Poisson brackets of the constraints in the Hamiltonian formulation of tetrad gravity

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The Poisson brackets of the constraints are explicitly computed for various equivalent forms of canonical gravity with tetrads. This study sheds useful insights on the supergravity "algebra" and is necessary for the Hamiltonian formulation of the path integral. The shortcomings of a method devised some time ago for this kind of calculation are clarified.

The quantization in an arbitrary gauge of a constrained Hamiltonian system with first-class constraints requires the explicit knowledge, off the constraint hypersurface, of the "structure functions" of the theory.[1]

It is the purpose of this report to give a detailed calculation of these structure functions in the case of tetrad gravity, described by the canonical action[2]

$$S[h_{(\lambda)a},\pi^{(\lambda)a},N,N^k,\lambda^{(\lambda)(\mu)}] = \int dx^0 \int d^3x\,(\pi^{(\lambda)a}h_{(\lambda)a,0}-N\mathcal{H} - N^k\mathcal{H}_k-\lambda^{(\lambda)(\mu)}\mathcal{H}_{(\lambda)(\mu)}) \tag{1}$$

with

$$\mathcal{H}=G_{abcd}\pi^{ab}\pi^{cd}+\sigma Rg \ , \tag{2}$$

$$\mathcal{H}_k=-2\pi_k{}^m{}_{|m} \ , \tag{3}$$

$$\mathcal{H}_{(\lambda)(\mu)}=\tfrac{1}{2}(\pi_{(\lambda)}{}^k h_{(\mu)k}-\pi_{(\mu)}{}^k h_{(\lambda)k}) \ . \tag{4}$$

Here, G_{abcd} is DeWitt's supermetric rescaled, for later convenience, by the factor $g^{1/2}$ and σ is the Hamiltonian signature of spacetime.[3] The Lagrange multipliers N and N^k are the usual lapse (rescaled by $g^{-1/2}$) and shift, whereas the six multipliers $\lambda^{(\lambda)(\mu)} = -\lambda^{(\mu)(\lambda)}$ are associated with the possibility of performing arbitrary local Lorentz transformations on the tetrads $h_{(\lambda)a}$ in the course of the evolution (local indices are put in parentheses; Greek indices run from 0 to 3; Latin indices take on the values 1, 2, 3).

To our knowledge, the constraint "algebras" computed in the literature are incomplete—they are only weakly valid—and cannot be used to quantize the theory when the local Lorentz gauge is fixed by means of noncanonical gauge conditions (such as $\lambda^{(\lambda)(\mu)}=0$ or $\dot\lambda^{(\lambda)(\mu)}=0$). The need for such gauges has been advocated recently on practical grounds, when coupling to matter is considered.[4] It can also be understood geometrically, since there is no way to freeze the Lorentz gauge by canonical conditions $\phi^A(h_{(\lambda)a})=0$ which are simultaneously (i) local (i.e., the functions ϕ^A do not involve tetrad derivatives) and (ii) coordinate invariant, i.e., such that the tetrads obeying the gauge conditions $\phi^A(h_{(\lambda)a})=0$ in one coordinate system also obey these conditions in all other coordinate systems (at a given point, there is no preferred direction that can be determined by local means).

The point which has been overlooked in the literature is that the Poisson brackets of the metric g_{ab} and the "metric momenta" π^{ab},

$$\pi^{ab}=\tfrac{1}{4}(\pi^{(\lambda)a}h_{(\lambda)}{}^b+\pi^{(\lambda)b}h_{(\lambda)}{}^a) \ , \tag{5}$$

do not have the standard canonical values. A straightforward calculation indeed yields

$$[g_{ab}(x),g_{cd}(x')]=0 \ , \tag{6a}$$

$$[g_{ab}(x),\pi^{cd}(x')]=\tfrac{1}{2}(\delta_a^c\delta_b^d+\delta_a^d\delta_b^c)\delta(x,x') \ , \tag{6b}$$

as in ordinary metric gravity. But the brackets $[\pi^{ab}(x),\pi^{cd}(x')]$ only vanish weakly,[5]

$$[\pi^{ab}(x),\pi^{cd}(x')]=\mu^{abcd}\delta(x,x') \ , \tag{6c}$$

with

$$\mu^{abcd}=\tfrac{1}{8}(g^{ac}\mathcal{H}^{bd}+g^{ad}\mathcal{H}^{bc}+g^{bc}\mathcal{H}^{ad}+g^{bd}\mathcal{H}^{ac}) \tag{6d}$$

and

$$\mathcal{H}^{ab}=h_{(\lambda)}{}^a h_{(\mu)}{}^b\mathcal{H}^{(\lambda)(\mu)} \ . \tag{6e}$$

If coupling to nonzero-spin fields (described by their tetrad components) is included, the brackets (6c) do not even vanish.[6]

As a result, the Poisson brackets of the functions (2) and (3) differ from their metric gravity values by terms proportional to the new generators $\mathcal{H}_{(\lambda)(\mu)}$. In order to evaluate these modifications, we note that any two functionals F and G of the variables g_{ab} and π^{ab} obey

$$[F,G]=[F,G]_{\rm MG}+[F,G]_{\rm mod} \ .$$

Here, the bracket $[\,,\,]_{MG}$ is the metric gravity bracket, obtained by setting the right-hand side of (6c) equal to zero, and the symbol $[\,,\,]_{mod}$ stands for the modification of the bracket due to the nonvanishing of $[\pi^{ab},\pi^{cd}]$. It can be computed by setting the right-hand side of (6b) equal to zero.

Applied to the constraints $\mathcal{H}$ and $\mathcal{H}_k$, this rule yields[5]

$$[\mathcal{H}(x),\mathcal{H}(x')]=-\sigma[g(x)\mathcal{H}^k(x)+g(x')\mathcal{H}^k(x')]\delta_{,k}(x,x') \ , \tag{7a}$$

$$[\mathcal{H}_k(x),\mathcal{H}(x')]=[\mathcal{H}(x)+\mathcal{H}(x')]\delta_{,k}(x,x')+[\lambda_k{}^p\delta(x,x')]_{|p} \ , \tag{7b}$$

$$[\mathcal{H}_k(x),\mathcal{H}_m(x')]=\mathcal{H}_k(x')\delta_{,m}(x,x')+\mathcal{H}_m(x)\delta_{,k}(x,x')+4[\mu^a{}_{km'}{}^{b'}\delta(x,x')]_{|b'|a} \ , \tag{7c}$$

where the functions $\lambda_k{}^p$ are defined by

$$\lambda_k{}^p=-4G_{rsmn}\pi^{mn}\mu_k{}^{prs} \ . \tag{7d}$$

In the evaluation of the second covariant derivative in (7c), the indices a and k correspond to the point x, whereas m and b correspond to x'. Besides, there is a unit density weight at both x and x'. A similar prescription holds for (7b).

The remaining part of the constraint algebra is given by

$$[\mathcal{H}(x),\mathcal{H}_{(\rho)(\sigma)}(x')]=0 \ , \tag{8a}$$

$$[\mathcal{H}_k(x),\mathcal{H}_{(\rho)(\sigma)}(x')]=0 \ , \tag{8b}$$

$$[\mathcal{H}_{(\lambda)(\mu)}(x),\mathcal{H}_{(\rho)(\sigma)}(x')] =\tfrac{1}{2}(-\eta_{(\lambda)(\rho)}\mathcal{H}_{(\mu)(\sigma)}+\eta_{(\mu)(\rho)}\mathcal{H}_{(\lambda)(\sigma)} -\eta_{(\mu)(\sigma)}\mathcal{H}_{(\lambda)(\rho)}+\eta_{(\lambda)(\sigma)}\mathcal{H}_{(\mu)(\rho)})\delta(x,x') \tag{8c}$$

since $\mathcal{H}_{(\rho)(\sigma)}$ generates local Lorentz transformations and since $\mathcal{H}$ and $\mathcal{H}_k$ are both Lorentz invariant.

As pointed out in Ref. 5, there is a good geometrical reason why terms proportional to $\mathcal{H}_{(\lambda)(\mu)}$ should appear in the algebra (7). Indeed, the spacetime generators $\mathcal{H}$ and $\mathcal{H}_k$, which involve the momenta $\pi^{(\lambda)a}$ only through the symmetric combination π^{ab}, transport the tetrads $h_{(\lambda)a}$ according to the law

$$\delta h_{(\lambda)a}=\tfrac{1}{2}h_{(\lambda)}{}^b\delta g_{ab} \ . \tag{9}$$

Now, it is a key fact that this law is not integrable: Two successive variations $\delta_1 g_{ab}$ and $\delta_2 g_{ab}$ will not in general lead to the same final tetrad when performed in reverse order. The difference is an easily computed Lorentz rotation, which straightforwardly leads to (7) as a result of the argument developed in Ref. 7.

This suggests replacing $\mathcal{H}_k$ by $\bar{\mathcal{H}}_k$, the generators of ordinary Lie derivatives,

$$\bar{\mathcal{H}}_k=\pi^{(\lambda)m}h_{(\lambda)m,k}-(\pi^{(\lambda)m}h_{(\lambda)k})_{,m} \ . \tag{10}$$

Since $\bar{\mathcal{H}}_k$ and $\mathcal{H}_k$ differ by a combination of the constraints $\mathcal{H}_{(\lambda)(\mu)}$, this change is permissible. The new generators $\bar{\mathcal{H}}_k$ obey the algebra characteristic of the diffeomorphism group,

$$[\bar{\mathcal{H}}_k(x),\bar{\mathcal{H}}_m(x')]=\bar{\mathcal{H}}_k(x')\delta_{,m}(x,x')+\bar{\mathcal{H}}_m(x)\delta_{,k}(x,x'), \tag{7c'}$$

and are such that

$$[\bar{\mathcal{H}}_k(x),\mathcal{H}(x')]=[\mathcal{H}(x)+\mathcal{H}(x')]\delta_{,k}(x,x') \ , \tag{7b'}$$

$$[\bar{\mathcal{H}}_k(x),\mathcal{H}_{(\lambda)(\mu)}(x')]=\mathcal{H}_{(\lambda)(\mu)}(x)\delta_{,k}(x,x') \tag{8b'}$$

(the ordinary Lie derivative $\mathcal{L}_\xi^{\text{ord}}h_{(\lambda)a}$, although associated with an integrable transport law, is not Lorentz covariant; this is why $\bar{\mathcal{H}}_k$ and $\mathcal{H}_{(\lambda)(\mu)}$ do not commute). Besides, one easily finds

$$[\mathcal{H}(x),\mathcal{H}(x')]=-\sigma\{g(x)[\bar{\mathcal{H}}^k(x)+\mathcal{H}^{km}{}_{|m}(x)+\omega^{(\lambda)(\mu)k}(x)\mathcal{H}_{(\lambda)(\mu)}(x)] +g(x')[\bar{\mathcal{H}}^k(x')+\mathcal{H}^{km}{}_{|m}(x')+\omega^{(\lambda)(\mu)k}(x')\mathcal{H}_{(\lambda)(\mu)}(x')]\}\delta_{,k}(x,x') \ , \tag{7a'}$$

where $\omega_{(\lambda)(\mu)k}$ is the three-dimensional connection defined in Ref. 8.

The set of generators $(\mathcal{H},\bar{\mathcal{H}}_k,\mathcal{H}_{(\lambda)(\mu)})$ possesses two distinct advantages: (i) In the limit of zero signature ($\sigma=0$), its algebra becomes a true algebra (the structure coefficients no longer involve the fields); and (ii) it is of rank one according to the classification of Ref. 1 (the higher-order structure functions all vanish). This second property, which the reader can easily check, singles out this set of generators as the best starting point for the quantization.[9]

Another set of constraints which has been considered in the literature for geometrical reasons is defined as follows. If one expresses the Lagrange multipliers associated with the primary constraints $\mathcal{H}_{(\lambda)(\mu)}$ in terms of the tetrads and their derivatives by means of the first Hamiltonian equations (i.e., the equations $h_{(\lambda)a,0}=[h_{(\lambda)a},H]$), one finds[5]

$$\lambda_{(\lambda)(\mu)}=\tfrac{1}{2}[h_{(\lambda)a,0}h_{(\mu)}{}^a-n_{(\lambda),0}n_{(\mu)}-(\lambda\leftrightarrow\mu)] \ , \tag{11}$$

where $n_{(\lambda)}$ are the tetrad components of the normal to the hypersurfaces $x^0=\text{const}$. These multipliers are thus not equal to the zero components of the four-dimensional connection ${}^{(4)}\omega_{(\lambda)(\mu)0}$ and do not possess a direct geometrical meaning. However, by a redefinition of the generators $\mathcal{H}$ and $\mathcal{H}_k$, one can rewrite the canonical action (1) as

$$S[h_{(\lambda)a},\pi^{(\lambda)a},N,N^k,\omega^{(\lambda)(\mu)}] = \int dx^0 \int d^3x\,[\pi^{(\lambda)a}h_{(\lambda)a,0} - N\mathcal{H}' - N^k\mathcal{H}'_k - \omega^{(\lambda)(\mu)}\mathcal{H}_{(\lambda)(\mu)}] \tag{12a}$$

in such a way that the new multipliers $\omega^{(\lambda)(\mu)}$, when expressed in terms of the tetrads, their derivatives, the lapse and the shift by means of the first Hamiltonian equations, are precisely equal to ${}^{(4)}\omega^{(\lambda)(\mu)}{}_0$.

The link between $\mathcal{H}$ and $\mathcal{H}'$ on the one hand, and $\mathcal{H}_k$ and $\mathcal{H}'_k$ on the other hand, is well known,[8,10]

$$\mathcal{H}' = \mathcal{H} + 2\mathcal{H}^{\perp k}{}_{,k}\ , \tag{12b}$$

$$\mathcal{H}'_k = \mathcal{H}_k + \mathcal{H}^m{}_{k|m} - 2g^{-1/2}(\pi_{km} - \tfrac{1}{2}\pi g_{km})\mathcal{H}^{\perp m}\ . \tag{12c}$$

Here $\mathcal{H}^{\perp m}$ is equal to $-n_{(\lambda)}h_{(\mu)}{}^m\mathcal{H}^{(\lambda)(\mu)}$. It is noteworthy that the term added to $\mathcal{H}$ in (12b) is such that the tetrads remain fixed in a tilting of the hypersurfaces $x^0 = \text{const}$ (see Ref. 11 for a definition of these transformations).

A lengthy calculation leads to the following algebra,

$$[\mathcal{H}'(x),\mathcal{H}'(x')] = 0\ , \tag{7a''}$$

$$[\mathcal{H}'_k(x),\mathcal{H}'(x')] = [\mathcal{H}'(x)+\mathcal{H}'(x')]\delta_{,k}(x,x') - {}^{(4)}R_{k\perp(\lambda)(\mu)}g^{1/2}\mathcal{H}^{(\lambda)(\mu)}\delta(x,x') + \tfrac{1}{2}\mathcal{H}^{\perp}{}_n(x)\mathcal{H}^{\perp n}(x)\delta_{,k}(x,x')\ , \tag{7b''}$$

$$[\mathcal{H}'_k(x),\mathcal{H}'_m(x')] = \mathcal{H}'_m(x)\delta_{,k}(x,x') + \mathcal{H}'_k(x')\delta_{,m}(x,x') - {}^{(4)}R_{km(\lambda)(\mu)}\mathcal{H}^{(\lambda)(\mu)}\delta(x,x') + 2g^{-1/2}(x)g_{km}(x)\mathcal{H}^{\perp}_p(x)\mathcal{H}^{pn}(x)\delta_{,m}(x,x')\ , \tag{7c''}$$

where ${}^{(4)}R_{\alpha\beta(\lambda)(\mu)}$ reduce, on the constraint hypersurface, to the components of the four-dimensional Riemann tensor, expressed in terms of g_{ab} and π^{ab} as in Ref. 11 (their explicit form will not be needed here). Moreover, $\mathcal{H}'$ and $\mathcal{H}'_k$ both commute with the Lorentz generators $\mathcal{H}_{(\lambda)(\mu)}$ since they are Lorentz invariant.

The striking feature of the Poisson brackets (7b″) and (7c″) is that they differ from the constraint algebra computed in Refs. 8 and 12 by terms containing the squares of the constraints $\mathcal{H}_{(\lambda)(\mu)}$. This latter algebra is thus incorrect and only yields the structure functions on the constraint hypersurface. In order to understand the origin of the discrepancy, we have to analyze carefully the method used by the abovementioned authors. That method is extremely economical in providing important information on the constraint algebra. However, it suffers from one serious drawback: As we shall show, it is completely unable to give any information on the squared terms $\mathcal{H}_{(\lambda)(\mu)}\mathcal{H}$, $\mathcal{H}_{(\lambda)(\mu)}\mathcal{H}_k$, $\mathcal{H}_{(\lambda)(\mu)}\mathcal{H}_{(\rho)(\sigma)}$ and, for that reason, must be supplemented by additional considerations.

We shall consider from now on more general systems described by the following canonical action,

$$S_H[q^i,p_i,y^\alpha;\lambda^A,u^a] = \int dx^0[p_i\dot{q}^i + a_\alpha(q,y)\dot{y}^\alpha - \lambda^A\chi_A(q,p,y) - u^a\varphi_a(q,p,y)]\ . \tag{13}$$

We shall assume that the constraints $\chi_A \approx 0$ and $\varphi_a \approx 0$—referred to below as the secondary and primary constraints, respectively—are first class,

$$[\psi_\Delta,\psi_\Sigma] = C^\Gamma{}_{\Delta\Sigma}\psi_\Gamma \tag{14}$$

[the compact notation $\psi_\Delta \equiv (\chi_A,\varphi_a)$ has been used], and that the variational equations $\delta S_H/\delta p_i = 0$, $\delta S_H/\delta u^a = \varphi^a = 0$ can be solved for the momenta p_i and the multipliers u^a:

$$\frac{\delta S_H}{\delta p_i} = 0,\ \frac{\delta S_H}{\delta u^a} = 0 \Leftrightarrow \begin{cases} p_i = P_i(q,\dot{q},y,\lambda) \\ u^a = U^a(q,\dot{q},y,\lambda) \end{cases} \tag{15}$$

("Legendre transformation"). Pure gravity belongs to that class of models ($h_{(\lambda)a} \equiv q^i$; $\pi^{(\lambda)a} \equiv p_i$; $N,N^k \equiv \lambda^A$; $\lambda^{(\lambda)(\mu)} \equiv u^a$; no y^α) as well as gravity coupled to matter and supergravity.

If one replaces in the action S_H the variables p_i and u^a by the functions P_i and U^a, one gets the "Lagrangian action"

$$S_L[q^i,y^\alpha;\lambda^A] = \int L(q^i,\dot{q}^i,y^\alpha,\dot{y}^\alpha,\lambda^A)dx^0$$

which yields, as in well known, variational equations equivalent to those implied by S_H.

Now, as has been noticed in Ref. 12, the action S_H is invariant under the transformations

$$\delta_H q^i = \epsilon^\Delta[q^i,\psi_\Delta],\quad \delta_H p_i = \epsilon^\Delta[p_i,\psi_\Delta]\ ,$$
$$\delta_H y^\alpha = \epsilon^\Delta[y^\alpha,\psi_\Delta]\ , \tag{16a}$$
$$\delta_H\mu^\Delta = \dot{\epsilon}^\Delta + C^\Delta{}_{\Sigma\Gamma}\epsilon^\Sigma\mu^\Gamma$$

[$\mu^\Delta \equiv (\lambda^A,u^a)$]. This implies that the action S_L is invariant under

$$\delta_L q^i = \epsilon^\Delta[q^i,\psi_\Delta],\quad \delta_L y^\alpha = \epsilon^\Delta[y^\alpha,\psi_\Delta]\ ,$$
$$\delta_L\lambda^A = \dot{\epsilon}^A + C^A{}_{\Sigma\Gamma}\epsilon^\Sigma\mu^\Gamma\ , \tag{16b}$$

where the right-hand sides are evaluated "on shell," i.e., on the surface where (15) holds.

It so happens that the action S_L is in general the starting point of the theory and that all its invariances are known in closed form. Thus, (16b) must be one of these. This enables one to determine $C^A{}_{\Delta\Sigma}$ on shell.[13] By the relation (15)—which establishes a bijective correspondence between the velocities $\dot{q}^i$ on the one hand and the variables p_i [with $\varphi_a(q,p,y)=0$] and u^a on the other hand—the structure functions $C^A{}_{\Delta\Sigma}$ are determined as functions of q^i,p_i,y^α on the surface $\varphi_a(q,p,y)=0$. The method thus provides $C^A{}_{\Delta\Sigma}$ in the whole phase space (q,p,y) up to combinations of the primary constraints φ_a.

As to the other structure functions, they can be found by evaluating the variation $\delta_L U^a$ of the multipliers u^a viewed as functions of the Lagrangian variables. From the definition (15) of the functions P_i and U^a, one easily derives

$$\mu^\Delta \frac{\partial^2\psi_\Delta}{\partial p_j \partial p_i}\Delta P_j + \frac{\partial\varphi_a}{\partial p_i}\Delta U^a \approx \epsilon^\Delta \frac{\partial^2\psi_\Delta}{\partial p_j \partial p_i}\left[\mu^\Sigma \frac{\partial\psi_\Sigma}{\partial q_j} + \dot{P}_j\right] \tag{17a}$$

and

$$\frac{\partial\varphi_a}{\partial p_i}\Delta P_i \approx 0 \ , \tag{17b}$$

where ΔP_j and ΔU^a are the differences between $\delta_H p_i$, $\delta_H u^a$ [functions (16a)] evaluated on shell and δP_j, δU^a. If only one constraint is quadratic in the momenta—the other constraints being at most linear—these equations, which possess a unique solution, imply $\Delta U^a \approx 0$. ΔP_j is weakly proportional to $\mu^\Delta \cdot \partial\psi_\Delta/\partial q^j + \dot{P}_j$ and obeys (17b) because of the relations (15), $\partial\varphi_a/\partial p_i \cdot \dot{P}_i + \partial\varphi_a/\partial q_j \cdot \dot{q}^j = 0$ and $[\psi_\Delta, \psi_\Sigma] \approx 0$.[14] From the property

$$\delta U^a \approx \dot{\epsilon}^a + C^a{}_{\Delta\Sigma}\epsilon^\Delta\mu^\Sigma \ ,$$

the structure functions $C^a{}_{\Delta\Sigma}$ can be determined up to combinations of the functions ψ_Δ.

We have thus shown that the constraint algebra can be computed from the invariances of the Lagrangian action up to terms such as $\varphi_a\varphi_b$ or $\varphi_a\chi_B$, which vanish with the primary constraints. In order to evaluate these terms a finer analysis is required. It is not our purpose here to provide such an analysis for an arbitrary first-class theory (direct computation of the brackets—as was done above—is of course always a reliable method). Let us merely mention that in metric gravity there is no ambiguity since there is no φ_a constraint once the primary constraints $\pi^{0\mu} \approx 0$ have been eliminated. In the case of supergravity, the Lagrangian method can be easily supplemented by the appropriate analysis when the generators $\mathcal{H}$, $\mathcal{H}_k$, and $\mathcal{S}$ are taken to involve the gravitational momenta $\pi^{(\lambda)a}$ only through the symmetric combinations π^{ab}.

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[6] In fact, there exists no local, Lorentz-invariant tensor density $\overline{\pi}^{ab}(h_{(\lambda)c}, \pi^{(\lambda)c})$ which obeys simultaneously

$[g_{ab}(x), \overline{\pi}^{cd}(x')] = (\delta^c_a\delta^d_b + \delta^d_a\delta^c_b)\delta(x,x')/2$ and

$[\overline{\pi}^{ab}(x), \overline{\pi}^{cd}(x')] = 0$.

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[13] There is a little subtlety here. The first step in comparing (16b) with the known Lagrangian invariances requires the identification of the parameters of the infinitesimal invariance transformations in Lagrangian form with a suitable combination of the ϵ^A. Usually, this is done with the help of the first equations (16b) (containing $\delta_L q^i$ and $\delta_L y^\alpha$). Then, one has to equate $\delta_L \lambda^A$ [Eq. (16b)] with the known variation of λ^A, which reads $\delta_{\text{known}}\lambda^A = \dot{\epsilon}^A + \overline{C}^A{}_{\Delta\Gamma}\epsilon^\Delta\mu^\Gamma + \theta^{AB}\chi_B$ (plus, possibly, a term like $\theta^{A\alpha}\delta S/\delta y^\alpha$ if the equations for y^α are first order; this term is determined by examination of $\delta_L y^\alpha$). In the above formula, θ^{AB} is an arbitrary antisymmetric matrix—the λ's are assumed to be commuting for simplicity—and χ_B are the secondary constraints $(\chi_B = \delta S/\delta\lambda^B)$. The only thing that can be inferred from $\delta_L \lambda^A = \delta_{\text{known}}\lambda^A$ is

$$C^A{}_{\Delta\Gamma} = \overline{C}^A{}_{\Delta\Gamma} + F^{AB}{}_{\Delta\Gamma}\chi_B + M^A{}_{\Delta\Gamma\Sigma}\mu^\Sigma$$

with $F^{AB}{}_{\Delta\Gamma} = -F^{BA}{}_{\Delta\Gamma}$ and $M^A{}_{\Delta\Gamma\Sigma} = -M^A{}_{\Delta\Sigma\Gamma}$. $M^A{}_{\Delta\Sigma\Gamma}$ is determined by the requirement that $C^A{}_{\Delta\Gamma}$ be independent of the multipliers μ^Γ in the canonical formulation of the theory. $F^{AB}{}_{\Delta\Gamma}$ is undetermined, but since $C^A{}_{\Delta\Gamma}$ is multiplied by χ_A in the constraint algebra, this ambiguity plays no role here. I am indebted to P. van Nieuwenhuizen for calling my attention to the occurrence of terms such as $\theta^{AB}\delta S/\delta\lambda^B$ in the transformation laws.

[14] In the case of field theories, the weak equality $\Delta U^a \approx 0$ remains true when only one constraint per space point is quadratic in the momenta, provided its quadratic part does not involve spatial derivatives.