F_\pi(m_\rho{}^2)| \cong 3.67 \ , \\ a &= 5.74 \text{ GeV}^{-1} \ , \\ f_8 &\cong 0.175 \text{ GeV} \ , \end{aligned} \tag{28}$$

and

$$g^2 = g^2(m_{K^*}{}^2) = \frac{48\pi^2}{23\ln[m_{K^*}{}^2/(0.34 \text{ GeV})^2]} \cong 10.68 \tag{29}$$

we find

$$\Gamma(K^{*+} \to K^+\pi^0) \cong 16.44 \text{ MeV} \ . \tag{30}$$

Thus, from isospin considerations, we have

$$\Gamma(K^{*+} \to K\pi) \cong 49.3 \text{ MeV} \ , \tag{31}$$

to be compared with the experimental value of

$$\Gamma(K^{*+} \to K\pi) \cong 50.3 \text{ MeV} \ . \tag{32}$$

Evidently, our theoretical result is in reasonable agreement with observation. We are therefore encouraged to analyze the two remaining vector mesons in the lowest-lying vector-meson nonet of SU(3), the $\phi(1019.6)$ and the $\omega(782.4)$. We turn first to the ϕ in the next section.

IV. THE DECAY $\phi \to \bar{K}K$

In this section, we wish to apply our calculational methods to the decay $\phi \to \bar{K}K$ as realized by

$$\phi \to K^+K^- \ . \tag{33}$$

The analogous discussion for $\phi \to K_S K_L$ will be apparent from our discussion of (33), so we will analyze (33) and obtain the rate $\Gamma(\phi \to K_S K_L)$ from the standard simple isospin and phase-space considerations. We should emphasize that, in our calculation of (33), for the first time in our analysis[1] of the vector-meson decays, the full U(3) symmetry of the hadronic currents will be used. Since this symmetry is broken to a larger extent than is SU(3), most naively, we do not expect our methods to work as well as they did for the ρ and K^*. But, we may be in for a surprise.

Indeed, it is expected that, in the Bjorken-scaling region,[14] the hadronic currents would in fact reflect the entire U(3) symmetry. And, $m_\phi{}^2 = 1.0396 \text{ GeV}^2$ is very close to the boundary of this scaling region. Thus, there is reason for optimism here.

Turning now to the computation of (33) we have the amplitude

$$\mathscr{A}(\phi \to K^+K^-) = \langle K^+K^- | -i \int_{-\infty}^{\infty} dt\, H_{\text{eff}}^{(2)} | \phi \rangle \tag{34}$$

with $H_{\text{eff}}^{(2)}$ given by (11). The vacuum-insertion technique then allows us to write

$$\begin{aligned} \mathscr{A}(\phi \to K^+K^-) = \frac{-i}{a} \int_{-\infty}^{\infty} dt \sum_{\vec{j},n,\mu} \frac{\delta'(n)\delta'(-n)}{\frac{1}{2}g^2 |n| C_F} \times [\, & \langle K^+K^- | \psi_{\vec{j}}^{\dagger\alpha f} \alpha_\mu \psi_{\vec{j}+n\hat{\mu}}^{\beta f} | 0 \rangle \langle 0 | \psi_{\vec{j}+n\hat{\mu}}^{\dagger\beta f'} \alpha_\mu \psi_{\vec{j}}^{\alpha f'} | \phi \rangle \\ & + \langle K^+K^- | \psi_{\vec{j}+n\hat{\mu}}^{\dagger\beta f'} \alpha_\mu \psi_{\vec{j}}^{\alpha f'} | 0 \rangle \langle 0 | \psi_{\vec{j}}^{\dagger\alpha f} \alpha_\mu \psi_{\vec{j}+n\hat{\mu}}^{\beta f} | \phi \rangle \\ & - \langle K^+K^- | \psi_{\vec{j}\sigma_2}^{\dagger\alpha f} (\alpha_\mu)_{\sigma_2\sigma_1} \psi_{\vec{j}\nu_1}^{\alpha f'} | 0 \rangle \langle 0 | \psi_{\vec{j}+n\hat{\mu}\nu_2}^{\dagger\beta f'} (\alpha_\mu)_{\nu_2\nu_1} \psi_{\vec{j}+n\hat{\mu}\sigma_1}^{\beta f} | \phi \rangle \\ & - \langle K^+K^- | \psi_{\vec{j}+n\hat{\mu}\nu_2}^{\dagger\beta f'} (\alpha_\mu)_{\nu_2\nu_1} \psi_{\vec{j}+n\hat{\mu}\sigma_1}^{\beta f} | 0 \rangle \langle 0 | \psi_{\vec{j}\sigma_2}^{\dagger\alpha f} (\alpha_\mu)_{\sigma_2\sigma_1} \psi_{\vec{j}\nu_1}^{\alpha f'} | \phi \rangle] \ . \end{aligned} \tag{35}$$

To proceed, we need to specify the flavor-SU(3) properties of the ϕ.

For the purposes of our work, we will treat the ϕ as the ideally mixed state

$$\phi:\ \bar{s}s\ . \tag{36}$$

Thus, denoting by ϕ_0 and ϕ_8 the SU(3)-singlet and -octet states with $I=0$ and $Y=0$, we have

$$\phi=-(\tfrac{2}{3})^{1/2}\phi_8+\frac{1}{\sqrt{3}}\phi_0 \tag{37}$$

so that the ω will be identified subsequently, in Sec. V, as

$$\omega=(\tfrac{2}{3})^{1/2}\phi_0+\frac{1}{\sqrt{3}}\phi_8\ . \tag{38}$$

From (37), it follows that the nonlocal terms in (35), diagonal in f,f', cannot contribute. The ϕ_8 part of ϕ gives zero by the Wigner-Eckart theorem in f,f' space whereas the ϕ_0 part of ϕ gives zero by SU(3) parity (the analog of G parity for pions). Thus, we need only consider the ff' local terms in (35).

Considering these local terms, we may proceed in complete analogy with our discussion in I and the preceding section. In this way, we arrive at (the kinematics is summarized in Fig. 2)

$$\begin{aligned}\mathscr{A}(\phi\to K^+K^-)=\frac{-i}{a}\int_{-\infty}^{\infty}dt\sum_{\vec{j},n,\mu}\frac{\delta'(n)\delta'(-n)}{\frac{1}{2}g^2\,|\,n\,|\,C_F}\\ \times(-\tfrac{1}{4})\{\exp[i(q_2+q_3-q_1)\cdot x_{\vec{j}}]\\ +\exp[i((q_2+q_3)\cdot x_{\vec{j}+n\hat{\mu}}-q_1\cdot x_{\vec{j}})]\}(-1)^{\delta_{i\mu}-1}\\ \times\langle K^+K^-\,|\,\bar{\psi}^{\alpha f}(0)\gamma^i\psi^{\alpha f'}(0)\,|\,0\rangle\langle 0\,|\,\bar{\psi}^{\beta f'}(0)\gamma^i\psi^{\beta f}(0)\,|\,\phi\rangle\ .\end{aligned} \tag{39}$$

Hence, we need to evaluate the current matrix elements in this last expression (39).

In evaluating the current matrix elements in (39), we note that, by the rule of Zweig, only the term

$$f'=s,\quad f=s\ , \tag{40}$$

will contribute to (39), in an obvious notation. For this choice of f' and f, we have

$$\begin{aligned}\langle K^+K^-\,|\,\bar{\psi}^{\alpha f}(0)\gamma^i\psi^{\alpha f'}(0)\,|\,0\rangle\langle 0\,|\,\bar{\psi}^{\beta f'}(0)\gamma^i\psi^{\beta f}(0)\,|\,\phi\rangle=\langle K^+K^-\,|\,\bar{\psi}^{\alpha}(0)\gamma^i\left[-\frac{2}{\sqrt{3}}\lambda_8+\left[\frac{2}{3}\right]^{1/2}\lambda_0\right]\psi^{\alpha}(0)\,|\,0\rangle\\ \times\langle 0\,|\,\bar{\psi}^{\beta}(0)\gamma^i\left[-\frac{2}{\sqrt{3}}\lambda_8+\left[\frac{2}{3}\right]^{1/2}\lambda_0\right]\psi^{\beta}(0)\,|\,\phi\rangle\ ,\end{aligned} \tag{41}$$

where λ_0 and λ_8 are Gell-Mann's matrices.[6] Using the full U(3) symmetry we have (here ϵ_μ is the ϕ polarization vector)

$$\langle 0\,|\,\bar{\psi}^{\beta}(0)\gamma^i\left[-\frac{2}{\sqrt{3}}\lambda_8+\left[\frac{2}{3}\right]^{1/2}\lambda_0\right]\psi^{\beta}(0)\,|\,\phi\rangle=\sqrt{2}f'_8\,\epsilon^i m_\phi/(2q_1^0)^{1/2} \tag{42}$$

and, from SU(3),

$$\begin{aligned}\langle K^+K^-\,|\,\bar{\psi}^{\alpha}(0)\gamma^i\left[-\frac{2}{\sqrt{3}}\lambda_8+\left[\frac{2}{3}\right]^{1/2}\lambda_0\right]\psi^{\alpha}(0)\,|\,0\rangle&=\langle K^+K^-\,|\,\bar{\psi}^{\alpha}(0)\gamma^i\left[-\frac{2}{\sqrt{3}}\lambda_8\right]\psi^{\alpha}(0)\,|\,0\rangle\\ &=-2\langle K^+K^-\,|\,J^i_{\rm EM}(0)\,|\,0\rangle\\ &=-2F_K(m_\phi{}^2)(q_3^i-q_2^i)/(2q_2^0 2q_3^0)^{1/2}\ ,\end{aligned} \tag{43}$$

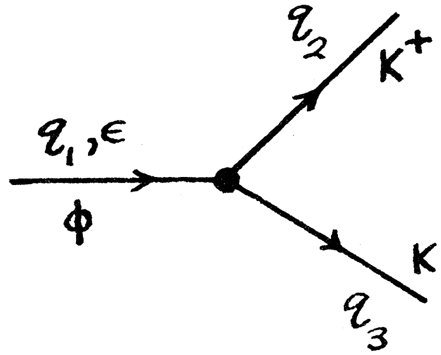

FIG. 2. The decay $\phi \to K^+K^-$.

where $J_{\rm EM}$ is the electromagnetic current and F_K is the familiar isoscalar kaon electromagnetic form factor. In writing (42) and (43), we have taken the view in Ref. 1 in which we may use a latticed PCAC condition for kaons in conjunction with MIT bag ideas[15] to determine the effective lattice spacing in (39). This is done in the Appendix, where we find

$$a \cong 1.81 \text{ GeV}^{-1} \tag{44}$$

for kaonic lattice transitions. This result (44) is substantially smaller than the radius of the vector mesons $\{\rho,\omega,\phi,K^*\}$ in the MIT bag model,[15] for example, where it was found that these SU(3) nonet mesons all have radii in the range 4.61–5.47 GeV^{-1}. Continuing with the strategy in Ref. 1, we argue that the physical decay constants for the SU(3)-nonet vector mesons, such as f_8 in (21), are adequately represented by the Lorentz-group-reimplemented Heisenberg current and latticed vector-meson states for the lattice spacing found in Ref. 1, namely,

$$a \cong 5.74 \text{ GeV}^{-1} . \tag{45}$$

Our position here, of course, is also supported by our results for the K^* in Sec. III. On this view, the constant f_8' in (42), which refers to the respective SU(3) decay constant of lattice spacing $a = 1.81$ GeV^{-1}, must be corrected for the difference in physics between the scales (44) and (45), in order to express it in terms of a parameter such as f_8 in (21). Fortunately, since we are working only to leading order in the SLAC order-$1/g^2$ effective Hamiltonian (11), this latter correction is straightforward.

Indeed, in this order of the interaction (11), we have reimplemented, in the spirit of Gell-Mann, the Lorentz group so that the current in (42) is the fully Lorentz-covariant Heisenberg operator current. This current acts on the state $|\phi\rangle$, which, due to (44), is the SLAC lattice ϕ state at lattice spacing 1.81 GeV^{-1}. As we found in Ref. 1, in order to account for the size of ϕ, we need to relate the state $|\phi\rangle$ at $a = 1.81$ GeV^{-1} to the state $|\phi\rangle$ at 5.74 GeV^{-1}. Let $|\phi,a\rangle$ denote the ϕ state at rest at lattice spacing a. (We continue to suppress polarization vectors.) Then, since the expression (39) already contains the leading strong-interaction effects to the order to which we are working, we may treat the renormalization Z factor[7] for the state $|\phi,a\rangle$ as trivial. Thus, the renormalization-group equation[16] for the state $|\phi,a\rangle$ is clearly

$$\Lambda_{\rm QCD}\frac{\partial}{\partial\Lambda_{\rm QCD}}|\phi,a\rangle = 0 . \tag{46}$$

Using naive dimensional scaling (the violations[17] of this naive scaling are suppressed to the order that we are working), we have

$$\left[-a\frac{\partial}{\partial a} + \Lambda_{\rm QCD}\frac{\partial}{\partial\Lambda_{\rm QCD}}\right]|\phi,a\rangle = -\tfrac{3}{2}|\phi,a\rangle . \tag{47}$$

The solution of (46) and (47) is, for $a = \lambda a_0$, $\lambda > 0$,

$$|\phi,\lambda a_0\rangle = \lambda^{3/2}|\phi,a_0\rangle . \tag{48}$$

Using $a = 1.81$ GeV^{-1}, we have $\lambda = 5.74/1.81$ so that the desired state is

$$|\phi,5.74 \text{ GeV}^{-1}\rangle = \left(\tfrac{5.74}{1.81}\right)^{3/2}|\phi,1.81 \text{ GeV}^{-1}\rangle . \tag{49}$$

Combining (42) and (49), we find

$$f_8' = \left(\tfrac{5.74}{1.81}\right)^{-3/2} f_8 \cong (3.17)^{-3/2} f_8 \tag{50}$$

as an approximation consistent with our general framework.

At this point it is appropriate to note the following. From Refs. 1 and 13, one can easily derive three different values for f_8: $f_8 = 0.161$ GeV from $e^+e^- \to \phi \to$hadrons, $f_8 = 0.154$ GeV from $e^+e^- \to \omega \to$hadrons, and $f_8 = f_\rho = 0.14$ GeV from $\bar{\tau} \to \rho^+ + \bar{\nu}_\tau$. On averaging these values of f_8 we find

$$f_8 \cong 0.15 \text{ GeV} . \tag{51}$$

Henceforward, we will use this average value of f_8 in an effort to compensate for the SU(3) breaking effects that are self-evident in these differing values of f_8. We emphasize that (51) can only be an estimate. In (28), we have used the value $f_8 = 0.175$ GeV. As we explain in Ref. 9, the use of $f_8 = 0.175$ GeV in (28) is justified on the basis that the K^* is expected to have much smaller finite-width effects compared to the ϕ. Thus, these ϕ finite-width effects, as estimated in Ref. 13, have been removed from f_8 in (28), where we use $e^+e^-\to\phi\to$hadrons to determine the non-finite-width corrected value of f_8. Our reason for using the process $e^+e^-\to\phi\to$hadrons is explained in Ref. 9. We may now proceed with the computation of $\phi\to\bar{K}K$.

Returning to this general development, we have, on introducing (42) and (43) into (39), the amplitude

$$\mathcal{A}(\phi\to K^+K^-)=\frac{-i}{a}\int_{-\infty}^{\infty}dt\,a^3\sum_{\vec{j},n,\mu}\frac{\delta'(n)\delta'(-n)}{\frac{1}{2}g^2|n|C_F}\times(-\tfrac{1}{4})\{\exp[i(q_2+q_3-q_1)\cdot x_{\vec{j}}]+\exp[i((q_2+q_3)\cdot x_{\vec{j}+n\hat{\mu}}-q_1\cdot x_{\vec{j}})]\}$$
$$\times(-1)^{\delta_{i\mu}-1}\sqrt{2}f'_8\frac{\epsilon^i m_\phi}{(2q_1^0)^{1/2}}(-2)F_K(m_\phi{}^2)\frac{(q_3^i-q_2^i)a^3}{(2q_2^0 2q_3^0)^{1/2}}$$
$$=\frac{(2\pi)^4\delta^4(q_1-q_2-q_3)[2\zeta(3)]}{g^2C_F}i\sqrt{2}f'_8 m_\phi(-2)F_K(m_\phi{}^2)\frac{\vec{\epsilon}\cdot(\vec{q}_3-\vec{q}_2)a^2}{(2q_1^0 2q_2^0 2q_3^0)^{1/2}}\,, \tag{52}$$

where, here, $a\cong 1.81$ GeV^{-1}, the lattice constant for kaonic transitions as computed in the Appendix using the kaonic PCAC condition and MIT bag ideas, to repeat. The standard methods then give, using (50),

$$\Gamma(\phi\to K^+K^-)=\frac{4}{3\pi}\frac{[2\zeta(3)]^2|F_K(m_\phi{}^2)|^2|f_8|^2a^4(m_\phi{}^2/4-m_K{}^2)^{3/2}}{C_F{}^2g^4}\left[\frac{1.81}{5.74}\right]^3 . \tag{53}$$

For the evaluation of (53), we need experimental and/or theoretical values for $F_K(m_\phi{}^2)$ and $g^2(m_\phi{}^2)$. The parameter $g^2(m_\phi{}^2)$ is easily determined from our formula (3) to be

$$g^2(m_\phi{}^2)=9.38\,. \tag{54}$$

Thus, there remains only the determination of the value of $F_K(m_\phi{}^2)$ for the evaluation of (53).

Turning now to the evaluation of $F_K(m_\phi{}^2)$, we note that, from Refs. 18 and 19, the cross section for $e^+e^-\to K^+K^-$ at the ϕ resonance is, approximately,

$$\langle\sigma\rangle_{K^+K^-}=2.3\ \mu\text{b}\,. \tag{55}$$

The formula (here α is the fine-structure constant)

$$\langle\sigma\rangle_{K^+K^-}=\frac{\pi\alpha^2}{3m_\phi{}^2}|F_K(m_\phi{}^2)|^2(1-4m_K{}^2/m_\phi{}^2)^{3/2} \tag{56}$$

then allows us to identify

$$|F_K(m_\phi{}^2)|=84.16\,. \tag{57}$$

We will suppress the finite-width correction here,[11,18] for we feel the effective large-distance currents should include the full effects of the strong interactions. This completes the determination of the parameters required to evaluate (53).

On introducing (51), (54), and (57) into (53) we find

$$\Gamma(\phi\to K^+K^-)\cong 1.73\text{ MeV}\,. \tag{58}$$

This result should be compared to the experimental result[12]

$$\Gamma(\phi\to K^+K^-)\cong 1.99\text{ MeV}\,. \tag{59}$$

Further, using the standard isospin considerations and correcting the momentum magnitude of the kaons in (53) for the mass difference between charged and neutral kaons (we ignore $m_{K_L}-m_{K_S}$), we find the theoretical result

$$\Gamma(\phi\to K_SK_L)\cong 1.14\text{ MeV} \tag{60}$$

to be compared to the experimental result

$$\Gamma(\phi\to K_SK_L)\cong 1.44\text{ MeV}\,. \tag{61}$$

Again, our lattice estimates are reasonable estimates. Encouraged by this, we turn to the decay $\omega\to\pi\pi\pi$ in the next section.

V. THE DECAY $\omega\to\pi\pi\pi$

In view of our general success in computing the ρ, K^*, and ϕ hadronic decays, we now turn naturally to the remaining member of the lowest flavor-SU(3) vector-meson nonet—the ω meson. We will treat specifically the decay

$$\omega\to\pi^+\pi^-\pi^0 . \tag{62}$$

We will do this in the by now familiar way—that is, by combining the SLAC-lattice-theory interaction (11) with the vacuum-insertion technique[3] and our variant of Gell-Mann's current-algebra methods.[4]

More specifically, for the decay (62), we require the amplitude

$$\begin{aligned}\mathscr{A}(\omega\to\pi^+\pi^-\pi^0)=-i\int_{-\infty}^{\infty}dt\,\frac{1}{a}\sum_{\vec{j},n,\mu}\frac{\delta'(n)\delta'(-n)}{\frac{1}{2}g^2|n|C_F}\\ \times[\langle\pi^+\pi^-\pi^0|\psi^{\dagger\alpha f}_{\vec{j}}\alpha_\mu\psi^{\beta f}_{\vec{j}+n\hat{\mu}}|0\rangle\langle 0|\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}}\alpha_\mu\psi^{\alpha f'}_{\vec{j}}|\omega\rangle\\ +\langle\pi^+\pi^-\pi^0|\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}}\alpha_\mu\psi^{\alpha f'}_{\vec{j}}|0\rangle\langle 0|\psi^{\dagger\alpha f}_{\vec{j}}\alpha_\mu\psi^{\beta f}_{\vec{j}+n\hat{\mu}}|\omega\rangle\\ -\langle\pi^+\pi^-\pi^0|\psi^{\dagger\alpha f}_{\vec{j}\sigma_2}(\alpha_\mu)_{\sigma_2\sigma_1}\psi^{\alpha f'}_{\vec{j}\nu_1}|0\rangle\langle 0|\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}\nu_2}(\alpha_\mu)_{\nu_2\nu_1}\psi^{\beta f}_{\vec{j}+n\hat{\mu}\sigma_1}|\omega\rangle\\ -\langle\pi^+\pi^-\pi^0|\psi^{\dagger\beta f'}_{\vec{j}+n\hat{\mu}\nu_2}(\alpha_\mu)_{\nu_2\nu_1}\psi^{\beta f}_{\vec{j}+n\hat{\mu}\sigma_1}|0\rangle\langle 0|\psi^{\dagger\alpha f}_{\vec{j}\sigma_2}(\alpha_\mu)_{\sigma_2\sigma_1}\psi^{\alpha f'}_{\vec{j}\nu_1}|\omega\rangle]\,,\end{aligned}\tag{63}$$

where the kinematics is summarized in Fig. 3. We would emphasize here that, contrary to the analyses of the ρ, K^*, and ϕ, the nonlocal terms in (63) do not vanish by SU(3) symmetry arguments. We may argue presently, however, that in the framework of the SLAC lattice QCD theory,[2] these nonlocal terms again vanish due to the space-time structure of the ω-meson state in this theory.

More precisely, in the SLAC theory, the light hadrons such as the ω are represented as the action of the appropriate spin projections of a quark field and its adjoint on a single site $\vec{j}$. Thus, due to the quark field commutation relations on the SLAC lattice, all of the nonlocal vacuum to $|\omega\rangle$ matrix elements in (63) vanish. We are left with only the local terms in (63) to analyze.

Turning to these local terms, we recall that, since the ω is the combination of the ϕ_0 and ϕ_8 states which is orthogonal to the ϕ, we may write the contribution of the local terms in (63) to complete analogy with (39). In this way, we find

$$\begin{aligned}\mathscr{A}(\omega\to\pi^+\pi^-\pi^0)=\frac{-i}{a}\int_{-\infty}^{\infty}dt\sum_{\vec{j},n,\mu}\frac{\delta'(n)\delta'(-n)}{\frac{1}{2}g^2|n|C_F}\\ \times(-\tfrac{1}{4})\{\exp[i(q_2+q_3+q_4-q_1)\cdot x_{\vec{j}}]\\ +\exp[i((q_2+q_3+q_4)\cdot x_{\vec{j}+n\hat{\mu}}-q_1\cdot x_{\vec{j}})]\}(-1)^{\delta_{i\mu}-1}\\ \times\langle\pi^+\pi^-\pi^0|\bar{\psi}^{\alpha f}(0)\gamma^i\psi^{\alpha f'}(0)|0\rangle\langle 0|\bar{\psi}^{\beta f'}(0)\gamma^i\psi^{\beta f}(0)|\omega\rangle\,.\end{aligned}\tag{64}$$

We must now evaluate the various matrix elements on the right-hand side of (64).

Toward this evaluation, we use the standard flavor matrix manipulations to write

$$\begin{aligned}\langle\pi^+\pi^-\pi^0|\bar{\psi}^{\alpha f}(0)\gamma^i\psi^{\alpha f'}(0)|0\rangle\langle 0|\bar{\psi}^{\beta f'}(0)\gamma^i\psi^{\beta f}(0)|\omega\rangle&=2\langle\pi^+\pi^-\pi^0|\bar{\psi}^{\alpha}(0)\gamma^i\lambda_a\psi^{\alpha}(0)|0\rangle\langle 0|\bar{\psi}^{\beta}(0)\gamma^i\lambda_a\psi^{\beta}(0)|\omega\rangle\\ &=\frac{2\epsilon^i m_\omega}{(2q_1^0)^{1/2}}f_8^\omega\left[\left[\frac{2}{3}\right]^{1/2}\langle\pi^+\pi^-\pi^0|\bar{\psi}^{\alpha}(0)\gamma^i\lambda_0\psi^{\alpha}(0)|0\rangle\right.\\ &\quad\left.+\frac{1}{\sqrt{3}}\langle\pi^+\pi^-\pi^0|\bar{\psi}^{\alpha}(0)\gamma^i\lambda_8\psi^{\alpha}(0)|0\rangle\right]\\ &=\frac{2\epsilon^i m_\omega}{(2q_1^0)^{1/2}}f_8^\omega\left[\frac{1}{\sqrt{3}}\right]\langle\pi^+\pi^-\pi^0;\underline{8}|\bar{\psi}^{\alpha}(0)\gamma^i\lambda_8\psi^{\alpha}(0)|0\rangle\\ &=\frac{2\epsilon^i m_\omega}{(2q_1^0)^{1/2}}f_8^\omega\langle\pi^+\pi^-\pi^0;\underline{8}|J^i_{\rm EM}(0)|0\rangle\,,\end{aligned}\tag{65}$$

where $\vec{\epsilon}$ is the polarization vector of the ω. A few comments about (65) are in order. We have used the U(3) result $f_8 = f_0$ (Zweig's rule) and the standard normalization for the SU(3) flavor matrices

$$tr(\lambda_a \lambda_b) = \tfrac{1}{2}\delta_{ab} \ . \qquad (66)$$

The notation f_a^ω, $a = 0,8$, simply emphasizes that the decay constants are determined in our latticed ω state. Further, we use the notation $\langle \pi^+\pi^-\pi^0;\underline{8}\,|$ to denote that we have presumed octet dominance[20] for the ω decay so that the final $\pi^+\pi^-\pi^0$ state is an octet state. On this octet-dominance view, the singlet term in (65) vanishes, as we have indicated. The result (65) allows us to continue with our evaluation of $\omega \to \pi^+\pi^-\pi^0$.

Indeed, on introducing (65) into (64), we find

$$\mathscr{A}(\omega \to \pi^+\pi^-\pi^0) = (2\pi)^4 \delta^4(q_2+q_3+q_4-q_1) 2i[2\zeta(3)] \frac{a^2}{g^2 C_F} f_8^\omega \frac{\epsilon^i m_\omega}{(2q_1^0)^{1/2}} \langle \pi^+\pi^-\pi^0 \,|\, J_{\rm EM}^i(0)\,|\,0\rangle \ . \qquad (67)$$

The rate for $\omega \to \pi^+\pi^-\pi^0$ is now readily seen to be

$$\Gamma(\omega \to \pi^+\pi^-\pi^0) = \frac{4a^4[2\zeta(3)]^2}{g^4 C_F{}^2} \,|\, f_8^\omega \,|^2 m_\omega{}^2 (-\tfrac{1}{3}) \frac{g^{ij}}{2m_\omega} H_{ij}(m_\omega{}^2) \ , \qquad (68)$$

where

$$H^{\mu\nu} \equiv \int \frac{d^3q_2 d^3q_3 d^3q_4}{(2\pi)^9} (2\pi)^4 \delta^4(q_1-q_2-q_3-q_4) \langle \pi^+\pi^-\pi^0 \,|\, J_{\rm EM}^\mu(0)\,|\,0\rangle \langle 0\,|\,J_{\rm EM}^\nu(0)\,|\,\pi^+\pi^-\pi^0\rangle$$
$$= (q_1^\mu q_1^\nu - m_\omega{}^2 g^{\mu\nu}) F(m_\omega{}^2) \ . \qquad (69)$$

On introducing (69) into (68), we find the result

$$\Gamma(\omega \to \pi^+\pi^-\pi^0) = \frac{2[2\zeta(3)]^2 \,|\, f_8^\omega \,|^2 m_\omega{}^3 F(m_\omega{}^2) a^4}{g^4 C_F{}^2} \ . \qquad (70)$$

To make contact with observation, we note that the appropriate value of a in (70) is, according to the final pion state, the pionic value found in Ref. 1

$$a = 5.74 \ \text{GeV}^{-1} \ . \qquad (71)$$

However, from our discussion of $\phi \to \bar{K}K$, which also proceeded through the ϕ_8 state, we found that the strange quarks in ϕ_8 required an a of 1.81 GeV^{-1} for their transformation to light hadrons. Thus, while the final state would dictate (71), the initial state ϕ_8 would require $a = 1.81\ \text{GeV}^{-1}$. Here, we use (71) for a in (70) and follow (50) and take

$$f_8^\omega = (\tfrac{5.74}{1.81})^{-3/2} f_8 \ , \qquad (72)$$

where f_8 is determined now from (51).

Further, at $m_\omega{}^2 = 0.612\,15\ \text{GeV}^2$, we have

$$g^2(m_\omega{}^2) \cong 12.36 \ . \qquad (73)$$

Thus, the lone remaining parameter required for the evaluation of (70) is the parameter $F(m_\omega{}^2)$. We now turn to the evaluation of this parameter.

More specifically, by the standard methods, we have the cross section (s is the squared center of momentum energy)

$$\sigma(e^+e^- \to \pi^+\pi^-\pi^0)\,|_{s=m_\omega{}^2} = \frac{8\pi^2\alpha^2}{m_\omega{}^2} F(m_\omega{}^2) \ , \qquad (74)$$

where $\alpha \cong \frac{1}{137}$ is the fine-structure constant. Experimentally,[21] we have

$$\sigma(e^+e^- \to \pi^+\pi^-\pi^0)\,|_{s=m_\omega{}^2} \cong 1.8\ \mu\text{b} \ . \qquad (75)$$

This allows us to compute

$$F(m_\omega{}^2) \cong 0.673 \ . \qquad (76)$$

On introducing (51), (71)—(73), and (76) into (70), we find the result

$$\Gamma(\omega \to \pi^+\pi^-\pi^0) \cong 10.5\ \text{MeV} \qquad (77)$$

to be compared with the experimental value

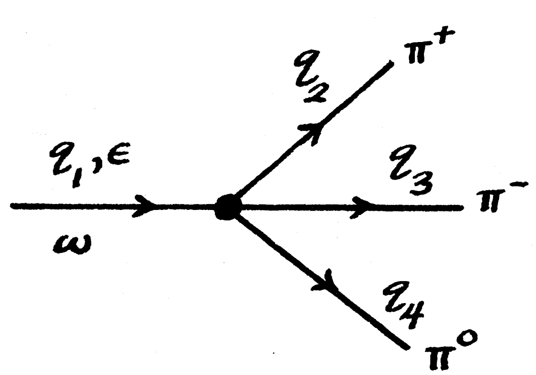

FIG. 3. The decay $\omega \to \pi^+\pi^-\pi^0$.

$$\Gamma(\omega\rightarrow\pi^{+}\pi^{-}\pi^{0})\cong 9.07 \text{ MeV} . \quad (78)$$

The agreement between the theoretical result and observation is again a reasonable agreement.

This completes our discussion of the decay $\omega\rightarrow\pi\pi\pi$. We are indeed encouraged that it, too, like $\rho\rightarrow\pi\pi$, $K^{*}\rightarrow K\pi$, and $\phi\rightarrow\bar{K}K$, lends itself to analysis in the SLAC lattice QCD theory augmented with vacuum insertion and a natural extension of Gell-Mann's work in current algebra.

IV. DISCUSSION

It is evidently a reasonable position to take that the SLAC lattice QCD theory can account for the essential aspects of the large-distance-dominated hadronic decays by the lowest-lying vector-meson nonet of ordinary SU(3) flavor symmetry. That this is true argues for the deep dynamical significance of the entire lattice framework in general and of the SLAC formulation of this framework in particular. The stage is therefore set for the further application of this SLAC theory to large-distance hadron dynamics. Such applications will be taken up elsewhere.[22]

It should be emphasized that, in all of the calculations which we have performed for the vector-meson nonet, the PCAC idea augmented with MIT bag ideas in kaonic transitions, vacuum insertion, and our variant of Gell-Mann's idea to abstract to the hadron currents the properties of the currents of quasirealistic model field theories were all necessary. Thus, operationally it appears that the lattice idea is in fact intimately related with these various techniques, which are already well tested in other areas of theoretical particle physics.

That such a relationship exists is also consistent with the recent results of the Monte Carlo approach[23] to lattice QCD. Note however that, while the work on hadronic masses in Ref. 23 is effected at finite lattice spacings, the work on the transition from weak- to strong-coupling behavior is done in the limit $a\rightarrow 0$, the continuum limit. That no blatant contradiction with observation was found by the authors in Ref. 23 then shows that, insofar as the static properties of the light hadrons are concerned, the lattice theory is a theory which describes the large-distance confining aspect of QCD and which joins on smoothly and abruptly to the weak-coupling asymptotically free[5] region of this QCD theory. Thus, the results of Ref. 23 should be viewed as complementary to the dynamical decay results obtained in I and in the present analysis. Taken together then, the results of our work and of Ref. 23 would suggest that the lattice framework is indeed a complete theory of large-distance hadron dynamics.[24] To repeat, further checks of this suggestion are in order.

We would like to recall, at this time, that the hadronic decays of the vector mesons of the SU(3) nonet have been discussed, historically, from the point of view of the "current-field identity," as used by Sakurai,[25] and the ρ-dominance type of idea as used by Gell-Mann, Sharp, and Wagner.[26] These two approaches to the vector-meson decays are not unrelated. We would note that, in general, we have obtained results which are comparable to, and in some cases better than, the results of Refs. 25 and 26 for the corresponding large-distance-dominated vector-meson decays. Thus, in this regard our results show that the SLAC lattice has in fact allowed us to improve on our understanding of these large-distance processes—quantitatively.

We would like to end this paper with the following remark. The lattice was introduced into hadron dynamics to represent the confining large-distance aspect of hadron dynamics in a gauge-invariant manner. This apparently artificial cutoff, in view of our results and the results in Ref. 2, has now been shown to describe, in addition to confinement, the actual details of large-distance light-hadron decay dynamics. Such a situation is clearly a situation of progress in our understanding of the fundamental aspects of light-hadron dynamics.

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APPENDIX: LATTICED KAON NORMALIZATION CONDITION

In this appendix, we wish to present our determination of the effective lattice constant for purely kaonic processes such as $\phi\rightarrow\bar{K}K$. This determination will be effected by using a latticed kaon normalization condition together with the PCAC and MIT bag ideas as they apply to kaons.[6,15] We understand that the application of the PCAC idea to kaons is much more of an approximation than is the application of this same idea to purely pionic processes. Nonetheless, since we are trying to determine whether the SLAC theory, as represented by (11), contains the essence of the large-distance physics in the process $\phi\rightarrow\bar{K}K$, we will be content to obtain an estimate of the appropriate lattice constant a. It should be remembered that, since our rate for $\phi\rightarrow\bar{K}K$ depends, at the least, on the fourth power of a, this rate may easily differ from the actual observed rate for this decay by a factor of 2—3 if we make an error of 25% in a. With this understanding, we will use the limit on a implied for kaons in the MIT bag model[15] as a check of our respective PCAC estimate. In this way, we may hope to obtain an appropriate value for a in kaonic transitions.

Thus, to determine the effective lattice spacing for kaonic processes, we first appeal to the normalization condition

$$\langle 0 | \psi_{\vec{0}}^{\dagger}\gamma_5 \otimes \lambda_{4-i5}\psi_{\vec{0}} | K^+ \rangle = i\sqrt{2} f_K p_0 / \sqrt{2p_0} \ , \tag{A1}$$

where f_K is the $K_{\mu\nu}$ decay parameter and will be taken to be

$$f_K \cong 0.114 \text{ GeV} \ . \tag{A2}$$

In (A1), the K^+ is at rest so that its four-momentum is $p=(m_{K^+},\vec{0})$. The notation λ_{4-i5} is an obvious notation:

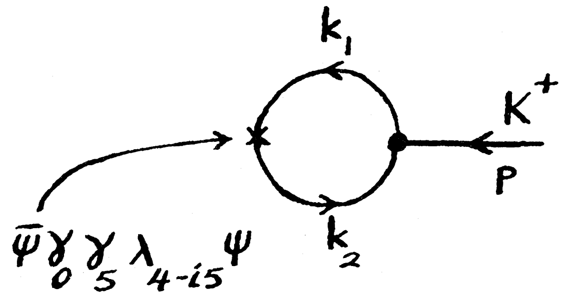

FIG. 4. PCAC equation for f_K on a lattice.

$$\lambda_{4-i5} = \lambda_4 - i\lambda_5 \ , \tag{A3}$$

where λ_a are Gell-Mann's SU(3) matrices with the normalization condition $\text{tr}(\lambda_a\lambda_b)=\frac{1}{2}\delta_{ab}$, $a,b=1,2,\ldots,8$.

To proceed, we follow our work in I and use the PCAC idea to write

$$\frac{\sqrt{2} f_K p^0 i}{(2p^0)^{1/2}} = -3\int_{\text{lattice}} \frac{d^4k_1}{2\pi} \text{tr}\left[i(\not{k}_1+m_q)\frac{1}{k_1{}^2-m_q{}^2+i\epsilon}\gamma^0\gamma_5\lambda_{4-i5} i(\not{k}_2+m_q)\frac{1}{k_2{}^2-m_q{}^2+i\epsilon} \times \gamma^0\gamma_5\lambda_{4+i5}\right] \frac{im_K{}^2}{-i\sqrt{2} f_K p_0 (2p^0)^{1/2}} \ , \tag{A4}$$

where $k_1 = k_2 + p$ and the kinematics is summarized in Fig. 4. As in I, we have defined the lattice integral by

$$\int_{\text{lattice}} d^4k_1 = \int_{-\infty}^{\infty} dk_1^0 \int_{-\pi/a}^{\pi/a} \frac{dk_1^x}{2\pi} \int_{-\pi/a}^{\pi/a} \frac{dk_1^y}{2\pi} \int_{-\pi/a}^{\pi/a} \frac{dk_1^z}{2\pi} \ . \tag{A5}$$

The restriction to latticed Fourier components is forced by the lattice current in (A1). Further, continuing with our analogy from the analysis in I, we have replaced the exact quark propagators in Fig. 4 with the effective free propagators

$$\frac{i}{\not{k}_1 - m_q + i\epsilon}, \ \frac{i}{\not{k}_2 - m_q + i\epsilon} \tag{A6}$$

with

$$m_q \equiv \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} , \tag{A7}$$

where we take[27]

$$m_u = m_d = 0.343 \text{ GeV} \ , \tag{A8}$$

$$m_s = 0.52 \text{ GeV} \ . \tag{A9}$$

Indeed, on effecting the integral over dk_1^0 in (A4), we find the result

$$2 f_K{}^2 m_K{}^2 = 3 i m_K{}^2 \mathscr{I} \ , \tag{A10}$$

where

$$\mathscr{I}=4i\int_{-\pi/a}^{\pi/a}\frac{dk_1^x}{2\pi}\int_{-\pi/a}^{\pi/a}\frac{dk_1^y}{2\pi}$$

$$\times\int_{-\pi/a}^{\pi/a}\frac{dk_1^z}{2\pi}\left\{\frac{-(\vec{k}_1^{\,2}+m_s^2)^{1/2}[-(\vec{k}_1^{\,2}+m_s^2)^{1/2}-m_K]+\vec{k}_1^{\,2}}{-2(\vec{k}_1^{\,2}+m_s^2)^{1/2}[-m_K-(\vec{k}_1^{\,2}+m_u^2)^{1/2}-(\vec{k}_1^{\,2}+m_s^2)^{1/2}][-m_K+(\vec{k}_1^{\,2}+m_u^2)^{1/2}-(\vec{k}_1^{\,2}+m_s^2)^{1/2}]}\right.$$

$$+\frac{[m_K-(\vec{k}_1^{\,2}+m_u^2)^{1/2}][-(\vec{k}_1^{\,2}+m_u^2)^{1/2}]+\vec{k}_1^{\,2}}{-2(\vec{k}_1^{\,2}+m_u^2)^{1/2}[m_K-(\vec{k}_1^{\,2}+m_u^2)^{1/2}+(\vec{k}_1^{\,2}+m_s^2)^{1/2}][m_K-(\vec{k}_1^{\,2}+m_u^2)^{1/2}-(\vec{k}_1^{\,2}+m_s^2)^{1/2}]}$$

$$-\left[\frac{m_sm_u}{-2(\vec{k}_1^{\,2}+m_s^2)^{1/2}[-m_K-(\vec{k}_1^{\,2}+m_s^2)^{1/2}-(\vec{k}_1^{\,2}+m_u^2)^{1/2}][-m_K+(\vec{k}_1^{\,2}+m_u^2)^{1/2}-(\vec{k}_1^{\,2}+m_s^2)^{1/2}]}\right.$$

$$\left.\left.+\frac{m_sm_u}{-2(\vec{k}_1^{\,2}+m_u^2)^{1/2}[m_K-(\vec{k}_1^{\,2}+m_u^2)^{1/2}+(\vec{k}_1^{\,2}+m_s^2)^{1/2}][m_K-(\vec{k}_1^{\,2}+m_u^2)^{1/2}-(\vec{k}_1^{\,2}+m_s^2)^{1/2}]}\right]\right\}. \tag{A11}$$

On making the approximation of replacing the integration volume

$$\{-\pi/a\le k_1^x\le\pi/a,\ -\pi/a\le k_1^y\le\pi/a,\ -\pi/a\le k_1^z\le\pi/a\}$$

by a sphere of the same volume, we obtain

$$2f_K{}^2=\frac{3}{2\pi^2}[I(m_K,m_u,m_s)+I(-m_K,m_s,m_u)]\ , \tag{A12}$$

where

$$I(m_K,m_u,m_s)=\left[\frac{2m_s^3}{m_K}\left[\frac{r}{2}-r^3\right]+m_s^2(r^2-\tfrac{1}{2})-\frac{rm_s^2(-m_s-m_u)}{m_K}\right]\ln|\sec\theta_0+\tan\theta_0|$$
$$+\left[\frac{2m_s^3}{m_K}(r^2-\tfrac{1}{3})+m_s^2(-r)+\frac{m_s^2(-m_s-m_u)}{m_K}\right]\tan\theta_0$$
$$+\left[\frac{2m_s^3}{m_K}(-r/2)+\frac{m_s^2}{2}\right]\frac{\sin\theta_0}{\cos^2\theta_0}+\frac{2}{3}\frac{m_s^3}{m_K}\frac{\sin\theta_0}{\cos^3\theta_0}$$
$$+\left\{\frac{2m_s^3}{m_K}[-2(r^2-r^4)]+m_s^2[2(r-r^3)]+\frac{2m_s^2(-m_s-m_u)(r^2-1)}{m_K}\right\}$$
$$\times\frac{1}{(1-r^2)^{1/2}}\tan^{-1}\left[\frac{(1-r^2)^{1/2}\tan(\theta_0/2)}{1+r}\right] \tag{A13}$$

with

$$r=\frac{m_K{}^2+m_s{}^2-m_u{}^2}{2m_Km_s} \tag{A14}$$

and

$$\theta_0=\tan^{-1}\left[\left[\frac{6}{\pi}\right]^{1/3}\frac{\pi}{am_s}\right]. \tag{A15}$$

Using an average value of

$$m_K=0.495\,669\,5\ \text{GeV} \tag{A16}$$

we find, by the Newton method, that

$$a\cong 5.75\ \text{GeV}^{-1}\ . \tag{A17}$$

This is the kaonic PCAC lattice parameter.

This value of a is very close to the pionic value of 5.74 GeV^{-1} found in I using these same latticed PCAC arguments as applied to pions. Thus, since we know that (A17) is correct for pions, we can expect that it will be at most an estimate for kaons—for PCAC is not expected to give accurate values for kaonic transitions. Thus in view of our results for the $\rho\to\pi\pi$ and $K^*\to K\pi$ widths, we are naturally invited to assess how far off (A17) is for kaons.

To do this, we appeal to a model which describes kaons with relative case while it has trouble with pions (and PCAC)—the complementary situation to our lattice PCAC calculations. We have reference to the MIT bag model.[15] In the two exemplary mass spectra worked out in Ref. 15, the kaon mass varies from 0.497 GeV when the kaonic radius is 3.26 GeV^{-1} to 0.371 GeV when this radius is 0.73 GeV^{-1}. The nonstrange-quark mass is 0 for $m_K=0.497$ GeV and is 0.108 GeV for $m_K=0.371$ GeV. The corresponding strange-quark masses are 0.279 and 0.353 GeV, respectively. Hence, from Ref. 15, we find the average strange-quark momentum p_s in the kaon in the two fits to be ~ 1.86 GeV whereas the average nonstrange quark momentum p_{ns} in these two fits is 1.74 GeV. We will use these two average values as the respective internal momenta of the kaon. The requirement that the kaon be consistent with the lattice of spacing a is simply that all momenta in the Brillouin zone $-\pi/a \le k \le \pi/a$, for a given direction in space, are essentially disjoint from the momenta which describe the internal dynamics of the kaon. Evidently, this requires

$$\frac{\pi}{a} \le 1.74\ \mathrm{GeV}\ . \tag{A18}$$

Since we wish to describe all large-distance effects associated with these kaons, we would use the equality in (A18) to find

$$a = 1.81\ \mathrm{GeV}^{-1}\ . \tag{A19}$$

As we expected, this a is within an order of magnitude of the PCAC result (A17). We argue that (A19) is to be preferred in the treatment of transitions involving kaons (and strange quarks) on the lattice—transitions such as those discussed in the text.

The result (A19) agrees with the result (44) in the text.

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$$|F_\pi(m_{K^*}^{\ 2})|^2 = |F_\pi(m_\rho^{\ 2})|^2 \times \frac{m_\rho^{\ 2}\Gamma_\rho^{\ 2}}{(m_{K^*}^{\ 2}-m_\rho^{\ 2})^2+m_\rho^{\ 2}\Gamma_\rho^{\ 2}(p_{K\pi}/p_{\pi\pi})^6 m_\rho^{\ 2}/m_{K^*}^{\ 2}}\ ,$$

where we take (Ref. 12) $\Gamma_\rho=0.158$ GeV, $m_\rho=0.776$ GeV, and $p_{\pi\pi}=(m_\rho^{\ 2}/4-m_\pi^{\ 2})^{1/2}$ with $m_\pi=0.139\,566\,9$ GeV; $p_{K\pi}$ is the magnitude of the π^0 and K^+ momenta in the K^{*+} rest frame. The use of $p_{K\pi}$ here is an effort to include the effect of the difference in rest masses between $K^+\pi^0$ and $\pi^+\pi^-$. Our value for the lattice constant a for pionic transitions (i.e., for final states such as that described by F_π) was derived in Ref. 1 from a latticed PCAC condition for pions. The value of f_8 in (28) is derived from the value of the ϕ decay constant g_ϕ defined by

$$\langle 0|J^\mu_{\mathrm{EM}}(0)|\phi\rangle = m_\phi^{\ 2}\epsilon^\mu/[g_\phi(2E)^{1/2}]\ ,$$

where ϵ^μ is the ϕ polarization vector and E is the ϕ energy. From Ref. 13, we find $|g_\phi| \cong 12.39$, considering finite-width effects,[13] so that

$$-f_8 = 3m_\phi/(\sqrt{2}g_\phi) \cong -0.175\ \mathrm{GeV}\ .$$

We use the ϕ decay constant because, like the K^*, ϕ is composed of strange quarks; neither the ρ nor the ω contains explicitly strange quarks. We feel justified in removing the finite-width effect from f_8 here because, using the simplest generalization of the results of Ref. 11 to the K^*, one expects the finite-width effects to be reduced substantially in K^* decay as compared to ϕ decay. Note that, instead of making an SU(3) transformation to the ρ channel in (21), we could have worked directly with the K-π transition matrix element. This would entail a corresponding change in a (due to the kaon) and would not change our results in an essential way. We thank Professor P. M. Zerwas for helpful discussion on this point.

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