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Comparison of Monte Carlo data with strong-coupling expansions for U(N) lattice gauge theories

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The average action per plaquette is calculated using strong-coupling expansions up to 15th order for the pure U(N), $N=2$, 3, 4, 5, and 6, four-dimensional Euclidean lattice gauge theory. We compare these expansions with Monte Carlo-generated data and find agreement to be better than 1% over the whole strong-coupling region.

The techniques for evaluating the strong-coupling expansion were outlined in a series of papers.[1-4] In the present paper the results of a numerical study of U(N), $N=2$, 3, 4, 5, and 6, lattice gauge theories in four Euclidean space-time dimensions are reported. The free energy and the average action per plaquette are investigated through strong-coupling expansions and Monte Carlo simulations.

The action for the U(N) Wilson theory is

$$S=BN\sum_{p}(\mathrm{Tr}\,U_p+\mathrm{Tr}\,U_p^{\dagger}) \ , \tag{1}$$

where $\sum_p$ is a sum over each elementary plaquette in the lattice. Equation (1) defines our normalization of B; the factor of N in front of the sum guarantees that the theory will be a smooth function of B in the large-N limit.

The free energy F per volume V per N^2 is defined as

$$F=\frac{1}{N^2}\frac{1}{V}\ln Z \ , \tag{2}$$

where V is the number of sites in the lattice. The reduced free energy f or free energy per plaquette in D dimensions is

$$f=\frac{2F}{D(D-1)} \ . \tag{3}$$

The quantity f has been calculated in a strong-coupling B expansion to order B^{16} and the results are tabulated in Appendix A. Here is a brief summary of the method of calculation. First of all, we used Eq. (2) in Ref. 5. For U(N) theories to order B^{16} it is not necessary to compute any of the group integrals in Eq. (8) of Ref. 5. It is, however, necessary to know (i) the dimensions of the representations of the U(N) groups, (ii) the character coefficients for the Wilson action, and (iii) the group product coefficients N_{rst} and N_{rstu} of Eqs. (5), (6), and (7) in Ref. 5. The first two can be found, for example, in Appendix A of Ref. 6. The latter, namely N_{rst} and N_{rstu}, can, respectively, be found in Tables I and II. Tables I and II are valid for any U(N) group with $N\geq 2$. For $N=1$, all coefficients in Table II are one instead of two, and all coefficients involving the representations $(1^2;0)$, $(0;1^2)$, and $(1;1)$ in Table I are zero instead of one. The notation for the U(N) representations is defined in Sec. II of Ref. 6. The calculations of the character coefficients and their B expansions were performed with the use of MACSYMA. For $N\geq 3$ it

TABLE I. The N_{rst} coefficients.

r	s	t	N_{rst}
(1;0)	(1;0)	$(0;1^2)$	1
(0;1)	(0;1)	$(1^2;0)$	1
(1;0)	(1;0)	(0;2)	1
(0;1)	(0;1)	(2;0)	1
(1;0)	(0;1)	(1;1)	1
(0;1)	(1;0)	(1;1)	1
$(1^2;0)$	(0;1)	(0;1)	1
$(0;1^2)$	(1;0)	(1;0)	1
(0;1)	$(1^2;0)$	(0;1)	1
(1;0)	$(0;1^2)$	(1;0)	1
(2;0)	(0;1)	(0;1)	1
(0;2)	(1;0)	(1;0)	1
(0;1)	(2;0)	(0;1)	1
(1;0)	(0;2)	(1;0)	1
(1;1)	(0;1)	(1;0)	1
(1;1)	(1;0)	(0;1)	1
(0;1)	(1;1)	(1;0)	1
(1;0)	(1;1)	(0;1)	1

TABLE II. The n_{rstu} coefficients ($N \geq 2$).

r	s	t	u	N_{rstu}
(1;0)	(1;0)	(0;1)	(0;1)	2
(1;0)	(0;1)	(1;0)	(0;1)	2
(1;0)	(0;1)	(0;1)	(1;0)	2
(0;1)	(1;0)	(1;0)	(0;1)	2
(0;1)	(1;0)	(0;1)	(1;0)	2
(0;1)	(0;1)	(1;0)	(1;0)	2

is time consuming to compute these character coefficients using Eq. (A4) of Ref. 6 because of the size of the determinants; instead, the formulas in Eq. (A19) of Ref. 6 were used.

There were two nontrivial checks on our free-energy calculations. First, we checked that for $N=1$ we obtained the U(1) result of Ref. 2. Secondly, we checked that as $N \rightarrow \infty$ f had a smooth large-N limit; this would not have been the case if the subtractions involved in taking the connected pieces of graphs had not been correctly done. The large-N results are also presented in Appendix A.

The average plaquette action P is defined as

$$P^{U(N)} = \frac{1}{N} \langle \mathrm{Tr}\, U_p \rangle \ , \tag{4}$$

and is related to $f^{U(N)}$ by

$$P^{U(N)} = \frac{1}{2} \frac{\partial}{\partial B} f^{U(N)} \ . \tag{5}$$

Appendix B tabulates the $P^{U(N)}$ in $D=4$ as computed via Eq. (5).

The Monte Carlo simulation data were generated on 6^4 lattices. We carried out 300 iterations through the lattice and averaged over the last 100 iterations. Mixed-phase starting lattices were used throughout our calculations except in the extremely large coupling region. Periodic boundary conditions were used in all our calculations. The method of Metropolis *et al.*[7] was used to achieve statistical equilibrium with 20 Monte Carlo updates per link of the lattice. In order to eliminate rounding errors due to the finite work length on the computer, we renormalize our matrices after every 50 iterations through the lattice.

In Figs. 1(a)–1(e) our Monte Carlo simulation data on 6^4 lattices for $U(N)$, $N=2$, 3, 4, 5, and 6,

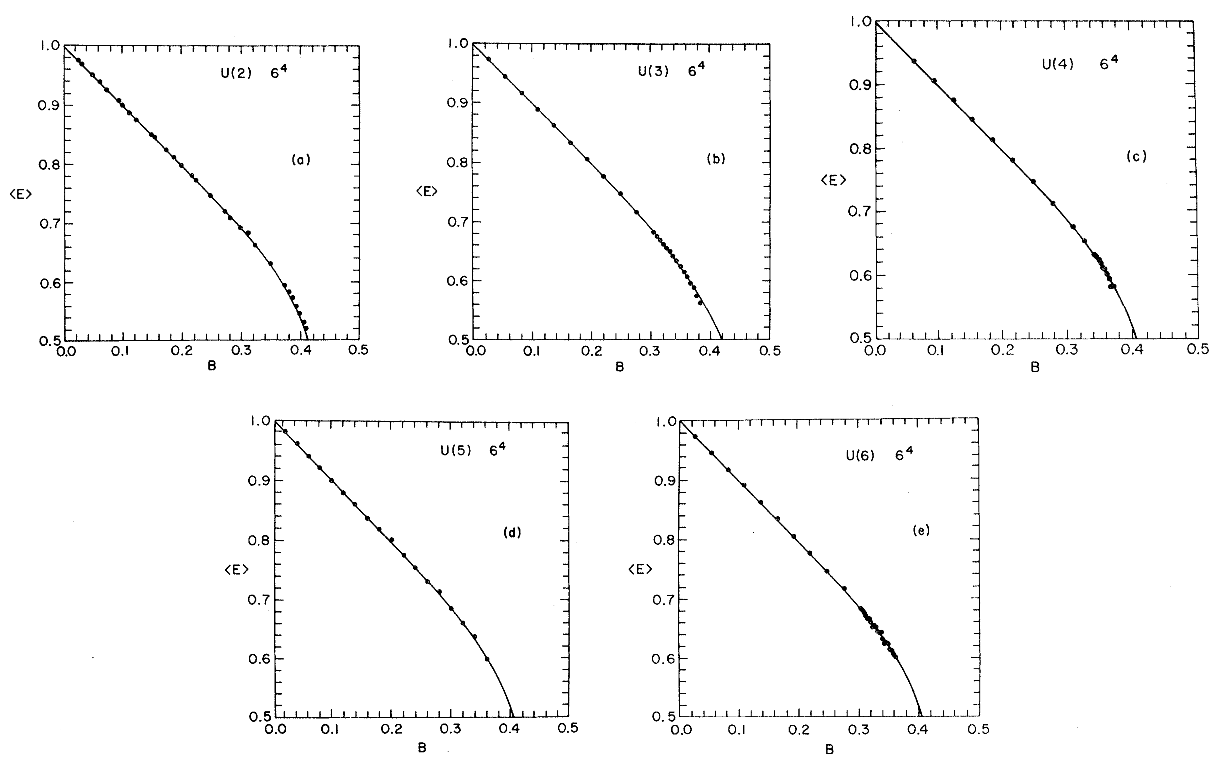

FIG. 1. The average action per plaquette $\langle E \rangle$ as a function of the inverse coupling constant squared β on a 6^4 lattice for (a) U(2), (b) U(3), (c) U(4), (d) U(5), and (e) U(6). The solid lines represent the 15th-order strong-coupling expansions derived in the text. The statistical error in the Monte Carlo data is less than the size of the Monte Carlo data points.

respectively, are presented together with the strong-coupling expansions of Appendix B. To ensure accuracy in our diagrams our results were plotted by computer.[8] The Monte Carlo-simulated data and the strong-coupling expansions agree over the whole strong-coupling region with an error of less than 1%. The critical couplings are $B_c = 0.416$, 0.392, 0.375, 0.375, and 0.375 for $N = 2$, 3, 4, 5, and 6, respectively.

In the case of a continuous phase transition, series analysis such as the ratio test and Padé approximations can sometimes be used to locate the transition. Unfortunately, for $N \geqslant 2$ U(N) gauge theories have first-order transitions.[9] Nevertheless, we examined Padé approximations to the free energy. They proved to be uninteresting. All free-energy coefficients are positive for the U(∞) theory (at least to 16th order) which curiously also happens in the U(∞) chiral models[6] and which suggests a singularity in the series on the positive B axis. Ratio test estimates of this singularity do not yet give consistent results; apparently more terms in the series are needed. In any case, this singularity is unlikely to be the large-N phase-transition point, again because the transition is expected to be first order.

Summarizing, strong-coupling techniques appear to be able to accurately calculate quantities on the strong-coupling side of the transition, but unfortunately appear to be of little use in obtaining the phase transition's location.

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APPENDIX A

The reduced free energies f for various U(N) theories to 16th order are

$$f^{U(2)} = B^2 + \left[\frac{2D}{3} - \frac{16}{9}\right]B^6 + \frac{2}{3}B^8 + \left[4D^2 - \frac{70D}{3} + \frac{6788}{225}\right]B^{10} + \left[\frac{20D^2}{3} - \frac{836D}{81} - \frac{2504}{405}\right]B^{12}$$
$$+ \left[40D^3 - 372D^2 + \frac{391\,718D}{405} - \frac{15\,187\,468}{19\,845}\right]B^{14}$$
$$+ \left[168D^3 - \frac{200\,933D^2}{216} + \frac{2\,221\,069D}{1080} - \frac{19\,707\,343}{11\,340}\right]B^{16} + O(B^{18}) \;,$$

$$F^{U(3)} = B^2 + \left[\frac{2D}{3} - \frac{4}{3}\right]B^6 - \frac{81}{64}B^8 + \left[4D^2 - 18D + \frac{4729}{200}\right]B^{10} + \left[\frac{20D^2}{3} - \frac{573\,967D}{12\,288} + \frac{631\,057}{10\,240}\right]B^{12}$$
$$+ \left[40D^3 - 292D^2 + \frac{7829D}{10} - \frac{11\,182\,933}{15\,680}\right]B^{14}$$
$$+ \left[168D^3 - \frac{6\,853\,127D^2}{4608} + \frac{1\,406\,292\,923D}{368\,640} - \frac{26\,104\,872\,011}{8\,601\,600}\right]B^{16} + O(B^{18}) \;,$$

$$f^{U(4)} = B^2 + \left[\frac{2D}{3} - \frac{4}{3}\right]B^6 + \left[4D^2 - 18D + \frac{3476}{225}\right]B^{10} + \left[\frac{20D^2}{3} - \frac{1\,396\,012D}{50\,625} + \frac{822\,008}{16\,875}\right]B^{12}$$
$$+ \left[40D^3 - 292D^2 + \frac{27\,854D}{45} - \frac{14\,339\,572}{33\,075}\right]B^{14}$$
$$+ \left[168D^3 - \frac{632\,690\,197D^2}{540\,000} + \frac{1\,740\,305\,449D}{540\,000} - \frac{3\,450\,623\,791}{1\,134\,000}\right]B^{16} + O(B^{18}) \;,$$

$$f^{U(5)}=B^2+\left(\frac{2D}{3}-\frac{4}{3}\right)B^6+(4D^2-18D+20)B^{10}+\left(\frac{20D^2}{3}-\frac{9\,291\,925D}{331\,776}+\frac{1\,743\,245}{165\,888}\right)B^{12}$$
$$+\left(40D^3-292D^2+710D-\frac{38\,666\,759}{84\,672}\right)B^{14}$$
$$+\left(168D^3-\frac{408\,777\,613D^2}{345\,600}+\frac{322\,906\,001D}{138\,240}-\frac{377\,650\,503\,541}{232\,243\,200}\right)B^{16}+O(B^{18}) ,$$

$$F^{U(6)}=B^2+\left(\frac{2D}{3}-\frac{4}{3}\right)B^6+(4D^2-18D+20)B^{10}+\left(\frac{20D^2}{3}-\frac{127\,050\,796D}{4\,501\,875}+\frac{44\,683\,864}{1\,500\,625}\right)B^{12}$$
$$+\left(40D^3-292D^2+710D-\frac{805\,676}{1225}\right)B^{14}$$
$$+\left(168D^3-\frac{128\,400\,013\,871D^2}{108\,045\,000}+\frac{303\,963\,890\,107D}{108\,045\,000}-\frac{5\,554\,216\,651}{3\,601\,500}\right)B^{16}+O(B^{18}) ,$$

$$f^{U(\infty)}=B^2+\left(\frac{2D}{3}-\frac{4}{3}\right)B^6+(4D^2-18D+20)B^{10}+\left(\frac{20D^2}{3}-\frac{86D}{3}+\frac{92}{3}\right)B^{12}$$
$$+(40D^3-292D^2+710D-572)B^{14}+(168D^3-1200D^2+2866D-2276)B^{16}+O(B^{18}) ,$$

where D is the dimension of space-time and B is the (inverse) coupling constant in the Wilson theory.

APPENDIX B

The average plaquettes P for various U(N) theories in $D=4$ to 15th order are

$$P^{U(2)}=B+\frac{8B^5}{3}+\frac{8B^7}{3}+\frac{188B^9}{45}+\frac{1776B^{11}}{5}-\frac{1\,144\,996B^{13}}{567}+\frac{10\,688\,606B^{15}}{567}+O(B^{17}) ,$$

$$P^{U(3)}=B+4B^5-\frac{81B^7}{16}+\frac{3129B^9}{40}-\frac{569\,699B^{11}}{5120}+\frac{960\,879B^{13}}{448}-\frac{21\,139\,567\,193B^{15}}{3\,225\,600}+O(B^{17}) ,$$

$$P^{U(4)}=B+4B^5+\frac{1676B^9}{45}+\frac{4\,563\,952B^{11}}{16\,875}-\frac{2\,303\,212B^{13}}{4725}+\frac{10\,511\,596\,807B^{15}}{708\,750}+O(B^{17}) ,$$

$$P^{U(5)}=B+4B^5+60B^9+\frac{284\,705B^{11}}{9216}+\frac{22\,974\,457B^{13}}{12\,096}-\frac{105\,820\,185\,397B^{15}}{29\,030\,400}+O(B^{17}) ,$$

$$P^{U(6)}=B+4B^5+60B^9+\frac{212\,096\,816B^{11}}{1\,500\,625}+\frac{86\,124B^{13}}{175}+\frac{156\,528\,678\,962B^{15}}{13\,505\,625}+O(B^{17}) ,$$

$$P^{U(\infty)}=B+4B^5+60B^9+136B^{11}+1092B^{13}+5920B^{15}+O(B^{17}) .$$

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