Comments

Comments are short papers which comment on papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on derivation of anomalous Ward-Takahashi identities in the functional integration formulation of field theory

Douglas W. McKay
Department of Physics and Astronomy, University of Kansas, Lawrence, Kansas 66045

Bing-Lin Young
Ames Laborabory and Department of Physics, Iowa State University, Ames, Iowa 50011

(Received 10 January 1983)

We point out that in the calculations of the triangle anomaly there is a discrepancy between the result of the functional method applied to a chiral model by Fujikawa and the result of the conventional perturbation approach applied to this model. Possible sources for the discrepancy are briefly discussed.

Recently Fujikawa[1] showed that the Adler-Bell-Jackiw[2,3] anomaly and related field-theory phenomena can be simply treated by the functional-integration formulation of field theory, without appeal to perturbation-expansion calculations. It was subsequently shown[4] that Fujikawa's method can be used to solve the two-dimensional Schwinger model, correctly producing the gauge-boson mass term.

The key step in Fujikawa's calculation is the use of solutions to the full Dirac equation of the theory under consideration, including source terms, to expand the functional-integral measure. The measure thus expanded is generally not invariant under chiral transformation on the fermions fields, and the triangle anomaly is a result of the nontrivial Jacobian of the transformation. This situation can be contrasted with the approach in the early literature, for instance, Refs. 5 and 6, in which the fermion functional-integration measure is essentially expanded in terms of the eigenfunctions of the free Dirac equation. When fermion fields undergo chiral transformation in this latter case, the resultant Jacobian of the transformation is unity, and so only the canonical form of the axial-vector Ward-Takahashi identities (WTI's) are obtained.

Attracted by the elegance and simplicity of Fujikawa's method, and intrigued by his comment that the Pauli-Villars regularization[7] can be considered the perturbative realization of his path-integral analysis, we tried to apply it to a theory which has been regularized by the Pauli-Villars method and the results of the anomaly analysis reported in the literature.[8] In this theory, fermions are coupled to vector, axial-vector, scalar, and pseudoscalar sources, with the fermions assigned to the fundamental representation of chiral $SU(3)\times SU(3)$ and the sources to the adjoint representation. The anomalies of the WTI's of this theory are calculated in one-loop order in Ref. 8. If Fujikawa's method with a particular cutoff procedure is indeed equivalent, as is mentioned in Ref. 1, to the Pauli-Villars regularization, the set of naive anomalies listed in Table I of Ref. 8 should be reproduced.[9] We have found that Fujikawa's method applied to the theory defined by the Lagrangian

$$L=\bar{\psi}(i\partial\!\!\!/+V\!\!\!/^{a}T^{a}+A\!\!\!/^{a}\gamma_5T^{a}+S^{a}T^{a}+i\gamma_5P^{a}T^{a})\psi-m\bar{\psi}\psi \tag{1}$$

shows that the Jacobian of the $SU(3)\times SU(3)$ chiral transformation contains exactly the same types of terms as those in Table I of Ref. 8 (e.g., $\langle AVV\rangle$, $\langle APS\rangle$, etc.).

If we drop the axial-vector and pseudoscalar external sources of Eq. (1), so that the fermions couple only to the external vector and scalar sources, we find that the anomalies in this latter theory as calculated in the manner of Ref. 1 are identical to those of the theory containing vector interaction only, which agrees with the corresponding anomalies of Refs. 8 and 10.[11] It appears that including γ_5 couplings leads to differences between the method of Ref. 1 and the standard perturbation-theory approach, and we turn to a detailed discussion of a simple chiral theory considered by Fujikawa in the third article of Ref. 1 in order to reinforce this point.

The Lagrangian which we consider is given by

$$\mathscr{L}=\bar{\psi}_L(i\partial\!\!\!/+A\!\!\!/)\psi_L\ , \tag{2}$$

where we take the simple case of $U(1)$ and drop the kinetic energy term of the vector field A_μ, which is considered here to be an external source. The left-hand spinor is defined in the usual way: $\psi_L=\frac{1}{2}(1-\gamma_5)\psi$. Writing

$$j_\mu\equiv-\bar{\psi}_L\gamma_\mu\psi_L=\bar{\psi}\gamma_\mu\frac{(-1+\gamma_5)}{2}\psi\ , \tag{3}$$

Eq. (2) can be rewritten as

$$\mathscr{L}=\mathscr{L}_0-A^\mu j_\mu\ ,$$

with $\mathscr{L}_0=\bar{\psi}_Li\partial\!\!\!/\psi_L$. Following the argument of Ref. 1, one finds that the generating functional of the WTI for the divergence $\partial_\mu j^\mu$ is given by

$$0=\int D\psi_LD\bar{\psi}_L\left[\partial^\mu j_\mu(x)+\frac{1}{16\pi^2}F^{*\mu\nu}(x)F_{\mu\nu}(x)\right]\times\exp\left[i\int L_0dx'-i\int A^\mu j_\mu dx'\right]\ , \tag{4}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad F^*_{\mu\nu}=\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}\ .$$

Let us remark that all of the WTI's which follow from Eq. (4) are obtained by differentiating with respect to A_μ. In particular, functional differentiation of Eq. (4) with respect to $A_\nu(y)$ and $A_\lambda(z)$ gives, setting $A_\mu=0$,

$$0=\int D\psi_L D\bar{\psi}_L\left[-\partial^\mu j_\mu(x)j_\nu(y)j_\lambda(t)+\frac{1}{16\pi^2}\frac{\delta}{\delta A^\nu(y)}\frac{\delta}{\delta A^\lambda(z)}F^*_{\rho\sigma}F^{\rho\sigma}\right]\exp\left(i\int\mathscr{L}dx'\right)\ ,$$

which is the anomalous WTI

$$\partial^\mu\langle T(j_\mu(x)j_\nu(y)j_\lambda(z))\rangle_0=\frac{1}{16\pi^2}\frac{\delta}{\delta A^\nu(y)}\frac{\delta}{\delta A^\lambda(z)}F^*_{\rho\sigma}(x)F^{\rho\sigma}(x)\ . \tag{5}$$

The Fourier transform of the expression (5) gives the momentum-space expression of the anomalous WTI, namely,

$$k^\mu\langle T(j_\mu(k)j_\nu(p)j_\lambda(q))\rangle_0=\frac{i}{2\pi^2}\epsilon_{\nu\lambda\rho\sigma}p^\rho q^\sigma\ , \tag{6}$$

where

$$(2\pi)^4\delta(p+q+k)\langle T(j_\mu(k)j_\nu(p)j_\lambda(q))\rangle_0\equiv\int dx\,dy\,dz\exp[-i(k\cdot x+p\cdot y+q\cdot z)]\langle T(j_\mu(x)j_\nu(y)j_\lambda(z))\rangle_0\ .$$

We wish to compare Eq. (6) with the corresponding expression calculated perturbatively. Substituting $j_\mu(x)$ from Eq. (3), we expand the left-hand side of Eq. (6) as follows:

$$\begin{aligned}k^\mu\langle T(j_\mu(k)j_\nu(p)j_\lambda(q))\rangle 0=\tfrac{1}{8}k^\mu[&\langle T(j_\mu^{(A)}(k)j_\nu^{(V)}(p)j_\lambda^{(V)}(q))\rangle_0+\langle T(j_\mu^{(A)}(k)j_\nu^{(A)}(p)j_\lambda^{(A)}(q))\rangle_0\\&-\langle T(j_\mu^{(A)}(k)j_\nu^{(A)}(p)j_\lambda^{(V)}(q))\rangle_0-\langle T(j_\mu^{(A)}(k)j_\nu^{(V)}(p)j_\lambda^{(A)}(q))\rangle_0\\&-\langle T(j_\mu^{(V)}(k)j_\nu^{(V)}(p)j_\lambda^{(V)}(q))\rangle_0+\langle T(j_\mu^{(V)}(k)j_\nu^{(A)}(p)j_\lambda^{(V)}(q))\rangle_0\\&+\langle T(j_\mu^{(V)}(k)j_\nu^{(V)}(p)j_\lambda^{(A)}(q))\rangle_0-\langle T(j_\mu^{(V)}(k)j_\nu^{(A)}(p)j_\lambda^{(A)}(q))\rangle_0]\ ,\end{aligned} \tag{7}$$

where

$$j_\mu^{(V)}\equiv\bar{\psi}\gamma_\mu\psi\ \text{and}\ j_\mu^{(A)}\equiv\bar{\psi}\gamma_\mu\gamma_5\psi\ .$$

The WTI can readily be evaluated in perturbation theory by considering the three-point function represented in Fig. 1. There the vertices are given arbitrary coefficients for the vector and axial-vector couplings. The results pertinent to the Lagrangian in Eq. (2) can be read off by the appropriate choice of a_j and b_j. Using dimensional regularization and the 't Hooft-Veltman γ_5 prescription,[12] one finds

$$k^\lambda\Gamma_{\lambda\mu\nu}=\frac{i}{2\pi^2}a_1(b_2b_3+\tfrac{1}{3}a_2a_3)\epsilon_{\mu\nu\lambda\rho}p^\lambda q^\rho\ , \tag{8}$$

which verifies that the axial-vector-current divergence is anomalous, but the vector-current divergence is not. Now setting $a_1=b_1=b_2=1$ and a_2 or $a_3=0$, one obtains

$$k^\mu\langle T(j_\mu^{(A)}(k)j_\nu^{(V)}(p)j_\lambda^{(V)}(q))\rangle_0=\frac{i}{2\pi^2}\epsilon_{\nu\lambda\rho\sigma}p^\rho q^\sigma\ , \tag{9a}$$

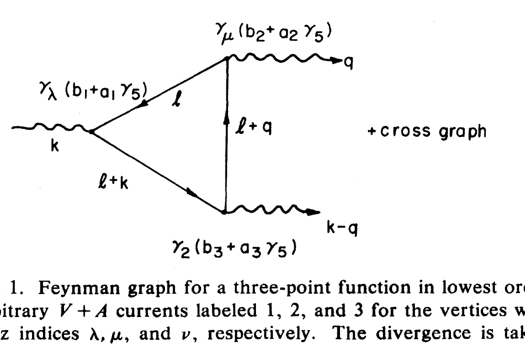

FIG. 1. Feynman graph for a three-point function in lowest order for arbitrary $V+A$ currents labeled 1, 2, and 3 for the vertices with Lorentz indices λ,μ, and ν, respectively. The divergence is taken on the λ index.

and with $a_1=a_2=a_3=1$, b_1 or $b_2=0$, one obtains

$$k^\mu\langle T(j_\mu^{(A)}(k)j_\nu^{(A)}(p)j_\lambda^{(A)}(q))\rangle_0=\frac{i}{6\pi^2}\epsilon\nu\lambda\rho\sigma p^\rho q^\sigma\ , \tag{9b}$$

results well known in perturbation theory.[6,8,10] Incidentally, the well-known ambiguity of the γ_5 matrix in n dimensions[13-15] does not cause any problems here, where the original prescription of Ref. 12 can be used unambiguously. Substituting Eqs. (9a) and (9b) into the first and second terms on the right-hand side of Eq. (7) (all the other terms are zero), we find the result

$$k^\mu\langle T(j_\mu(k)j_\nu(p)j_\lambda(q))\rangle_0=\frac{i}{12\pi^2}\epsilon_{\nu\lambda\rho\sigma}p^\rho q^\sigma\ , \tag{10}$$

which disagrees with the answer obtained by Fujikawa's method, Eq. (6). The latter calculation reproduces the anomalous WTI in the presence of a source coupled to the vector current, Eq. (9a), instead of that for the source coupled to the chiral current, which is the current defined in the theory.

As an additional check on the result (10), we use a completely different regularization method, the ϵ-splitting method described in Ref. 10. In this method, the local currents are made nonlocal by a symmetric splitting of the space-time coordinate of the bilinear currents. In the case that the vector and axial-vector currents are treated symmetrically, one obtains, following Bardeen,[10]

$$\begin{aligned}\partial^\mu j_\mu^{(V)}&=-\frac{1}{6\pi^2}F^{*\alpha\beta}F_{\alpha\beta}\ ,\\ \partial^\mu j_\mu^{(A)}&=-\frac{1}{6\pi^2}F^{*\alpha\beta}F_{\alpha\beta}\ ,\end{aligned} \tag{11}$$

for the theory of Eq. (2), which is parity violating and allows symmetric handling of $j_\mu^{(V)}$ and $j_\mu^{(A)}$. Using Eq. (11), one obtains equal contributions to the first, second, sixth, and seventh terms on the right-hand side of Eq. (7), and the result is, substituting Eq. (11) into (7),

$$k^{\mu}\langle T(j_{\mu}(k)j_{\nu}(p)jsub\lambda(q))\rangle_0=\frac{i}{12\pi^2}\epsilon_{\nu\lambda\rho\sigma}p^{\rho}q^{\sigma} \ , \tag{12}$$

in agreement with Eq. (10). When the anomalies are defined by further subtraction so that only the axial-vector current is anomalous,[10] then Eqs. (9a) and (9b) are reproduced and the result, Eq. (12), again follows, of course.

Several remarks concerning the Pauli-Villars regularization method and the anomalies are pertinent here. The naive anomalies, those present before counterterms are added, depend upon the regularization scheme employed to define the ultraviolet-divergent integrals.[11] In the Pauli-Villars[7] scheme, as in the dimensional scheme,[12] the naive canonical vector WTI's are always satisfied. In the Pauli-Villars regularization of a non-Abelian theory, the axial-vector WTI for $\langle T(j_{\mu}^{(A)}(x)j_{\nu}^{(A)}(y)j_{\lambda}^{(V)}(z))\rangle_0$ is anomalous before a suitable counterterm is introduced to cancel it.[8] The naive anomaly is proportional to a structure constant of the group so that in the present case of an Abelian theory, Eq. (2), the AAV three-point function has no naive anomalies. Therefore the anomalous divergence Eq. (10) is obtained in the Pauli-Villars scheme independently of the addition of counterterms.

Let us mention several points to consider in seeking the source of the discrepancy between the anomalous divergence equations obtained by the functional-measure argument of Fujikawa and the standard perturbation-theory calculations performed with a Pauli-Villars, dimensional regularization, or point-splitting prescription.

(1) As mentioned earlier, the results of the functional approach depend on the basis on which the integral measure is expanded. Fujikawa argues[1] that the result of the functional method, expanding the fermion field in terms of eigenfunctions of the full energy operator, directly relates the anomalous Ward-Takahashi identity to the local form of the index theorem. This suggests that the discrepancy between his result and that of perturbation theory might arise from nonperturbative effects included in the former.

(2) From the literature that is known to us,[16–19] it appears that the analytic continuation of fermion fields from Minkowski space to Euclidean space is not unique. The point is that one can choose chiral transformations of the usual sort,

$$\psi_1^{(E)}\rightarrow e^{i\alpha\gamma_5}\psi_1^{(E)}, \quad \psi_2^{(E)}\rightarrow\psi_2^{(E)}e^{i\alpha\gamma_5} \ , \tag{13}$$

if ψ and $\bar{\psi}$ are continued to independent fields $\psi_1^{(E)}$ and $\psi_2^{(E)}$, respectively, in Euclidean space.[20] On the other hand, Refs. 18 and 19 point out that when only one Euclidean four-component spinor ψ_E is defined by continuation of ψ, then the (noncompact) γ_5 transformation under which the massless theory is invariant reads

$$\psi_E\rightarrow e^{\alpha\gamma_5}\psi_E, \quad \psi_E^{\dagger}\rightarrow\psi_E^{\dagger}e^{\alpha\gamma_5} \ , \tag{14}$$

while the conventional γ_5 invariance of a massless theory does not hold in Euclidean space. The ambiguity illustrated in (13) and (14) perhaps deserves further study in relation to chiral conservation laws.

In view of the essential part played by the functional-integration formulation in studying properties of non-Abelian gauge field theories, the resolution of the question about the anomalies as calculated by Fujikawa's method compared with conventional perturbation methods would be welcome.

After completion of this work we found a recent paper by Adrianov, Bonora, and Gamboa-Saravi,[21] who have given a rigorous justification of Fujikawa's method in a pure vector theory. They too notice that additions to the usual anomalies occur for a theory with vector and axial-vector interactions. Their work has a different emphasis from ours, which is to examine the detailed differences between coefficients of anomalous WTI's in theories with extended interactions terms (S, P, V, and A) as computed by Fujikawa's functional-measure approach versus conventional perturbation theory.

We would like to thank H. Suura and H. Munczek for helpful discussions. This work was supported in part by the U.S. Department of Energy under Contracts No. W-7405-Eng-82, No. KA-01-01, and No. DE-AC02-79ER0528.

---

[1] K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); 44, 1733 (1980); Phys. Rev. D 21, 2848 (1980).

[2] S. Adler, Phys. Rev. 177, 2426 (1969); also in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser, M. Grisaru, and H. Pendelton (MIT, Cambridge, MA, 1970).

[3] J. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969); R. Jackiw, in *Lectures on Current Algebra and Its Applications,* edited by S. B. Treiman, R. Jackiw, and D. J. Gross, Princeton Series in Physics (Princeton Univ., Princeton, NJ, 1972).

[4] R. Roskies and F. Schaposnik, Phys. Rev. D 23, 558 (1981).

[5] H. Suura and B.-L. Young, Phys. Rev. D 8, 4353 (1973).

[6] D. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).

[7] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[8] R. W. Brown, C.-C. Shih, and B.-L. Young, Phys. Rev. 186, 1491 (1969).

[9] This set of anomalous WTI's includes the minimal set of Bardeen, Ref. 10, and 16 extra ones. The latter 16 anomalies can be eliminated by counterterms.

[10] W. A. Bardeen, Phys. Rev. 184, 1848 (1969).

[11] Note that in the Pauli-Villars regularization scheme, coefficients of the minimal set are not affected by the counterterms, Ref. 8.

[12] G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[13] W. A. Bardeen, R. Gastmans, and B. Lautrup, Nucl. Phys. B46, 319 (1972).

[14] M. Chanowitz, M. Furman, and I. Hinchliffe, Nucl. Phys. B159, 225 (1979).

[15] S. Gottlieb and J. T. Donohue, Phys. Rev. D 20, 3378 (1979). This article also contains a brief summary of other work on the problem of γ_5 in the general n-dimensional case.

[16] K. Osterwalder and R. Schrader, Helv. Phys. Acta 46, 277 (1973).

[17] J. Fröhlich and K. Osterwalder, Helv. Phys. Acta 47, 781 (1974).

[18] J. D. Stack, Report No. ILL-(TH)-77-11 (unpublished).

[19] P. Ramond, *Field Theory: A Modern Primer* (Benjamin-Cummings, Menlo Park, CA, 1981).

[20] This can be inferred from the treatment of ψ_L and $\bar{\psi}_L$ as independent variables in Ref. 1, where the transformations, Eq. (13), are assumed. References 16 and 17 discuss in detail the Euclidean theory formulation in terms of two independent spinors. This notion is indirectly touched on in Ref. 19 (see problem E, p. 227).

[21] A. Andrianov, L. Bonora, and R. Gamboa-Saravi, Phys. Rev. D 26, 2821 (1982).