Analysis of anomalies in higher space-time dimensions

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The anomaly for a dimension $D=2n$ gauge theory is calculated for arbitrary $n\geq 2$ from the l-agon Feynman diagram with $l=(n+1)$. The result is both finite and unique despite the nonrenormalizability of the theory. The contributions of higher polygons to the anomaly are deduced from gauge invariance and Lorentz invariance. The no-anomaly condition in $D=2n$ is the vanishing of a symmetrized trace over l generators written in the fermion representation. It is shown how to compute this trace for totally antisymmetric representations of SU(N).

I. INTRODUCTION

The triangle anomaly[1] holds a unique position in dimension $D=4$ gauge field theory since it must vanish for gauge symmetries to preserve renormalizability. The anomaly arises only from massless fermions if the gauge symmetry is exact. Conversely, the anomaly condition severely constrains the spectrum of massless fermions; in the standard low-energy $SU(3)\times SU(2)\times U(1)$ theory the quantum numbers of each family are such as to agreeably cancel the anomaly between quarks and leptons.

All interactions except gravity are apparently describable by $D=4$ gauge fields. Hence, the massless fermions appear to be unconstrained except by the triangle anomaly.

These considerations have generated interest[2] in dimensions $D>4$ because, in principle, the higher-D anomaly conditions could give further constraints on the fermion representation. We are very hesitant in invoking more than four dimensions since there is no direct experimental motivation. However, a leading candidate for a *finite* quantum gravity is provided by the superstring[3] which necessitates $D=10$. If, as appears to us unlikely, any $N=8$ supergravity should be finite in $D=4$, our motivation to consider $D>4$ would be correspondingly weakened.

Another important conceptual issue in considering $D>4$ is whether the higher dimensions are merely a mathematical device to derive an interesting $D=4$ theory which would not *necessitate* considering $D>4$ or are actual physical dimensions which are presumably compactified at a very small length scale so as to have so far escaped detection.[4] Only the latter alternative is of interest here, since otherwise higher-dimensional anomalies are surely irrelevant. Thus, we are not considering unphysical dimensions such as, e.g., the use[5] of $D=11$ as a device to arrive at $D=4$ supergravity.

Perhaps the most important point to emphasize in this introduction is that the calculations which follow are not tied to any specific notion of supergravity or supersymmetry, nor to any notion of grand unification or families. We have calculated the relevant anomalies in $D>4$ nonrenormalizable gauge theory with well-defined finite results, and without prejudice on their application.

The technical aspects of these anomaly calculations are interesting. The form of the anomaly in $D>4$ is completely constrained by considerations of differential geometry and topology. For example, in $D=10$, we must sum over $5!=120$ crossed Feynman diagrams to calculate the anomaly, but the tensor structure of the anomaly will be shown to be unique due to the fact that it can be written as the 5th Chern form of the curvature tensor for the Yang-Mills bundle.

The anomaly in $D=2n$ is controlled by the l-agon diagram with $l=(n+1)$. This is important because the polygons with $l+\Delta l$ sides with $1\leq\Delta l\leq(n-2)$ are all linearly divergent but contain uncontrolled counterterms for $\Delta l\geq 1$. The anomalous pieces are, however, required by Lorentz and gauge invariance to be directly proportional to the l-agon anomaly, as will be shown in this paper.

Why are we led to expect anomaly calculations in nonrenormalizable $D>4$ theories to give unique finite answers? The reason is simple to state as follows: Strings in $D=10$ provide cutoffs of ultraviolet divergences by massive Regge recurrences of the massless states. In the zero-slope limit, the cutoffs become infinite. But the anomaly must be independent of the Regge slope (and hence finite) because it is governed by only the massless fermions.[6]

The organization of the paper is as follows. In Sec. II, we establish notation and set up our procedure by the simple example of the $D=4$ triangle anomaly. The box anomaly in $D=6$ and the pentagon anomaly in $D=8$ are evaluated in Secs. III and IV, respectively. The hexagon in $D=10$, and generalization to the l-agon in $D=(2l-2)$ are treated in Sec. V. In Sec. VI, the consistency conditions on the anomaly for general D are derived from differential geometry and topology. Section VII illustrates how to evaluate the symmetrized traces for the case of fundamental representations of SU(N). Finally in Sec. VIII, there is some discussion. Two appendices are devoted to some generalization of Sec. VII, and to a speculative discussion of fermion families, respectively.

II. TRIANGLE ANOMALY IN $D=4$

To establish notation and demonstrate our calculational technique in a familiar setting, we first consider the triangle graph. The vertex function which is Bose symmetric is defined as

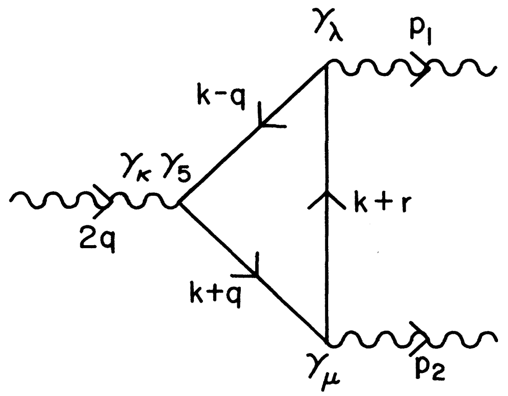

FIG. 1. Triangle diagram.

$$V_{\kappa\lambda\mu}(p_1p_2)=S\Gamma_{\kappa\lambda\mu}(p_1p_2)+\frac{1}{16\pi^2}a\epsilon_{\kappa\lambda\mu\alpha}(p_{2\alpha}-p_{1\alpha}) , \tag{2.1}$$

where S denotes the Bose symmetrization

$$S\Gamma_{\kappa\lambda\mu}(p_1p_2)=\frac{1}{2!}[\Gamma_{\kappa\lambda\mu}(p_1p_2)+\Gamma_{\kappa\mu\lambda}(p_2p_1)] \tag{2.2}$$

and $\Gamma_{\kappa\lambda\mu}(p_1p_2)$ is the Feynman amplitude for the graph of Fig. 1,

$$\Gamma_{\kappa\lambda\mu}(p_1p_2) = \int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma_\lambda(\not k-\not q)\gamma_\kappa\gamma_5(\not k+\not q)\gamma_\mu(\not k+\not r)]}{(k-q)^2(k+q)^2(k+r)^2} . \tag{2.3}$$

Here we have set the fermion mass to zero since the anomaly is mass independent. The kinematics has been chosen judiciously to facilitate generalization later. We have

$$q=\tfrac{1}{2}(p_1+p_2),\quad r=\tfrac{1}{2}(p_1-p_2) . \tag{2.4}$$

Very important is the final term in Eq. (2.1) which anticipates the linear divergence problem and is essential to ensure that the following Ward identities are satisfied:

$$p_{1\lambda}V_{\kappa\lambda\mu}=p_{2\mu}V_{\kappa\lambda\mu}=0 . \tag{2.5}$$

After imposing this requirement, we shall evaluate the anomaly X_3 in

$$2q_\kappa V_{\kappa\lambda\mu}=2^3X_3\epsilon_{\lambda\mu\alpha\beta}p_{1\alpha}p_{2\beta} . \tag{2.6}$$

It is important to realize that the asymmetry in Fig. 1 with a γ_5 at only one of the three vertices is not a genuine asymmetry. What we have in mind to calculate is a Weyl spinor circulating with coupling $\frac{1}{2}(1+\gamma_5)$ at *each* vertex. The anomaly calculated from Eq. (2.1) is obviously equivalent to this up to an overall factor 2^{-3} which we have restored in the normalization of Eq. (2.6).

One observation worth making is that

$$\Gamma_{\kappa\lambda\mu}(p_1p_2)=\Gamma_{\kappa\mu\lambda}(p_2p_1) . \tag{2.7}$$

This follows at once from Eq. (2.3) by using the change of variable $k'=-k$, anticommuting $\gamma_\kappa\gamma_5=-\gamma_5\gamma_\kappa$, and then reversing the order of factors by

$$\mathrm{Tr}(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_\epsilon\gamma_\xi)=+\mathrm{Tr}(\gamma_5\gamma_\xi\gamma_\epsilon\gamma_\delta\gamma_\gamma\gamma_\beta\gamma_\alpha) . \tag{2.8}$$

However, although the property Eq. (2.7) has generalizations equating Feynman amplitudes in pairs in higher dimensions, it will *not* play a central role in what follows.

To calculate the anomaly, we need to evaluate $p_{1\lambda}V_{\kappa\lambda\mu}=0$ to fix the coefficient a in Eq. (2.1), then compute the anomaly in Eq. (2.6) from $(2q_\kappa)V_{\kappa\lambda\mu}$.

To evaluate $p_{1\lambda}\Gamma_{\kappa\lambda\mu}$ we first put

$$\not p_1=-(\not k-\not q)+(\not k+\not r) . \tag{2.9}$$

The second term in Eq. (2.9) gives rise to a 2nd-rank pseudotensor depending on only one four-momentum and hence vanishes. The remaining term is

$$p_{1\lambda}\Gamma_{\kappa\lambda\mu}=-\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma_\kappa\gamma_5(\not k+\not q)\gamma_\mu(\not k+\not r)]}{(k+q)^2(k+r)^2} \tag{2.10}$$

$$\simeq\frac{4i}{(2\pi)^4}\int d^4k\frac{\epsilon_{\alpha\mu\beta\kappa}k_\alpha p_{2\beta}}{(k+q)^2(k+r)^2} , \tag{2.11}$$

where in Eq. (2.11) we have retained only the linearly divergent piece which will contribute the anomaly. Adding the crossed graph gives

$$p_{1\lambda}S\Gamma_{\kappa\lambda\mu}\simeq\frac{2i}{(2\pi)^4}\epsilon_{\alpha\mu\beta\kappa}\int d^4k\, k_\alpha p_{2\beta}\left[\frac{1}{(k+q)^2(k+r)^2}-\frac{1}{(k-q)^2(k-r)^2}\right] . \tag{2.12}$$

To extract the surface term in Eq. (2.12), we shift the momentum in the first term by defining $k'=(k-p_1)$. The two terms in Eq. (2.12) then cancel except for the surface term.

We then find using the generic Taylor expansion (with $k'=k+a$)

$$\int d^4k\, f(k)=\int d^4k' f(k'-a) \tag{2.13}$$

$$=\int d^4k' f(k') -a_\xi\int d^4k\frac{\partial}{\partial k_\xi}f(k)+\cdots \tag{2.14}$$

that the residual surface term in Eq. (2.12) gives

$$p_{1\lambda}S\Gamma_{\kappa\lambda\mu}=\frac{2i}{(2\pi)^4}p_{1\xi}p_{2\beta}\epsilon_{\alpha\mu\beta\kappa}\int d^4k\frac{d}{dk_\xi}\left[\frac{k_\alpha}{k^4}\right] . \tag{2.15}$$

Using $ik_0=k_4$ on going to Euclidean space and doing symmetric averaging gives

$$p_{1\lambda}S\Gamma_{\kappa\lambda\mu}=\frac{2i(-i)}{(2\pi)^4}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\mu\beta\kappa}\int d^4k_E\frac{d}{dk_\xi}\left[\frac{k_\xi}{4k^4}\right] \tag{2.16}$$

$$=\frac{1}{2(2\pi)^4}\epsilon_{\kappa\mu\alpha\beta}p_{1\alpha}p_{2\beta}\mathrm{Vol}(S^3) . \tag{2.17}$$

Since the "volume" of an n-sphere of unit radius is

$$\mathrm{Vol}(S^n)=\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \tag{2.18}$$

we have finally that

$$p_{1\lambda}S\Gamma_{\kappa\lambda\mu}=\frac{1}{16\pi^2}\epsilon_{\kappa\mu\alpha\beta}p_{1\alpha}p_{2\beta}\ . \tag{2.19}$$

Thus $p_{1\lambda}V_{\kappa\lambda\mu}$ requires $a=+1$ in Eq. (2.1). In generalizing Eq. (2.1) we shall use for $D=2n$ the coefficient

$$\frac{a}{n(2\pi)^n n!} \tag{2.20}$$

multiplying $\epsilon_{\kappa\lambda\mu\cdots\alpha\beta\gamma\cdots}(p_{2\alpha}p_{3\beta}p_{4\gamma}\cdots+\cdots)$. As we shall see, this will always lead to the condition $a=+1$ and hence provides a sign check.

Finally we need $(2q)_\kappa S\Gamma_{\kappa\lambda\mu}$. Using

$$2q\gamma_5=-(k\!\!\!/-q\!\!\!/)\gamma_5-\gamma_5(k\!\!\!/+q\!\!\!/) \tag{2.21}$$

in $\Gamma_{\kappa\lambda\mu}$ gives

$$(2q)_\kappa\Gamma_{\kappa\lambda\mu}=-\frac{1}{(2\pi)^4}\int d^4k\left\{\frac{\mathrm{Tr}[\gamma_\lambda\gamma_5(k\!\!\!/+q\!\!\!/)\gamma_\mu(k\!\!\!/+r\!\!\!/)]}{(k+q)^2(k+r)^2}+\frac{\mathrm{Tr}[\gamma_\lambda(k\!\!\!/-q\!\!\!/)\gamma_5\gamma_\mu(k\!\!\!/+r\!\!\!/)]}{(k-q)^2(k+r)^2}\right\} \tag{2.22}$$

$$\simeq\frac{4i}{(2\pi)^4}\epsilon_{\lambda\mu\alpha\beta}\int d^4k\,k_\alpha\left[\frac{p_{2\beta}}{(k+q)^2(k+r)^2}+\frac{p_{1\beta}}{(k-q)^2(k+r)^2}\right]. \tag{2.23}$$

The crossed diagram gives similarly, for the linearly divergent terms,

$$(2q)_\kappa\Gamma_{\kappa\mu\lambda}(p_2p_1)\simeq-\frac{4i}{(2\pi)^4}\epsilon_{\lambda\mu\alpha\beta}\int d^4k\,k_\alpha\left[\frac{p_{2\beta}}{(k-q)^2(k-r)^2}+\frac{p_{1\beta}}{(k+q)^2(k-r)^2}\right]. \tag{2.24}$$

Now observe that the sum of the $p_{2\beta}$ terms in Eqs. (2.23) and (2.24) are the same (within an overall sign) as those encountered in computing $p_{1\lambda}S\Gamma_{\kappa\lambda\mu}$. Likewise a momentum shift $k'=(k-p_2)$ allows us to combine the $p_{1\beta}$ terms. The result is

$$(2q)_\kappa S\Gamma_{\kappa\lambda\mu}=\frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}p_{1\alpha}p_{2\beta}\ . \tag{2.25}$$

Combining with Eqs. (2.1) where now $a=+1$ and (2.6) gives the result

$$X_3=\frac{1}{2^4\pi^2 2!}\ . \tag{2.26}$$

We have given many details of the triangle-anomaly calculation because the evaluation of the l-sided polygon in $(2l-2)$ dimensions will proceed analogously with only a few minor changes, most of which we shall summarize here.

First, since

$$\{\gamma_\alpha,\gamma_{2n+1}\}_+=0 \tag{2.27}$$

in $D=2n$ we have

$$2q\gamma_{2n+1}=-(k\!\!\!/-q\!\!\!/)\gamma_{2n+1}-\gamma_{2n+1}(k\!\!\!/+q\!\!\!/)\ , \tag{2.28}$$

where now $2q=p_1+p_2+\cdots+p_n$. For later use we also define kinematic variables

$$r_a=q-p_a\ , \tag{2.29}$$

$$s_{ab}=q-p_a-p_b\ , \tag{2.30}$$

$$t_{abc}=q-p_a-p_b-p_c\ , \tag{2.31}$$

and so on.

The four-dimensional momentum integration is replaced by

$$\int\frac{d^4k}{(2\pi)^4}\to\int\frac{d^{2n}k}{(2\pi)^{2n}}\ . \tag{2.32}$$

The symmetric averaging in $D=2n$ gives a factor of $(2n)^{-1}$ and the relevant spherical volume $\mathrm{Vol}(S^{2n-1})$ to be used in evaluating the surface term is given by Eq. (2.18). Finally, the terms in the symmetrized amplitude $S\Gamma_{\kappa\gamma\mu\cdots}(p_1p_2p_3\cdots)$ will all combine in pairs to yield surface terms when contracting with, for instance, $p_{1\lambda}$ or q_κ. Furthermore, as we shall see below, these surface terms will always *add* because of the Bose-symmetry property of the vertex function.

III. SQUARE ANOMALY IN $D=6$ (REF. 7)

In Fig. 2 a "square" Feynman diagram is depicted with external momenta $p_{a\lambda}$, $p_{b\mu}$, $p_{c\nu}$, and $(-2q_\kappa)$. (The use of a,b,c rather than 1,2,3 will provide a more convenient notation.) The amplitude for this diagram in $D=6$ is

$$\Gamma_{\kappa\lambda\mu\nu}(p_ap_bp_c)=\int\frac{d^6k}{(2\pi)^6}\frac{\mathrm{Tr}[\gamma_\lambda(k\!\!\!/-q\!\!\!/)\gamma_\kappa\gamma_7(k\!\!\!/+q\!\!\!/)\gamma_\nu(k\!\!\!/+r\!\!\!/_c)\gamma_\mu(k\!\!\!/-r\!\!\!/_a)]}{(k-q)^2(k+q)^2(k+r_c)^2(k-r_a)^2}\ . \tag{3.1}$$

The corresponding four-point vertex function is defined by

$$V_{\kappa\lambda\mu\nu}(p_ap_bp_c)=S\Gamma_{\kappa\lambda\mu\nu}(p_ap_bp_c)+\frac{a}{144\pi^3}\epsilon_{\kappa\lambda\mu\nu\alpha\beta}(p_{b\alpha}p_{c\beta}+p_{c\alpha}p_{a\beta}+p_{a\alpha}p_{b\beta}) \tag{3.2}$$

and the final term will be adjusted to give the appropriate Ward identities.

We first evaluate $p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu}$ by using

$$\not p_a = -(\not k - \not q) + (\not k - \not r_a) \ . \tag{3.3}$$

The second term in Eq. (3.3) leads to a third-rank pseudotensor depending on only two independent six-vectors and hence vanishes. The first term gives

$$p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu} = -\frac{1}{(2\pi)^6}\int d^6k \frac{\mathrm{Tr}[\gamma_\kappa\gamma_7(\not k+\not q)\gamma_\nu(\not k+\not r_c)\gamma_\mu(\not k-\not r_a)]}{D_+^{abc}(k)} \ , \tag{3.4}$$

where the denominator is written in the notation

$$D_\pm^{abc}(k) \equiv (k\pm q)^2(k+r_c)^2(k-r_a)^2 \ . \tag{3.5}$$

Using

$$\mathrm{Tr}(\gamma_{2n+1}\gamma_{\alpha^1}\gamma_{\alpha^2}\cdots\gamma_{\alpha^{2n}}) \equiv 2^n i\epsilon_{\alpha^1\alpha^2\cdots\alpha^{2n}} \tag{3.6}$$

gives for the linearly divergent term

$$p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu} \simeq -\frac{8i}{(2\pi)^6}\epsilon_{\alpha\nu\beta\mu\gamma\kappa}\int d^6k \frac{k_\alpha(-r_c r_a + q r_a + q r_c)_{\beta\gamma}}{D_+^{abc}(k)} \tag{3.7}$$

$$\simeq \frac{8i}{(2\pi)^6}\epsilon_{\alpha\nu\beta\mu\gamma\kappa}\int d^6k \frac{k_\alpha p_{b\beta}p_{c\gamma}}{D_+^{abc}(k)} \ . \tag{3.8}$$

Now we observe that shifting momentum according to

$$k' = k - p_a \tag{3.9}$$

shifts the denominator to

$$D_+^{abc}(k') = D_-^{bca}(k) = (k-q)^2(k+r_a)^2(k-r_b)^2 \ , \tag{3.10}$$

where D_-^{bca} is the denominator occurring in the contraction $p_{a\lambda}\Gamma_{\kappa\mu\nu\lambda}(p_b p_c p_a)$, since in the latter it is $(k'-q)^2$ which survives rather than $(k'+q)^2$ when one uses the identity

$$\not p_a = (\not k + \not q) - (\not k + \not r_a) \ . \tag{3.11}$$

Thus $p_{a\lambda}S\Gamma_{\kappa\lambda\mu}$ contains the partial contribution

$$p_{a\lambda}(\Gamma_{\kappa\lambda\mu\nu}+\Gamma_{\kappa\mu\nu\lambda}) \simeq \frac{8i}{(2\pi)^6}\epsilon_{\alpha\nu\beta\mu\gamma\kappa}\int d^6k\left[\frac{k_\alpha p_{b\beta}p_{c\gamma}}{D_+^{abc}(k)} - \frac{k_\alpha p_{b\beta}p_{c\gamma}}{D_-^{bca}(k)}\right] \tag{3.12}$$

$$\simeq \frac{8i}{(2\pi)^6}\epsilon_{\kappa\mu\nu\alpha\beta\gamma}p_{b\beta}p_{c\gamma}p_{a\delta}\int d^6k\frac{d}{dk_\delta}\left[\frac{k_\alpha}{k^6}\right] \tag{3.13}$$

$$= -\frac{1}{48\pi^3}\epsilon_{\kappa\mu\nu\alpha\beta\gamma}p_{a\alpha}p_{b\beta}p_{c\gamma} \ , \tag{3.14}$$

where we used $\mathrm{Vol}(S^5)=\pi^3$. Note that Eq. (3.14) is already symmetric in $p_{b\mu}$ and $p_{c\nu}$. Using this fact and noting that, because $\not p_a = (\not k + \not r_c) - (\not k - \not r_b)$,

$$p_{a\lambda}\Gamma_{\kappa\mu\lambda\nu} = p_{a\lambda}\Gamma_{\kappa\nu\lambda\mu} = 0 \tag{3.15}$$

gives eventually

$$p_{a\lambda}S\Gamma_{\kappa\lambda\mu\nu} = -\frac{1}{144\pi^3}\epsilon_{\kappa\mu\nu\alpha\beta\gamma}p_{a\alpha}p_{b\beta}p_{c\gamma} \ . \tag{3.16}$$

Thus, in Eq. (3.2), we find that $a=+1$ guarantees that $p_{a\lambda}V_{\kappa\lambda\mu\nu}=0$. Finally we need $2q_\kappa\Gamma_{\kappa\lambda\mu\nu}$ where we first use

$$2\not q\gamma_7 = -(\not k - \not q)\gamma_7 - \gamma_7(\not k + \not q) \tag{3.17}$$

giving for the linear divergence

$$2q_\kappa\Gamma_{\kappa\lambda\mu\nu} \simeq -\frac{8i}{(2\pi)^6}\int d^6k\,\epsilon_{\alpha\nu\beta\mu\gamma\lambda}\left[\frac{(-kr_c r_a - qkr_a + qr_c k)_{\alpha\beta\gamma}}{D_+^{abc}(k)} - \frac{(-kr_c r_a + qkr_a - qr_c k)_{\alpha\beta\gamma}}{D_-^{abc}(k)}\right] \tag{3.18}$$

$$= +\frac{8i}{(2\pi)^6}\int d^6k\,\epsilon_{\lambda\mu\nu\alpha\beta\gamma}k_\alpha\left[\frac{-(p_b p_c)_{\beta\gamma}}{D_+^{abc}(k)} + \frac{(p_a p_b)_{\beta\gamma}}{D_-^{abc}(k)}\right] \ . \tag{3.19}$$

Now, as we did above, we use $D_+^{abc}(k') = D_-^{bca}(k)$ where $k'=k-p_a$, and hence combine the first term in Eq. (3.19) with the "second" term in $2q_\kappa\Gamma_{\kappa\mu\nu\lambda}$ according to

$$\frac{8i}{(2\pi)^6}\int d^6k\,\epsilon_{\lambda\mu\nu\alpha\beta\gamma}k_\alpha\left[\frac{-(p_b p_c)_{\beta\gamma}}{D_+^{abc}(k)} + \frac{(p_b p_c)_{\beta\gamma}}{D_-^{bca}(k)}\right] \tag{3.20}$$

$$= -\frac{1}{24\pi^3}\epsilon_{\lambda\mu\nu\alpha\beta\gamma}p_{a\alpha}p_{b\beta}p_{c\gamma} \ . \tag{3.21}$$

Note that Eq. (3.21) has total Bose symmetry. Hence the other two pairs of terms *must* give the same answer (as we have checked explicitly) and therefore

$$2q_\kappa S\Gamma_{\kappa\lambda\mu\nu} = -\frac{1}{48\pi^3}\epsilon_{\lambda\mu\nu\alpha\beta\gamma}p_{a\alpha}p_{b\beta}p_{c\gamma} \ . \tag{3.22}$$

The anomaly defined by

$$2q_\kappa V_{\kappa\lambda\mu\nu}(p_a p_b p_c) = 2^4 X_4\epsilon_{\lambda\mu\nu\alpha\beta\gamma}p_{a\alpha}p_{b\beta}p_{c\gamma} \tag{3.23}$$

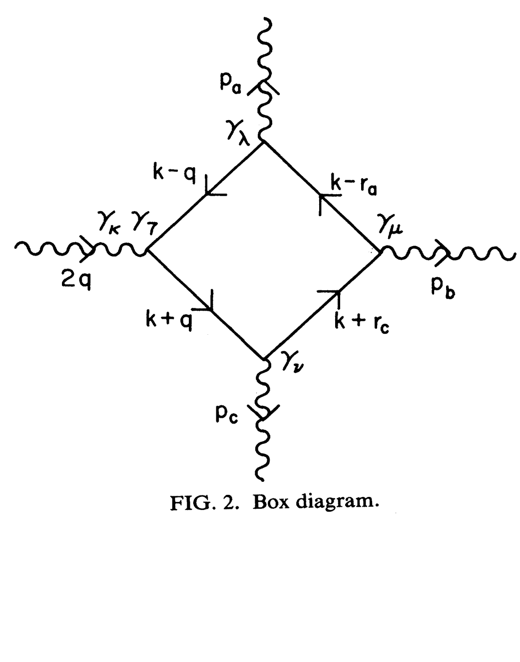

FIG. 2. Box diagram.

is thus

$$X_4 = -\frac{1}{2^6\pi^3 3!} \ . \tag{3.24}$$

IV. PENTAGON ANOMALY IN $D=8$

The diagram relevant for the anomaly in $D=8$ is the pentagon depicted in Fig. 3 for the external momenta with ordering $p_{a\lambda}$, $p_{b\mu}$, $p_{c\nu}$, $p_{d\rho}$, and $(-2q)_\kappa$. The anomaly for the Bose-symmetrized set of such diagrams is calculated in direct analogy with that for the triangle and square in $D=4$ and $D=6$, respectively. That is, we contract $S\Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d)$ first with $p_{a\lambda}$ then with $2q_\kappa$. The five-point vertex is given as

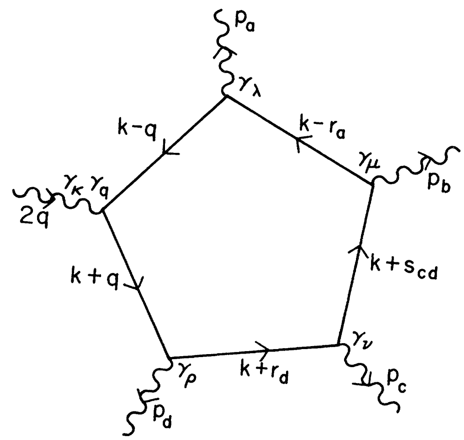

FIG. 3. Pentagon diagram.

$$V_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) = S\Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) + \frac{a}{1536\pi^4}\epsilon_{\kappa\lambda\mu\nu\rho\alpha\beta\gamma}(p_b p_c p_d - p_c p_d p_a + p_d p_a p_b - p_a p_b p_c)_{\alpha\beta\gamma} \ , \tag{4.1}$$

where the coefficient a will be adjusted (again $a=+1$, see below) such that the appropriate Ward identities hold.

The Feynman amplitude for the particular pentagon diagram in Fig. 3 is

$$\Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) = \int \frac{d^6k}{(2\pi)^6} \frac{\mathrm{Tr}[\gamma_\lambda(\not k - \not q)\gamma_\kappa\gamma_9(\not k + \not q)\gamma_\rho(\not k + \not r_d)\gamma_\nu(\not k + \not S_{cd})\gamma_\mu(\not k - \not r_q)]}{(k-q)^2(k+q)^2(k+r_d)^2(k+s_{cd})^2(k-r_a)^2} \ . \tag{4.2}$$

When $S\Gamma_{\kappa\lambda\mu\nu\rho}$ is contracted with $p_{a\lambda}$ we find, among other terms, the two contributions

$$p_{a\lambda}[\Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) + \Gamma_{\kappa\rho\lambda\mu\nu}(p_b p_c p_d p_a)] \simeq \epsilon_{\kappa\mu\nu\rho\alpha\beta\gamma\delta} p_{b\beta} p_{c\gamma} p_{d\delta} \int d^8k\, k_\alpha \left[\frac{1}{D_+^{abcd}(k)} - \frac{1}{D_-^{bcda}(k)}\right] , \tag{4.3}$$

where the denominators use the notation

$$D_\pm^{abcd}(k) = (k \pm q)(k+r_d)(k+s_{cd})(k-r_a) \tag{4.4}$$

and in Eq. (4.3) we have retained only the linearly divergent contribution. Substituting $k'=(k-p_a)$ in Eq. (4.4) gives

$$D_+^{abcd}(k') = D_-^{bcda}(k) \ , \tag{4.5}$$

so that in Eq. (4.3) only a surface term remains. We have

$$p_{a\lambda}[\Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) + \Gamma_{\kappa\rho\lambda\mu\nu}(p_b p_c p_d p_a)] = \frac{1}{384\pi^4}\epsilon_{\kappa\mu\nu\rho\alpha\beta\lambda\delta} p_{a\alpha} p_{b\beta} p_{c\lambda} p_{d\delta} \ . \tag{4.6}$$

This is one of six possible permutations of the three entities

$$(p_b, \mu; p_c, \nu; p_d, \rho) \tag{4.7}$$

that enter $p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu\rho}$. [Each permutation of bcd is both preceded and followed by a as in Eq. (4.6).] Note that any interchange of the entities in (4.7) leaves (4.6) invariant, hence all six permutations add. Also observe that when the p_a vertex is not adjacent to the q_κ vertex the contraction with $p_{a\lambda}$ vanishes. Therefore, we are left with

$$p_{a\lambda} S\Gamma_{\kappa\lambda\mu\nu\rho} = \frac{1}{1536\pi^4}\epsilon_{\kappa\mu\nu\rho\alpha\beta\lambda\delta} p_{a\alpha} p_{b\beta} p_{c\gamma} p_{d\delta} \ . \tag{4.8}$$

This requires that $a=+1$ in Eq. (4.1) such that $p_{a\lambda}V_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d)=0$.

Contracting $\Gamma_{\kappa\lambda\mu\nu\rho}$ next with $2q_\kappa$ we find

$$2q_\kappa \Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) = 16i\epsilon_{\lambda\mu\nu\rho\alpha\beta\gamma\delta} p_{b\beta} p_{c\gamma} p_{d\delta} \int \frac{d^8k}{(2\pi)^8} k_\alpha \left[\frac{1}{D_+^{abcd}(k)} - \frac{1}{D_-^{abcd}(k)}\right] \ . \tag{4.9}$$

All $2\times 4! = 48$ terms in $2q_\kappa S\Gamma_{\kappa\lambda\mu\nu\rho}$ combine in pairs to give similar expressions. Again, all are of the same sign and add. So we arrive at

$$2q_\kappa S\Gamma_{\kappa\lambda\mu\nu\rho}(p_a p_b p_c p_d) = \frac{1}{384\pi^4}\epsilon_{\lambda\mu\nu\rho\alpha\beta\gamma\delta} p_{a\alpha} p_{b\beta} p_{c\gamma} p_{d\delta} \ . \tag{4.10}$$

Substitution into Eq. (4.1) now with $a=+1$ gives for the pentagon anomaly

$$2q_\kappa V_{\kappa\lambda\mu\nu\rho}=2^5X_5\epsilon_{\lambda\mu\nu\rho\alpha\beta\gamma\delta}p_{a\alpha}p_{b\beta}p_{c\gamma}p_{d\delta} \tag{4.11}$$

with

$$X_5=+\frac{1}{2^8\pi^4 4!}\ . \tag{4.12}$$

V. HEXAGON ANOMALY IN $D=10$ (REF. 7) AND BEYOND

The six-point function in $D=10$ will be defined by

$$V_{\kappa\lambda\mu\nu\rho\sigma}(p_ap_bp_cp_dp_e)=S\Gamma_{\kappa\lambda\mu\nu\rho\sigma}(p_ap_bp_cp_dp_e)+\frac{a}{19200\pi^5}\epsilon_{\kappa\lambda\mu\nu\rho\sigma\alpha\beta\gamma\delta}(p_{b\alpha}p_{c\beta}p_{d\gamma}p_{e\delta}+p_{c\alpha}p_{d\beta}p_{e\gamma}p_{a\delta}$$
$$+p_{d\alpha}p_{e\beta}p_{a\gamma}p_{b\delta}+p_{e\alpha}p_{a\beta}p_{b\gamma}p_{c\delta}+p_{a\alpha}p_{b\beta}p_{c\gamma}p_{d\delta})\ . \tag{5.1}$$

The Feynman graph of Fig. 4 is the one contributing to $\Gamma_{\kappa\lambda\mu\nu\rho\sigma}(p_ap_bp_cp_dp_e)$. There is nothing new in the calculation of this hexagon anomaly. The relevant denominators are of the form

$$D_{\pm}^{abcde}(k)=(k\pm q)^2(k+r_e)^2(k+s_{de})^2\times(k-s_{ab})^2(k-r_a)^2 \tag{5.2}$$

and, with the substitution $k'=(k-p_a)$, one finds, e.g.,

$$D_{+}^{abcde}(k')=D_{-}^{bcdea}(k)\ , \tag{5.3}$$

the left-hand side of which enters in the computation of $p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu\rho\sigma}$. The other pair of denominators obtained from (5.3) by permuting the four 10-momenta $(p_{b\mu},p_{c\nu},p_{d\rho},p_{e\sigma})$ all contribute to the expression for $p_{a\lambda}S\Gamma_{\kappa\lambda\mu\nu\rho\sigma}$ with the same sign. The result is

$$p_{a\lambda}S\Gamma_{\kappa\lambda\mu\nu\rho\sigma}(p_ap_bp_cp_dp_e)=-\frac{1}{19200\pi^5}\epsilon_{\kappa\mu\nu\rho\sigma\alpha\beta\gamma\delta\epsilon}(p_ap_bp_cp_dp_e)_{\alpha\beta\gamma\delta\epsilon}\ . \tag{5.4}$$

Similarly all the $2\times 5!=240$ terms in $(2q_\kappa)S\Gamma_{\kappa\lambda\mu\nu\rho\sigma}$ combine to give equal-sign surface terms when we permute the five external momenta. The sum gives

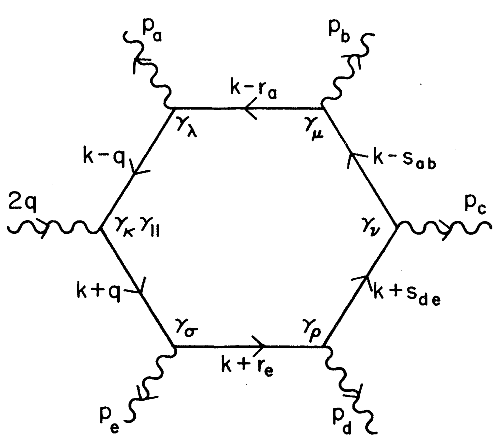

FIG. 4. Hexagon diagram.

$$2q_\kappa S\Gamma_{\kappa\lambda\mu\nu\rho\sigma}=-\frac{1}{3840\pi^5}\epsilon_{\lambda\mu\nu\rho\sigma\alpha\beta\gamma\delta\epsilon}(p_ap_bp_cp_dp_e)_{\alpha\beta\gamma\delta\epsilon}\ . \tag{5.5}$$

Equation (5.4) requires $a=+1$ in Eq. (5.1) to ensure $p_{a\lambda}V_{\kappa\lambda\mu\nu\rho\sigma}=0$. We then find from Eq. (5.5) that

$$2q_\kappa V_{\kappa\lambda\mu\nu\rho\sigma}(p_ap_bp_cp_dp_e)=2^6X_6\epsilon_{\lambda\mu\nu\rho\sigma\alpha\beta\gamma\delta\epsilon}p_{a\alpha}p_{b\beta}p_{c\gamma}p_{d\delta}p_{e\epsilon} \tag{5.6}$$

with

$$X_6=\frac{-1}{2^{10}\pi^5 5!}\ . \tag{5.7}$$

If one wishes to continue and calculate the one-loop graph associated with the anomaly in $D=2n=2l-2$ (l-agon, see Fig. 5) dimensions everything goes through as in the previous examples. The only nontrivial piece of information needed is the numerator when one contracts, e.g., $\Gamma_{\kappa\lambda_1\lambda_2\lambda_3\cdots\lambda_n}(p_1p_2\cdots p_n)$ with $(p_1)_{\lambda_1}$ which has the form

$$N^{+}_{\kappa\lambda_2\lambda_3\cdots\lambda_n}=\epsilon_{\kappa\lambda_2\lambda_3\cdots\lambda_n\alpha_1\alpha_2\cdots\alpha_n}(k+q)_{\alpha_1}\times(k+r_n)_{\alpha_2}(k+s_{n,n-1})_{\alpha_3}\cdots(k-r_1)_{\alpha_n}\ . \tag{5.8}$$

Recall that

$$(k+r_n)=k+\tfrac{1}{2}(p_1+p_2+\cdots+p_{n-1}-p_n) \tag{5.9}$$

and that $s_{n,n-1}$ has two negative signs, and so on. Going around the graph we finally reach $(k-r_1)$ which is

$$k+\tfrac{1}{2}(p_1-p_2-p_3-\cdots-p_n)\ . \tag{5.10}$$

Since the indices α_i are all contracted into the antisymmetric ϵ tensor, the term linear in k in the numerator is an n-form. Its value can conveniently be obtained by thinking of the contraction with n α indices as the $(n\times n)$ determinantal nth-rank tensor

$$(\mathrm{Det})_{\alpha_1\alpha_2\cdots\alpha_n}=(\tfrac{1}{2})^n\begin{vmatrix} (2k+p_1)_{\alpha_1} & (2k+p_1)_{\alpha_2} & \cdots & (2k+p_1)_{\alpha_{n-1}} & (2k+p_1)_{\alpha_n} \\ +p_{2\alpha_1} & +p_{2\alpha_2} & \cdots & +p_{2\alpha_{n-1}} & -p_{2\alpha_n} \\ \vdots & \vdots & \ddots & -p_{3\alpha_{n-1}} & \vdots \\ & & & \vdots & \\ & +p_{n-1\alpha_2} & & & \\ +p_{n\alpha_1} & -p_{n\alpha_2} & \cdots & -p_{n\alpha_{n-1}} & -p_{n\alpha_n} \end{vmatrix}, \tag{5.11}$$

where we have extracted a factor $\frac{1}{2}$ from each row. One finds then that the piece of Eq. (5.11) linear in k is equivalent to simply (here $[n/2]$ is the integer part of $n/2$)

$$(-1)^{[n/2]}k_{\alpha_1}p_{2\alpha_2}\cdots p_{n\alpha_n} \tag{5.12}$$

when contracted into the antisymmetric ϵ tensor. Thus

$$N^{+}_{\kappa\lambda_1\lambda_2\cdots\lambda_n}=(-1)^{[n/2]}\epsilon_{\kappa\lambda_1\lambda_2\cdots\lambda_n\alpha_1\alpha_2\cdots\alpha_n}k_{\alpha_1}p_{2\alpha_2}p_{3\alpha_3}\cdots p_{n\alpha_n}\ . \tag{5.13}$$

Similarly one finds

$$N^{-}_{\kappa\lambda_1\lambda_2\cdots\lambda_n}=\epsilon_{\kappa\lambda_1\lambda_2\cdots\lambda_n\alpha_1\alpha_2\cdots\alpha_n}(k-q)_{\alpha_1}(k+r_n)_{\alpha_2}\cdots(k-r_1)_{\alpha_n}\ . \tag{5.14}$$

The final result for the l-agon is that when we define $V_{\kappa\lambda_1\lambda_2\cdots\lambda_n}(p_1p_2\cdots p_n)$ to be Bose symmetrized and such that

$$(p_i)_{\lambda_i}V_{\kappa\lambda_1\lambda_2\cdots\lambda_n}=0 \tag{5.15}$$

for all i such that $1\le i\le n$ then

$$(2q)_\kappa V_{\kappa\lambda_1\lambda_2\cdots\lambda_n}=2^lX_l\epsilon_{\lambda_1\lambda_2\cdots\lambda_n\alpha_1\cdots\alpha_n}p_{1\alpha_1}p_{2\alpha_2}\cdots p_{n\alpha_n} \tag{5.16}$$

with

$$X_l=(-1)^n\frac{1}{2^{2n}\pi^n n!}\ . \tag{5.17}$$

This is the l-agon anomaly appropriate to $D=(2l-2)=2n$ dimensions, and is in agreement with the previous special cases we have considered.

VI. CONSISTENCY CONDITIONS[8]

We shall now study how the group-theoretic and Lorentz structure of the generalized anomalies is dictated by global differential geometric considerations. This is important because in the preceding sections we have explicitly evaluated only the lowest polygon for each D; the complete anomaly for a non-Abelian gauge group will have contributions from higher polygons but these need not be calculated separately.

Let us introduce notation[9] of p-forms and exterior derivatives in a D-dimensional manifold M. Let each point x of M have tangent space $T_x(M)$ which is a D-dimensional space with covariant basis vector $\{\partial/\partial x^i\}$ $(i=1,2,\ldots,D)$. The dual cotangent space $T_x^*(M)$ is defined with basis vectors dx^i such that $(\partial/\partial x^i,dx^j)=\delta_i^j$. A p-form is defined by

$$\omega_p(x)=f_{\alpha_1\alpha_2\cdots\alpha_p}(x)dx^{\alpha_1}\wedge dx^{\alpha_2}\cdots\wedge dx^{\alpha_p}\ . \tag{6.1}$$

Here $f_{\alpha_1\cdots\alpha_p}$ is totally antisymmetric on its p indices. The wedge product

$$dx^{\alpha_1}\wedge dx^{\alpha_2}\wedge\cdots\wedge dx^{\alpha_p} \tag{6.2}$$

denotes a contravariant antisymmetric pth-rank tensor. Examples are

$$\text{one-form: } A=A_\alpha dx^\alpha\ , \tag{6.3}$$

$$\text{two-form: } F=\tfrac{1}{2}F_{\alpha\beta}dx^\alpha\wedge dx^\beta\ . \tag{6.4}$$

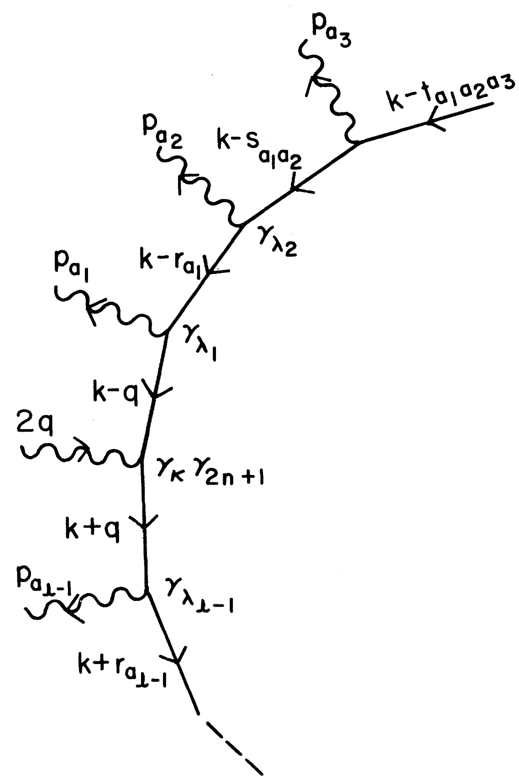

FIG. 5. General l-agon kinematics.

In general, a wedge product between a p-form and a q-form has the symmetry

$$\omega_p \wedge \omega_q = (-1)^{pq} \omega_q \wedge \omega_p \tag{6.5}$$

as can be seen by permuting the antisymmetrized indices. In particular, a wedge product of any number of two-forms is totally symmetric with respect to their interchange.

The exterior-derivative operation promotes a p-form to a $(p+1)$-form according to

$$d\omega_p = \frac{\partial f_{\alpha_1 \cdots \alpha_p}}{\partial x^{p+1}} dx^{p+1} \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} . \tag{6.6}$$

An example is to define, with $[\Lambda^a, \Lambda^b] = i f^{abc} \Lambda^c$,

$$A = A^a \left[\frac{\Lambda^a}{2i} \right] , \tag{6.7}$$

$$F = F^a \left[\frac{\Lambda^a}{2i} \right] , \tag{6.8}$$

then

$$F = dA + A \wedge A \tag{6.9}$$

is equivalent to the familiar expression

$$F^a_{\alpha\beta} = \partial_\alpha A^A_\beta - \partial_\beta A^a_\alpha + f^{abc} A^b_\alpha A^c_\beta . \tag{6.10}$$

Note that we will always set coupling strength $g=1$, since g can easily be restored by overall scaling arguments.

We need to introduce the Hodge star or duality transformation defined by

$$*\omega_p = *f_{\alpha_1\alpha_2 \cdots \alpha_p} (dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_p}) \tag{6.11}$$

$$= f_{\alpha_1\alpha_2 \cdots \alpha_p} * (dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_p}) \tag{6.12}$$

$$= f_{\alpha_1\alpha_2 \cdots \alpha_p} \frac{1}{(D-p)!} \epsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_{n-p}} \times dx^{\beta_1} \wedge dx^{\beta_2} \wedge \cdots \wedge dx^{\beta_{n-p}} . \tag{6.13}$$

Applying the Hodge star twice results in

$$**\omega_p = (-1)^{p(n-p)} \omega_p \tag{6.14}$$

through permutation of antisymmetrized indices.

In particular, when we write the equation

$$d*J = c \, \mathrm{Tr}(F \wedge F) \tag{6.15}$$

with

$$J = \tfrac{1}{2} J_\alpha dx^\alpha , \tag{6.16}$$

then it follows that

$$\partial_\mu J_\mu = c \, \mathrm{Tr}(F_{\alpha\beta} \widetilde{F}_{\alpha\beta}) \tag{6.17}$$

with

$$\widetilde{F}_{\alpha\beta} = \tfrac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta} . \tag{6.18}$$

The anomaly in $D=2n$ dimensions is determined by the highest Chern class. Using the curvature Ω given by

$$\Omega = -F \tag{6.19}$$

$$= -F^a \left[\frac{\Lambda^a}{2i} \right] \tag{6.20}$$

and where F is the two-form

$$F = \tfrac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu , \tag{6.21}$$

then the total Chern character is given by the invariant polynomial

$$c(\Omega) = \exp \left[\frac{i}{2\pi} \Omega \right] . \tag{6.22}$$

Expansion of the exponential now gives even Chern classes from 1 up to n; higher Chern classes vanish because of antisymmetry. We are interested in the sphere S^{2n-1} which has only one invariant relevant to the homotopy group for mapping, e.g., for SU(N) according to

$$\pi_{2n-1}(\mathrm{SU}(N)) = Z \tag{6.23}$$

for stable homotopy. This invariant is contained in the nth or highest Chern class in the (space-time) manifold of dimension $D=2n$, namely the D-form

$$C_n(\Omega) \approx \left[\frac{i}{2\pi} \right]^n \mathrm{Tr}(\Omega \wedge \cdots \wedge \Omega) \frac{1}{n!} \tag{6.24}$$

$$= \frac{(-1)^n}{2^{2n} \pi^n n!} (F^{a_1} \wedge F^{a_2} \wedge \cdots \wedge F^{a_n}) \times S \, \mathrm{Tr}(\Lambda^{a_1} \cdots \Lambda^{a_n}) , \tag{6.25}$$

where we have indicated a normalization suggestive of the anomalies found in previous subsections. The equality between the normalization of the nth Chern class in Eq. (6.25) and the $(n+1)$-agon anomaly quoted in Eq. (5.18) implies that one may cavalierly avoid evaluation of Feynman diagrams.

Note that in Eq. (6.25) we have exploited the fact, mentioned already, that a wedge product of two-forms is symmetric under interchange of the two-forms.

In an Abelian theory, the $(n+1)$-agon anomaly for $D=2n$ is complete and not modified by higher one-loop polygons. For a non-Abelian theory it is crucial to consider the higher polygons (this is already true for $D=4$). Nevertheless, it is unnecessary to compute the higher polygons because the anomaly is dictated by the differential geometry.

In $D=4$ we know that the axial-vector-current divergence must have the structure

$$d*J = c \, \mathrm{Tr}(F \wedge F) , \tag{6.26}$$

whereupon we may use Eq. (6.9) to rewrite

$$\mathrm{Tr}(F \wedge F) = d(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) . \tag{6.27}$$

Here we noted that $d\,dA \equiv 0$ and $A \wedge A \wedge A \wedge A \equiv 0$. Hence, up to a sign,

$$J = *c(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) . \tag{6.28}$$

Now we restore group indices, according to Eq. (6.7), to find that

$$J^a_\mu = C \epsilon_{\mu\alpha\beta\gamma} \mathrm{Tr} \Lambda^a (A_\alpha \partial_\beta A_\gamma + \tfrac{2}{3} A_\alpha A_\beta A_\gamma) \tag{6.29}$$

$$=CS\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c)\epsilon_{\mu\alpha\beta\gamma}(A^b_\alpha\partial_\beta A^c_\gamma+\tfrac{2}{3}if^{cde}A^b_\alpha A^d_\beta A^e_\gamma)\ . \tag{6.30}$$

The notation $S\,\mathrm{Tr}$ denotes the totally symmetrized trace

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\cdots\Lambda^{a_l})=\frac{1}{l!}\sum_{\substack{\text{perms}\\ \{i_k\}}}\mathrm{Tr}(\Lambda^{a_{i_1}}\Lambda^{a_{i_2}}\cdots\Lambda^{a_{i_l}}) \tag{6.31}$$

and the Λ^a are the generators of the gauge group written in an appropriate basis. Equation (6.30) is an important consistency condition[10] in $D=4$ and shows how the box diagram plays a role in the anomaly for a non-Abelian group.

In higher $D>4$, it becomes even more important to establish the consistency conditions because higher polygons are linearly divergent (unlike the box diagram in $D=4$). Thus, the effects of nonrenormalizability make explicit evaluation of, e.g., the pentagon in $D=6$, require extreme care; the consistency conditions mean that this piece of the anomaly may be deduced directly from the box in $D=6$.

We have looked at the higher even dimensions $D\geq 6$ and obtained the relevant consistency conditions.

In $D=6$, we have

$$d*J=c\,\mathrm{Tr}(F\wedge F\wedge F)\ . \tag{6.32}$$

Now we rewrite

$$F\wedge F\wedge F=(dA+A\wedge A)^3 \tag{6.33}$$

$$=d(A\wedge dA\wedge dA+A\wedge A\wedge A\wedge dA+\tfrac{3}{5}A\wedge A\wedge A\wedge A\wedge A)\ . \tag{6.34}$$

Restoration of Lorentz and group indices gives

$$\partial_\mu J^a_\mu=cS\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d)\epsilon_{\mu\alpha\beta\gamma\delta\epsilon}\partial_\mu(A^b_\alpha\partial_\beta A^c_\gamma\partial_\delta A^d_\epsilon+iA^b_\alpha A^k_\beta A^l_\gamma\partial_\delta A^d_\epsilon f^{ckl}-\tfrac{3}{5}A^b_\alpha A^k_\beta A^l_\gamma A^m_\delta A^n_\epsilon f^{ckl}f^{dmn})\ . \tag{6.35}$$

For $D=8$, the relevant eight-form is

$$F\wedge F\wedge F\wedge F=d(A\wedge dA\wedge dA\wedge dA+\tfrac{4}{3}A\wedge A\wedge A\wedge dA\wedge dA+\tfrac{6}{5}A\wedge A\wedge A\wedge A\wedge A\wedge dA+\tfrac{4}{7}A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A)\ , \tag{6.36}$$

which translates to

$$\partial_\mu J^a_\mu=S\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e)\epsilon_{\alpha\beta\gamma\delta\epsilon\phi\kappa\lambda}F^b_{\alpha\beta}F^c_{\gamma\delta}F^d_{\epsilon\phi}F^e_{\kappa\lambda} \tag{6.37}$$

$$=cS\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e)\epsilon_{\alpha\beta\gamma\delta\epsilon\phi\kappa\lambda}\partial_\alpha\left[A^b_\beta\partial_\gamma A^c_\delta\partial_\epsilon A^d_\phi\partial_\kappa A^e_\lambda+\frac{4i}{3}A^b_\beta A^f_\gamma A^g_\delta\partial_\epsilon A^d_\phi\partial_\kappa A^e_\lambda f^{cfg}-\tfrac{6}{5}A^b_\beta A^f_\gamma A^g_\delta A^h_\epsilon A^i_\phi\partial_\kappa A^e_\lambda f^{cfg}f^{dhi}\right.$$
$$\left.-\frac{4i}{7}A^b_\beta A^f_\gamma A^g_\delta A^h_\epsilon A^i_\phi A^j_\kappa A^k_\lambda f^{cfg}f^{dhi}f^{ejk}\right]\ . \tag{6.38}$$

In $D=10$,

$$F\wedge F\wedge F\wedge F\wedge F=d(A\wedge dA\wedge dA\wedge dA\wedge dA+\tfrac{5}{3}A\wedge A\wedge A\wedge dA\wedge dA\wedge dA\wedge+2A\wedge A\wedge A\wedge A\wedge A\wedge dA\wedge dA$$
$$+\tfrac{10}{7}A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge dA+\tfrac{5}{9}A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A) \tag{6.39}$$

whence

$$\partial_\mu J^a_\mu=\mathrm{Tr}[\Lambda^a(F\wedge F\wedge F\wedge F\wedge F)] \tag{6.40}$$

$$=S\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f)\epsilon_{\alpha\beta\gamma\delta\epsilon\kappa\lambda\mu\nu\rho}\partial_\alpha\left[A^b_\beta\partial_\gamma A^c_\delta\partial_\epsilon A^d_\kappa\partial_\lambda A^e_\mu\partial_\nu A^f_\rho+\frac{5i}{3}A^b_\beta A^i_\gamma A^j_\delta\partial_\epsilon A^d_\kappa\partial_\lambda A^e_\mu\partial_\nu A^f_\rho f^{cij}\right.$$
$$-2A^b_\beta A^i_\gamma A^j_\delta A^k_\epsilon A^l_\kappa\partial_\lambda A^e_\mu\partial_\nu A^f_\rho f^{cij}f^{dkl}$$
$$-\tfrac{10}{7}iA^b_\beta A^i_\gamma A^j_\delta A^k_\epsilon A^l_\kappa A^m_\lambda A^n_\mu\partial_\nu A^f_\rho f^{cij}f^{dkl}f^{emn}$$
$$\left.+\tfrac{5}{9}A^b_\beta A^i_\gamma A^j_\delta A^k_\epsilon A^l_\kappa A^m_\lambda A^n_\mu A^p_\nu A^q_\rho f^{cij}f^{dkl}f^{emn}f^{fpq}\right]\ . \tag{6.41}$$

Clearly, this procedure can be extended to give the consistency conditions for $D\geq 12$ but, at present, our motivation is mainly towards understanding $D=10$ relevant to the supersymmetric string theory.

In this subsection, we have seen that the index-free notation of differential geometry was of great practical convenience in deriving our consistency conditions.

The subject of the next subsection will be the study of the generalized no-anomaly condition which can now be cast into the form, for $D=2n$,

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\cdots\Lambda^{a_l})=0\ , \tag{6.42}$$

where $l=(n+1)$ and $\Lambda^{a_1},\Lambda^{a_2},\ldots$ are the gauge-group generators written in the fermion representation.

VII. SYMMETRIZED TRACE IDENTITIES[11]

The generalized no-anomaly condition involves a totally symmetrized trace condition. In this section we give some details of the calculation of these group-theoretic factors, such as Eq. (6.42), which involves for the l-agon

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\cdots\Lambda^{a_l})\ . \tag{7.1}$$

This enters the topological current for $D=2n=2(l-1)$. These traces are handled most easily in a diagrammatic notation (see, e.g., Ref. 12). Our discussion will center on the totally antisymmetric representations $[k]$ of SU(N) with k completely antisymmetrized tensor indices.

If Λ^a is in $[k]$ then it can be represented as

$$(\Lambda^a)^{i_1 i_2 \cdots i_k}_{i'_1 i'_2 \cdots i'_k} = \frac{1}{(k!)^2} \sum_{\substack{\{i_m\} \\ \{i'_n\}}} (-1)^P (\lambda^a)^{i_1}_{i'_1} \delta^{i_2}_{i'_2} \cdots \delta^{i_k}_{i'_k} , \tag{7.2}$$

where the sum is over permutations of the upper and lower indices with the sign (± 1) for even and odd permutations, respectively.

This may be represented diagrammatically as in Fig. 6. The product of $l\Lambda$'s is then obtained by connecting l objects of this type together. Finally, the trace in (7.1) amounts to connecting the incoming and outgoing lines in all possible ways with plus (minus) signs for each even (odd) permutation of the lines. By this time, the diagram may be drawn as a circular "target" with k rings and l spokes; this provides a convenient visualization for the trace.

Consider first $l=2$. Here there are $(k!)^2$ terms in the trace which may be divided into those where the "target" must be ringed $(k-p)$ times to pick up the two λ's and p times for the δ's; here $0 \le p \le (k-1)$. For a given p the δ trace gives the binomial coefficient

$$\begin{bmatrix} N \\ p \end{bmatrix} = \Pi^{a_1 a_2 \cdots a_k}_{a_1 a_2 \cdots a_k} , \tag{7.3}$$

where

$$\Pi^{a_1 a_2 \cdots a_k}_{b_1 b_2 \cdots b_k} = \frac{1}{k!} \sum_{\{b_i\}} (-1)^P \delta^{a_1}_{b_1} \delta^{a_2}_{b_2} \cdots \delta^{a_k}_{b_k} . \tag{7.4}$$

The λ trace gives, for $(k-p)$ connected rings,

$$(-1)^{k-p+1}(k-p)\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}) . \tag{7.5}$$

The factor $(k-p)$ arises because there are $(k-p)!$ choices at each contraction of indices, and a $[(k-p-1)!]^{-1}$ normalization. One arrives at $(k-p)^l$ in this way, but the starting point is arbitrary and one has overcounted by one factor $(k-p)$. Hence the magnitude is $(k-p)^{l-1}=(k-p)$ for $l=2$. The sign $(-1)^{k-p+1}$ arises from parity of the permutation involving the (λ^a).

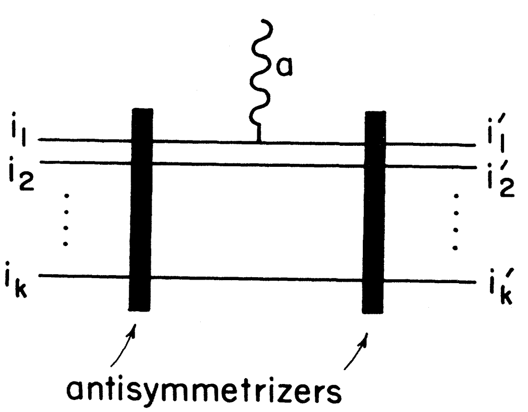

FIG. 6. Generator in basis of kth-rank antisymmetric tensor.

Hence we find

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}) = A_2(N,k) S\,\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}) \tag{7.6}$$

with

$$A_2(N,k) = \sum_{p=0}^{k} (-1)^{k-p+1}(k-p) \begin{bmatrix} N \\ p \end{bmatrix} \tag{7.7}$$

$$= \begin{bmatrix} N-2 \\ k-1 \end{bmatrix} , \tag{7.8}$$

where Eq. (7.8) is the familiar quadratic Casimir invariant and to make the step from (7.7) to (7.8) one compares coefficients of x^{k-1} on the two sides of the identity

$$(1+x)^{N-2} = (1+x)^N (1+x)^{-2} . \tag{7.9}$$

Now consider $l=3$; this gives

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\Lambda^{a_3}) = A_3(N,k) S\,\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) , \tag{7.10}$$

$$A_3(N,k) = \sum_{p=0}^{k} (-1)^{k-p+1}(k-p)^2 \begin{bmatrix} N \\ p \end{bmatrix} \tag{7.11}$$

$$= \begin{bmatrix} N-3 \\ k-1 \end{bmatrix} - \begin{bmatrix} N-3 \\ k-2 \end{bmatrix} \tag{7.12}$$

$$= \frac{(N-3)!(N-2k)}{(k-1)!(N-k-1)!} . \tag{7.13}$$

This is the well-known result for the triangle anomaly.

Our new results are for $l \ge 4$. For $l=4$ we find

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\Lambda^{a_3}\Lambda^{a_4})$$
$$= A_4(N,k) S\,\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4})$$
$$+ A_4^{2,2}(N,k) S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2})\mathrm{Tr}(\lambda^{a_3}\lambda^{a_4})] , \tag{7.14}$$

where, as before,

$$A_4(N,k) = \sum_{p=0}^{k} (-1)^{k-p+1}(k-p)^3 \begin{bmatrix} N \\ p \end{bmatrix} \tag{7.15}$$

and

$$A_4^{2,2}(N,k) = 3A_2(N-2,k-1) . \tag{7.16}$$

To derive $A_4^{2,2}(N,k)$ we note that

$$A_4^{2,2}(N,k) = 3 \sum_{p=0}^{k-2} \begin{bmatrix} N \\ p \end{bmatrix} (-1)^{k-p} \sum_{q-p}^{k-p-1} (k-p-q)q \tag{7.17}$$

as follows by considering p δ rings of the "target," q rings for one pair of λ^a, and $(k-p-q)$ rings for the other pair of λ^a. The factor 3 arises from partitioning the four λ^a into two pairs. Simple algebra then allows us to go from Eq. (7.17) to (7.16).

For $l=5$ we find

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\Lambda^{a_3}\Lambda^{a_4}\Lambda^{a_5})$$
$$=A_5(N,k)S\,\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}\lambda^{a_5})$$
$$+A_5^{3,2}(N,k)S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\mathrm{Tr}(\lambda^{a_4}\lambda^{a_5})] \quad (7.18)$$

with $A_5(N,k)$ given by

$$A_5(N,k)=\sum_{p=0}^{k-1}(-1)^{k-p+1}(k-p)^4\begin{bmatrix}N\\p\end{bmatrix} \quad (7.19)$$

and

$$A_5^{3,2}(N,k)=\frac{5!}{2!3!}\sum_{p=0}^{k-2}\begin{bmatrix}N\\p\end{bmatrix}(-1)^{k-p}\sum_{q=0}^{k-p-1}(k-p-q)q^2 \quad (7.20)$$

from p δ rings, q λ rings with 3 λ's, and $(k-p-q)$ rings with 2 λ's; again the combinatorial coefficient is from partitioning $5\rightarrow(2+3)$. It is straightforward to rewrite Eq. (7.20) as

$$A_5^{3,2}(N,k)=10A_3(N-2,k-1)\ . \quad (7.21)$$

For the hexagon $l=6$

$$S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\Lambda^{a_3}\Lambda^{a_4}\Lambda^{a_5}\Lambda^{a_6})=A_6(N,k)S\,\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}\lambda^{a_5}\lambda^{a_6})+A_6^{4,2}(N,k)S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4})\mathrm{Tr}(\lambda^{a_5}\lambda^{a_6})]$$
$$+A_6^{2,2,2}(N,k)S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2})\mathrm{Tr}(\lambda^{a_3}\lambda^{a_4})\mathrm{Tr}(\lambda^{a_5}\lambda^{a_6})]$$
$$+A_6^{3,3}(N,k)S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\mathrm{Tr}(\lambda^{a_4}\lambda^{a_5}\lambda^{a_6})]\ , \quad (7.22)$$

where

$$A_6(N,k)=\sum_{p=0}^{k-1}(-1)^{k-p+1}(k-p)^5\begin{bmatrix}N\\p\end{bmatrix} \quad (7.23)$$

and

$$A_6^{4,2}(N,k)=15\sum_{p=0}^{k-2}\begin{bmatrix}N\\p\end{bmatrix}(-1)^{k-p}\sum_{q=0}^{k-p-1}(k-p-q)q^3 \quad (7.24)$$

$$=15A_4(N-2,k-1)\ , \quad (7.25)$$

$$A_6^{2,2,2}(N,k)=15\sum_{p=0}^{k-3}\begin{bmatrix}N\\p\end{bmatrix}(-1)^{k-p}\times\sum_{q=0}^{k-p-2}q\sum_{r=0}^{k-p-q-1}r(k-p-q-r) \quad (7.26)$$

$$=15\begin{bmatrix}N-6\\k-3\end{bmatrix}\ . \quad (7.27)$$

Finally,

$$A_6^{3,3}(N,k)=10\sum_{p=0}^{k-2}\begin{bmatrix}N\\p\end{bmatrix}(-1)^{k-p}\sum_{q=0}^{k-p-1}q^2(k-p-q)^2 \quad (7.28)$$

$$=\tfrac{2}{3}A_6^{4,2}(N,k)+\tfrac{4}{3}A_6^{2,2,2}(N,k)\ . \quad (7.29)$$

The general case $l>6$ is treated in Appendix A.

VIII. SUMMARY

We have calculated the anomaly in $D=2n$ for arbitrary integer $n\geq 3$. The answer is always finite and unique despite the nonrenormalizability of the gauge theory for $D>4$.

The result is not surprising, at least for $D=10$, since there is a $D=10$ finite string theory. The ultraviolet divergences of the string theory are cutoff by the high-mass recurrences. But the hexagon anomaly receives contributions only from zero-mass fermions. Consider now the zero-slope limit. In this limit, the high-mass recurrences are removed and the resulting field theory is highly singular. Nevertheless, the zero-mass spectrum remains unaltered and hence the anomaly is independent of the Regge slope. Hence it is not surprising that the anomaly may be successfully calculated directly in the nonrenormalizable field theory.

The anomaly arises from the l-agon Feynman diagram with $l=(n+1)$. In the non-Abelian case higher polygons also contribute but these contributions have been shown to follow from gauge and Lorentz invariance, and hence need not be calculated separately. Differential geometry was useful in arriving at such consistency conditions.

The no-anomaly condition for the non-Abelian case in $D=2n=2(l-1)$ requires vanishing of a totally symmetrized group trace over l generators. In Sec. VII and Appendix A the calculation of this trace was made for arbitrary antisymmetric representations of SU(N).

In Appendix B there are speculations about the use of this no-anomaly condition for $D>4$ in constraining the massless fermion representation of a $D=4$ theory. We relegated this material to an appendix to emphasize that the calculations of the text, though motivated by such considerations, are actually independent of the choice of model.

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APPENDIX A: SYMMETRIZED TRACE FOR GENERAL l

In this appendix, we give the general form of the trace of l antisymmetric generators of the irreducible representation $[k]$ of SU(N). To do this we first define

$$A_N(k)=A_N(N,k)=(-1)^{k+1}A_{N-1,k}\ , \tag{A1}$$

where the $A_{N,k}$ are the Eulerian numbers of combinatorial analysis[13] and $A_N(N,k)$ is the generalization of Eqs. (7.19) and (7.23). That is,

$$A_l(N,k)=\sum_{p=0}^{k-1}(-1)^{k-p-1}(k-p)^{l-1}\begin{bmatrix}N\\ p\end{bmatrix}\ . \tag{A2}$$

This can be expressed in the useful form

$$A_l(N,k)=\sum_{i=1}^{l=1}A_l(i)\begin{bmatrix}N-l\\ k-i\end{bmatrix}\ . \tag{A3}$$

The coefficient of a general term in the expression for $S\,\mathrm{Tr}(\Lambda^{a_1}\cdots\Lambda^{a_l})$ is easily seen to be (in analogy with the results of Sec. VII)

$$C_{l_1 l_2\cdots l_s}\sum_{P_s=1}^{k-l}\cdots\sum_{P_2}\sum_{P_1}\begin{bmatrix}N\\ P_s\end{bmatrix}(-1)^{k-P_s}P_1^{l_1-1}P_2^{l_2-1}\cdots P_s^{l_s-1}\left[k-\sum_{i=1}^{s}P_i\right]^{l_s}\ . \tag{A4}$$

Here $C_{l_1\cdots l_n}$ is a combinatorial factor. It is equal to the number of ways of partitioning l distinguishable objects into $(l_1,l_2,\ldots,l_n)$ reduced by a factor equal to the number of equivalent changes of summation variables that can be made in (A4), where we arrange the l's such that $l_i\le l_{i+1}$. Hence

$$C_{l_1\cdots l_n}=\left[\frac{l!}{l_1!l_2!\cdots l_n!}\right]\left[\frac{1}{q_1!q_2!\cdots q_r!}\right]\ , \tag{A5}$$

where q_1 is the number of l's equal to l_1; q_2 is the number of l's equal to l_{q_1+1}, etc.

With the use of a generalization of (A3) we can write (A4) as

$$C_{l_1\cdots l_n}\sum_{i_n=1}^{l_n-1}\cdots\sum_{i_1=1}^{l_1-1}A_{l_1}(i_1)A_{l_2}(i_2)\cdots A_{l_n}(i_n)\begin{bmatrix}N-l\\ k-\sum_{\alpha=1}^{n}i_\alpha\end{bmatrix}\ . \tag{A6}$$

Our final expression for the general symmetrized trace l generator of the irrep$[k]$ is

$$\begin{aligned}S\,\mathrm{Tr}(\Lambda^{a_1}\cdots\Lambda^{a_l})=&\sum_i A_l(i)\begin{bmatrix}N-l\\ k-i\end{bmatrix}S\,\mathrm{Tr}(\lambda^{a_1}\cdots\lambda^{a_l})\\
&+\sum C_{l_1l_2}\sum_{i_2}\sum_{i_1}A_{l_1}(i_1)A_{l_2}(i_2)\begin{bmatrix}N-l\\ k-(i_1+i_2)\end{bmatrix}S[\mathrm{Tr}(\lambda^{a_1}\cdots\lambda^{a_{l_1}})\mathrm{Tr}(\lambda^{a_{l_1+1}}\cdots\lambda^{a_{l_2}})]\\
&+\cdots+\sum C_{l_1\cdots l_n}\sum_{i_n}\cdots\sum_{i_1}A_{l_i}(i_1)\cdots A_{l_n}(i_n)\begin{bmatrix}N-l\\ k-\sum_\alpha i_\alpha\end{bmatrix}\\
&\times S[\mathrm{Tr}(\lambda^{a_1}\cdots\lambda^{a_{l_1}})\cdots\mathrm{Tr}(\lambda^{a_{l_{n-1}+1}}\cdots\lambda^{a_{l_n}})]+\cdots\ .\end{aligned} \tag{A7}$$

In the sum over partitions, note that

$$\sum_{i=1}^{n}l_i=l,\quad l_1\ge 2,\ \text{and}\ l_i\le l_{i+1}\ .$$

APPENDIX B: POSSIBLE RELATIONSHIP TO QUARK-LEPTON FAMILIES

A possible motivation for the study of anomalies in higher dimensions is provided by the general idea that it might be related to the occurrence of the replication of quarks and leptons. After all, the minimal set of nontrivial fermions in $D=4$ with vanishing triangle anomaly under SU(3)×SU(2)×U(1) is similar to one family (u,d,e,ν_e).

In order to check out this idea, the results of this paper will be useful. However, to be honest, it is necessary to know the answers to additional questions which we have not yet even addressed. Here, we shall mention what those additional questions are and even guess their answers in order to hint at how things *may* evolve. At least, these speculations will provide motivation to tackle the questions of dimensional compactification.

A basic ingredient[2] is the generalized survival hypothesis: that, starting with vanishing hexagonal anomaly in $D=10$, leads after compactification to a subset of massless, or near massless, fermions in $D=4$ which have separately a vanishing hexagon anomaly. That is,

$$A_6(\rho)=0\rightarrow A_6(\rho_L)=A_6(\rho_H)=0\ , \tag{B1}$$

where the total fermion representation separates into $\rho=\rho_H+\rho_L$ (i.e., heavy and light components) as a result of dimensional compactification. In the simplest dimensional compactification schemes (e.g., on a torus) the generalized survival hypothesis is trivially satisfied since $\rho=\rho_H$ and there are no light fermions, hence no hexagon anomaly. In a physically interesting compactification, the hypothesis needs proof.

Let us assume here that this hypothesis is valid in some physically interesting dimensional compactification from $D=10$ to $D=4$ and that the light $D=4$ chiral fermions satisfy a new constraint due to $A_6(\rho_L)=0$. This leads to a second question: Does only the connected group factor (A_6) vanish or does the entire symmetrized trace $S\,\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\Lambda^{a_3}\Lambda^{a_4}\Lambda^{a_5}\Lambda^{a_6})$ vanish? The latter condition leads, in general, to four new no-anomaly conditions additional to the triangle anomaly and, as we shall see, this is very restrictive. Vanishing of only the connected part is less restrictive (being only one new no-anomaly condition), but more weakly motivated—though, this depends on the specific details of the compactification mechanism.

First, assume that the totally symmetrized trace vanish. Then we can demonstrate that for totally antisymmetric representations of SU(N), $N\geq 2$, there is no solution. Consider $N\geq 6$. Then $A_6^{2,2,2}(N,k)$ is positive definite for $3\leq k\leq(N-3)$ and vanishing otherwise. Hence, $A_6^{2,2,2}=0$ allows only $k=1$, 2, $N-2$, $N-1$. But now $A_6^{4,2}$ is positive definite for $k=2$, $N-2$ and vanishing for $k=1$, $N-1$. Hence we have only defining and antidefining representations of SU(N). But these contribute positive definitely to A_6 and are hence eliminated. If $N<6$, the tensors on the right-hand side of Eq. (7.22) become linearly dependent. For $N=5$ we find

$$
\begin{aligned}
S\,\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}\lambda^{a_5}\lambda^{a_6})
&=\tfrac{3}{4}S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4})\mathrm{Tr}(\lambda^{a_5}\lambda^{a_6})]\\
&\quad-\tfrac{1}{8}S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2})\mathrm{Tr}(\lambda^{a_3}\lambda^{a_4})\mathrm{Tr}(\lambda^{a_5}\lambda^{a_6})]\\
&\quad+\tfrac{1}{3}S[\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\mathrm{Tr}(\lambda^{a_4}\lambda^{a_5}\lambda^{a_6})]\,, \qquad \text{(B2)}
\end{aligned}
$$

so that there are only three no-anomaly conditions. The formulas of Sec. VII apply here and give[14]

	k	$A_6(5,k)$	$A_6^{4,2}(5,k)$	$A_6^{2,2,2}(5,k)$	$A_6^{3,3}(5,k)$
$\underline{5}$	1	1	0	0	0
$\underline{10}$	2	-27	15	0	$+10$
$\overline{\underline{10}}$	3	$+93$	-75	$+15$	-30
$\overline{\underline{5}}$	4	-119	90	-15	$+40$

On using Eq. (B1), a model of the form

$$a(\underline{10}+\overline{\underline{5}})+b(\underline{10}+\overline{\underline{10}})+c(\underline{5}+\overline{\underline{5}}) \qquad \text{(B3)}$$

is easily seen to be impossible for SU(5).

In SU(4), the relation (B2) is still valid and the relevant coefficients are[14]

	k	$A_6(4,k)$	$A_6^{4,2}(4,k)$	$A^{2,2,2}(4,k)$	$A_6^{3,3}(4,k)$
$\underline{4}$	1	1	0	0	0
$\underline{6}$	2	-28	15	0	10
$\overline{\underline{4}}$	3	121	-90	15	-40

After reduction using Eq. (B2), we find that $A_6^{3,3}$ is positive definite and hence cannot vanish.

In SU(3), we have only $\underline{3}$ and $\underline{3}^*$ with equal hexagon anomaly and hence no opportunity for cancellation. Finally in SU(2), only $\underline{2}$ is fully antisymmetric and it has nonvanishing hexagon anomaly.

This completes the proof that the fully symmetrized hexagon trace cannot vanish for antisymmetric representations of SU(N), any $N\geq 2$.

In view of this, let us instead assume only the connected piece $A_6(N,k)$ must vanish. The values of $A_6(N,k)$ are given by[14]

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
SU(5)	1	-27	$+93$	-119				
SU(6)	1	-26	$+66$	-26	1			
SU(7)	1	-25	40	40	-25	1		
SU(8)	1	-24	15	80	15	-24	1	
SU(9)	1	-23	-9	95	95	-9	-23	1

Let us take SU(8), as it occurs already in $N=8$ supergravity. The most general combination without triangle anomaly is

$$m(\underline{28}+4(\overline{\underline{8}})]+n[\underline{56}+5(\overline{\underline{8}})]\,. \qquad \text{(B4)}$$

This has $f=(m+2n)$ families and from the above table *connected* hexagon anomaly $A_6=(-20m+20n)$ which vanishes for $m=n$ and hence $f=3m$ is a multiple of three. To avoid residual replication (superfamilies?) f must be precisely three.

As another exemplar, slightly less motivated, consider SU(9). The general form is

$$m[\underline{36}+5(\overline{\underline{9}})]+n[\underline{84}+9(\overline{\underline{9}})]+p[(\underline{126}+5(\overline{\underline{9}})] \qquad \text{(B5)}$$

with $f=(m+3n+2p)$ families. The above table gives for the connected part $A_6=-2(9m-50p)$, independent of n. The simplest possibility is then $m=p=0$, $n=1$ giving the three-family model considered in Ref. 15.

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