PHYSICAL REVIEW D VOLUME 27, NUMBER 4 15 FEBRUARY 1983

Classical gluon dynamics and condensates

Dennis Sivers
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
(Received 7 September 1982)

Time-independent, spherically symmetric solutions are constructed for the classical equations of motion for SU_2 gluon dynamics in the presence of infinitely extended sources. These solutions can be used to investigate the concept of gluon condensation and to model the physical vacuum as a color-repelling medium.

I. INTRODUCTION

Quantum chromodynamics is widely believed to possess the property of color confinement and this property is usually associated with the vacuum structure of the theory. The realization that the vacuum in a confining theory can be quite different from that of a nonconfining theory developed over years of trying to understand confinement in simple field-theory models. The idea forms the basis for the MIT bag model[1] where hadrons are pictured as "bubbles" of a perturbative vacuum surrounded by the medium of a nonperturbative, color-repelling vacuum.[2]

There have been many attempts in recent years to establish confinement in QCD by identifying a mechanism for constructing a vacuum state with color-confining properties. For example, Callan, Dashen, and Gross[3] have suggested that it is possible to describe the QCD vacuum as a meron/instanton plasma. Other efforts to model the QCD vacuum by identifying the mechanism responsible for producing it have started with such diverse building blocks as monopoles,[4] flux tubes,[5] vortices,[6] or glueballs.[7]

Among the motives for the construction of models of this type has been the hope that an explicit representation of the vacuum would aid in the formulation of a "proof" of confinement. This hope has not, as yet, been realized. Although these models have proved informative and have provided significant physical insight, they have not made possible any such proof. In fact, the issue of confinement has been more satisfactorily treated by other theoretical methods. For example, Monte Carlo simulations on lattices suggest strongly that Wilson loops in SU_2 and SU_3 exhibit an area-law behavior (signaling confinement) even in the weak-coupling regime.[8] The fact that lattice regularization can be made completely gauge independent so that the calculations involve only the physical degrees of freedom and do not require the invention of specialized field configurations in intermediate steps makes these results particularly compelling.

It is possible to criticize lattice calculations. Since they are carried out with discrete, rather than continuous, space-time variables we can imagine that there are some subtleties in passing to the continuum limit which make the results of lattice calculations difficult to interpret. However, lattice calculations provide a significant advance in our understanding of nonperturbative effects in non-Abelian gauge theories and it is appropriate to build on the experience gained from these calculations to formulate new goals for semiclassical models. While it does not now seem so important to uncover a mechanism for confinement, it is still important to have a quantitative, analytic model for a confining vacuum. While lattice calculations support the general picture in which the physical vacuum can be described as a system of mobile charges forming an extended medium they reveal little of its local structure.[9] It would be desirable, for example, to have analytic expressions for "background" fields in the vacuum to formulate perturbation theory[10] in a manner which connects to work done by Lee and others[11] on the corrections to the Feynman rules in field theories subject to confining boundary conditions.

One element which makes this goal possible involves the important series of papers by Shifman, Vainshtein, and Zakharov.[12] These papers explore in considerable detail the phenomenology of experimental observables associated with the issue of a nontrivial vacuum. The difference between the physical vacuum $|\,\mathrm{vac}\rangle$ and the hypothetical perturbative vacuum $|\,\phi\,\rangle$, where quarks and gluons propagate freely, can be described by various order parameters,

$$\sigma_i = \langle \mathrm{vac}\,|\,O_i\,|\,\mathrm{vac}\rangle - \langle \phi\,|\,O_i\,|\,\phi\,\rangle \ , \tag{1.1}$$

giving the difference between the expectation values of scalar operators in the two states. In language borrowed from many-body physics, the appearance of a nonvanishing σ_i is characterized as a "condensate" associated with the appropriate operator.

Two operators, $O_g = G_a^{\mu\nu} G_{a,\mu\nu}$ and $O_q = m_q \bar{q}_i q_i$ (each flavor), which are dimension-4 scalars associated, respectively, with the gluon and the quark fields, play a key role in the bulk description of the QCD vacuum. It has long been accepted that the existence of a quark condensate is necessary to understand the breaking of chiral symmetry and that current-algebra sum rules[13] are able to attach a number to this condensate,

$$\sigma_q \cong -\frac{m_\pi{}^2 f_\pi{}^2}{4} + \cdots \quad (q=u,d)$$
$$\cong -0.84\times 10^{-4}\ \mathrm{GeV}^4\ . \qquad (1.2)$$

The importance of the work of Shifman, Vainshtein, and Zakharov is that, using sophisticated sum-rule techniques, they are able to extract from experiment a comparable number for gluons,

$$\sigma_g = \langle \mathrm{vac} \,|\, (\alpha_s/\pi) G_a^{\mu\nu} G^a_{a,\mu\nu} \,|\, \mathrm{vac} \rangle$$
$$\cong 1.2\times 10^{-2}\ \mathrm{GeV}^4\ . \qquad (1.3)$$

These numbers then turn the exercise of understanding the vacuum structure in a non-Abelian gauge theory from one of primarily formal interest to one which makes direct contact with experiment. A useful model of the vacuum would be able to incorporate numbers such as (1.2) and (1.3) and relate them to other quantities of experimental interest.

An important additional impetus to the idea of reexamining models for the non-Abelian vacuum can be traced to work started by Savvidy.[14] Savvidy calculates the one-loop quantum corrections to the field configuration $G^3_{12} = H$ (constant). He argues that, under certain conditions, the state with zero field is unstable against the effects of quantum fluctuations. This argument has been taken up and improved by Fukuda and Kazama[15] so that it can be formulated in a gauge-invariant way using an effective potential parametrized in terms of the gluon condensate σ_g. This line of reasoning does not necessarily lead to a *proof* that condensation necessarily occurs in a non-Abelian gauge theory such as QCD (Refs. 16—18), but when combined with the phenomenological evidence of a gluon condensate such as that found in Ref. 12, it does give insight into the reasons behind the formation of a nontrivial vacuum state.

The ansatz of Savvidy, $G^3_{12} = H(\mathrm{const})$, serves as the starting point for the construction by Olesen and Nielsen[5] of the most ambitious and complete model for the QCD vacuum thus far in existence. Nielsen and Olesen call their construction a "quantum liquid" model but it has come to be known more simply as the "Copenhagen vacuum." We will compare the properties of the Copenhagen vacuum with our models in the discussion of Sec. IV. The construction of Nielsen and Olesen is very imaginative. However, after the early stages of construction, there are no explicit representations of the fields or vector potential. This makes it difficult, for example, to show directly that the Copenhagen vacuum confines.[19] Therefore, it is possible that using the classical equations of motion to suggest a different starting point would eventually yield a model more useful for calculations.

The above discussion should serve to indicate that the idea of the QCD vacuum as an extended medium containing color fields is well motivated even if it falls short of being firmly established. To represent this medium and its effects on physical processes, we can model it by a set of extended classical sources. We then look at solutions of the classical equations of motion in the presence of such sources. The study of the classical equations of motion for a non-Abelian gauge theory is by now a well-developed topic.[20] Most analyses, however, look for "particlelike" solutions which are either localized or have finite classical energy. This will not be the approach in this paper. If we label the classical energy density $\epsilon_c(x)$ we will be looking at field configurations for which

$$E_c = \int d^3x\, \epsilon_c(x) = \infty\ . \qquad (1.4)$$

The motivation for this relies on the unknown 0-point fluctuations assumed in the work of Savvidy,[14] Fukuda and Kazama[15] and on the results of Nielsen and Olesen[5] discussed above. While the formation of a condensate is a process which may require the full quantum theory in order to be understood, we merely assume that it is possible to represent the final state in some fashion using classical fields.

The remainder of this paper is organized as follows. Section II defines our terminology and focuses on some of the arguments for the existence of a gluon condensate. Section III makes some simplifying assumptions and constructs some "sample" solutions of the equations of motion for SU_2 gluon dynamics which have the property of a nonvanishing gluon condensate. Section IV discusses briefly the problem of using these solutions to construct a model for the vacuum and compares our approach with that used in the construction of the Copenhagen vacuum.

II. CONDENSATE AND THE QCD VACUUM AS A MEDIUM

In order to fix conventions, consider an SU_N gauge theory to be defined by the Lagrangian

$$\mathscr{L}=-\tfrac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}+\sum_f \bar{q}_f\,(i\not{D}-m_f)q_f\ , \qquad (2.1)$$

where

$$G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\ ,\qquad a,b=1,\ldots,N^2-1\ . \qquad (2.2)$$

If T_a,T_b form a representation of SU_N,

$$[T_a,T_b]=if^{abc}T_c\ , \qquad (2.3)$$

and the gauge-covariant derivative is given by

$$D^\mu_{ij}=(\partial^\mu\delta_{ij}-igA^\mu_a T^a_{ij})\ . \qquad (2.4)$$

Equations (2.1) and (2.2) describe the interactions of quarks and gluons. Since they play no direct role in our discussion, ghost or gauge-fixing terms are not included above although they may be necessary to describe the quantized theory.

The appearance of nontrivial vacuum structure in the SU_N gauge theory defined above has been an accepted feature of theoretical analyses for many years. An early approach which indicates the need for such structure involves the problem of chiral-symmetry breaking. The chiral limit of (2.1) can be defined by setting $m_q=0$ for the "light" quarks ($q=u,d,s$) and $m_q=\infty$ for the "heavy" quarks ($q=c,t,b,\ldots$). In this limit there are fermionic currents associated with the light quarks:

$$j^a_{V\mu}=\bar{q}\gamma_\mu T^a q\ ,\qquad j^a_{A\mu}=\bar{q}\gamma_\mu\gamma_5 T^a q\ . \qquad (2.5)$$

The invariance of the Lagrangian under an $SU^L_3\otimes SU^R_3$ transformation in the chiral limit can be used to show

$$\langle\phi\,|\,j^a_{V\mu}(x)j^b_{V\mu}(y)\,|\,\phi\rangle=\langle\phi\,|\,j^a_{A\mu}(x)j^b_{A\nu}(y)\,|\,\phi\rangle\ , \qquad (2.6)$$

where $|\,\phi\,\rangle$ is the dressed Fock state representing the vacuum in perturbation theory and the equality is true to arbitrary order in the perturbative expansion parameter. The observation that this equality does not seem to be realized experimentally (for example, the mass of the ρ is not equal to the mass of the A_1) leads to the hypothesis that the physical vacuum $|\,\text{vac}\,\rangle$ does not correspond to the perturbative vacuum $|\,\phi\,\rangle$. As mentioned in the Introduction, to characterize the difference between the two states, we can use the value of various order parameters

$$\sigma_i=\langle\text{vac}\,|\,O_i(x)\,|\,\text{vac}\rangle-\langle\phi\,|\,O_i(x)\,|\,\phi\,\rangle \qquad (2.7)$$

associated with different scalar operators. In our discussion, we will primarily be interested in three operators of dimension 4,

$$G^a_{\mu\nu}G^{a,\mu\nu},\quad m_q\bar{q}q\ \text{(each flavor), and}\ \theta^\mu_\mu\ ,$$

where $G^a_{\mu\nu}$ and $m_q\bar{q}q$ are defined in (2.1)–(2.4) and $\theta_{\mu\nu}$ is the energy-momentum tensor. Of course, operators of higher dimension, such as

$$f^{abc}g^{\sigma\mu}G^a_{\mu\nu}G^{b,\nu\rho}G^c_{\rho\sigma}\ \text{and}\ m_q{}^2\bar{q}\Gamma^\mu q\bar{q}\Gamma_\mu q$$

may also be necessary to characterize the physical vacuum. In fact, it may require an infinite number of these order parameters to describe the difference between $|\,\text{vac}\,\rangle$ and $|\,\phi\,\rangle$. However, the assumption is usually made that only a few of the low-dimension operators are important and that it is possible to write the other σ's in terms of this small set using the equations of motion and various averaging processes.[12] One reason for seeking an explicit "model" of the vacuum is to test this assumption for reasonableness or, if possible, to obtain other estimates of the high-dimension operators.

The three dimension-4 operators mentioned above are not independent. They are related by the trace anomaly equation

$$\theta^\mu_\mu=\frac{\beta(g)}{2g}G^a_{\mu\nu}G^{a,\mu\nu}+\sum_f m_q\bar{q}q+\lambda\Pi\ , \qquad (2.8)$$

where the constant λ accounts for vacuum energy renormalization. The β function can be expanded

$$\beta(g)=-\beta_0\frac{g^3}{(4\pi)^2}-\beta_1\frac{g^5}{(4\pi)^4}+\cdots\ . \qquad (2.9)$$

If we then normalize

$$\begin{aligned}
&\sigma_q=-\langle\text{vac}\,|\,m_q\bar{q}q\,|\,\text{vac}\rangle\ ,\\
&0=\langle\phi\,|\,m_q\bar{q}q\,|\,\phi\,\rangle\ ,\\
&\sigma_G=\langle\text{vac}\,|\,(\alpha_s/\pi)G^a_{\mu\nu}G^{a\mu\nu}\,|\,\text{vac}\rangle\ ,\\
&0=\langle\phi\,|\,(\alpha_s/\pi)G^a_{\mu\nu}G^{a,\mu\nu}\,|\,\phi\,\rangle\ ,\\
&0=\langle\text{vac}\,|\,\theta^\mu_\mu\,|\,\text{vac}\rangle\ ,\\
&\sigma_\theta=\langle\phi\,|\,\theta^\mu_\mu\,|\,\phi\,\rangle\ ,
\end{aligned} \qquad (2.10)$$

the consequences of (2.8) can be written

$$\sigma_\theta=\frac{\beta_0}{8}\sigma_G\left[1+\frac{\beta_1}{\beta_0}\frac{\alpha_s}{4\pi}+\cdots\right]+\sum_f\sigma_q\ . \qquad (2.11)$$

As shown originally by Gell-Mann, Oakes, and Renner,[13] when chiral-symmetry breaking occurs through the Goldstone mechanism, the size of the quark condensate can be determined in terms of the pseudoscalar meson masses. With the normalization (2.10) this can be written

$$\sigma_q \cong \tfrac{1}{4} m_\pi{}^2 f_\pi^{02} + \cdots \quad (q = u,d,) , \tag{2.12}$$

where f_π^0 is the pion decay constant in the chiral limit. The nonvanishing quark condensate provides direct confirmation of the idea that the QCD vacuum can be considered as an extended medium. Since the main topic of this paper is the gluon condensate we will not go into the various applications of quark condensation such as those involving the determination of quark mass parameters.[21] It is interesting, however, to consider the possible implications for the gluon condensate of the "accepted" existence of quark condensation. In defining the chiral limit of (2.1), we made a sharp distinction between "light" quarks, with $m_q/\Lambda \cong 0$, and "heavy" quarks, with $m_q/\Lambda \cong \infty$. In practice, this distinction is not clear cut and there exist quarks which can be considered both light and heavy. In renormalization prescriptions (such as versions of minimal subtraction[22]) where all quarks contribute to the β function, the contribution of a specific quark flavor to the energy-momentum tensor anomaly is

$$(\Delta\theta_\mu^\mu)_f = \tfrac{1}{8}(\Delta\beta_0)_f \sigma_G + \sigma_f . \tag{2.13}$$

For large masses, the requirement that heavy quarks not contribute to the vacuum energy in Eq. (2.13) leads to

$$\sigma_G = \lim_{m_f \to \infty} 12\pi\sigma_f . \tag{2.14}$$

In terms of the vast energy scales associated with grand unified theories, the fermion mass spectrum does not seem to span such a large energy range. Since pions can evidently be understood as $\bar{q}q$ states in a way not too different from other pseudoscalars, it seems plausible that the value of the quark condensate should be nonzero for all flavors of quarks. Eq. (2.14) then implies that the gluon condensate be nonvanishing.

A more convincing argument that σ_G is nonvanishing can be found in the phenomenology of Shifman, Vainshtein, and Zakharov.[12] Using dispersion techniques, they are able to derive sum rules relating the masses and couplings of hadronic resonances to σ_q and to σ_G. If we define the vacuum polarization function

$$\Pi^{AB}(q^2) = i \int d^4x \, e^{iq\cdot x} \times \langle \text{vac} \,|\, T[j_V^A(x) j_V^B(x)] \,|\, \text{vac} \rangle , \tag{2.15}$$

where A and B are flavor indices. Using the operator-product expansion on the right-hand side,

$$\sum C_n(q^2) \langle \text{vac} \,|\, O_n \,|\, \text{vac} \rangle = C_0(q^2)\pi + G_G(q^2)\frac{1}{(q^2)^2}\sigma_G + C_A(q^2)\frac{1}{(q^2)^2}\sigma_q + (\text{dim 6 and higher}) . \tag{2.16}$$

The coefficients C_n can be calculated using perturbation theory while various averaging processes (such as vacuum dominance[12]) are used to estimate the higher-dimension order parameters. Using weighted dispersion relations, Shifman, Vainshtein, and Zakharov[12] form sum rules by saturating the left-hand side of (2.15) with narrow resonances. The estimate for σ_G obtained phenomenologically in this way is

$$\sigma_G \cong 0.012 \text{ GeV}^4 \cong (0.33 \text{ GeV})^4 . \tag{2.17}$$

Alternative analyses using different approaches suggest that the value may be a factor of 2 higher[21] but we will use the values in Eq. (2.17). The importance of this work is that it shows that experimental numbers are sensitive to the vacuum structure.

An independent chain of reasoning suggesting the existence of a nonvanishing gluon condensate is based on the possibility of showing that the perturbative vacuum is unstable. This approach has been pursued, for example, by Savvidy,[14] who has calculated the effect through one loop of quantum fluctuations on the vacuum energy density. Starting with a constant field configuration $G_{12}^3 = H$ all other $G_{\mu\nu}^a = 0$ then gets the result for the real part of the energy density,

$$\text{Re}\epsilon(H) = \tfrac{1}{2}H^2 + \frac{11N}{192\pi^2} g^2 H^2 \left[\ln \frac{g^2H^2}{\mu^2} - 1 \right] , \tag{2.18}$$

where the N refers to SU_N and μ^2 is a renormalization point. This expression (for large enough g^2) can have a nontrivial minimum away from $H = 0$. Thus it appears that the non-Abelian state $H = 0$ should be unstable against the development of a finite magnetic field. With the constant fields above, the energy density also has an imaginary part signaling further instabilities. These are discussed in further detail by Ambjørn and Olesen,[5] and by Chang and Ni[18] and are used by Nielsen and Olesen in the construction of the Copenhagen vacuum.

The argument concerning the instability of the "naive" perturbative vacuum has also been formulated by Fukuda and Kazama[15] directly in terms of the order parameter associated with a gluon condensate. The approach of Fukuda and Kazama has the immediate advantage of dealing with Lorentz-invariant quantities. For our purposes, it is not important whether or not these arguments prove that gluon condensation must take place in SU_N gauge theories. One thing which is important, however, is that such condensation is not a purely classical effect. The classical energy density for the Savvidy vacuum

$$\epsilon_c = \frac{1}{2}H^2$$

has the obvious, trivial, minimum at $H=0$.

Therefore, if we are to make a quasiclassical model of the QCD vacuum, we should look for field configurations which are nonvanishing almost everywhere even if they have infinite classical energy and have no simple interpretation in terms of localized particles. We now turn to the question of whether the equations of motion admit such solutions.

III. THE CLASSICAL EQUATIONS OF MOTION FOR SU_2 GLUON DYNAMICS

The fact that the classical equations of motion for a non-Abelian gauge theory are nonlinear leads to difficulties in studying their solutions. Many useful techniques for finding solutions to Maxwell's equations in ordinary classical electrodynamics involve linear expansions or the superposition of elementary solutions. Such techniques can simply not be carried over into the study of chromodynamics.

In spite of the difficulties, there has been a great deal learned about the classical equations of motion. For example, Coleman[21] has demonstrated the existence of plane-wave solutions. In addition, the self-dual sector of the theory, where

$$G^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma} = \pm G_{\mu\nu} \tag{3.1}$$

has been extensively studied utilizing the special topological properties of the solutions.[22] There has also been a lot of special interest in generalizations of Coulombic solutions[23] and in solutions with the configuration of magnetic multipoles.[24] We are now interested in looking for a completely new type of solutions, those with field configurations consistent with gluon condensate in the vacuum.

For the analysis in this section, we will take the liberty of making extensive simplifications to QCD. The first simplification involves the neglect of fermionic degrees of freedom. Thus, strictly speaking, we will be dealing with gluon dynamics rather than chromodynamics, with all the influence of fermions absorbed into the parametrization of the classical sources. The second simplification involves the change of the color gauge group from SU_3 to SU_2. In the solutions, the vector potential plays the role of a connection between the manifold of space-time and the manifold of the group space. The mappings involved utilize the projection of an O(3) or SU_2 subgroup of the original group. By dealing with the gauge group SU_2 we can perhaps avoid some possible problems involving the degeneracy of solutions without seriously changing the physical content.

Another modification of the equations is that we will be working in Euclidean space rather than in Minkowski space. Generalized Lorentz covariance requires

$$\langle \text{vac} \,|\, G^a_{\mu\nu}G^a_{\rho\sigma} \,|\, \text{vac} \rangle = c\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{3.2}$$

so that if the vacuum is to be a Lorentz scalar,

$$\langle \text{vac} \,|\, E^a_iE^a_i \,|\, \text{vac} \rangle = -\langle \text{vac} \,|\, B^a_iB^a_i \,|\, \text{vac} \rangle \tag{3.3}$$

and if $\sigma_G > 0$, the "classical" E fields are to be represented as imaginary numbers. By using a Euclidean metric where

$$G^a_{\mu\nu}G^a_{\mu\nu} = \frac{1}{2}(E^a_iE^a_i + B^a_iB^a_i) \tag{3.4}$$

instead of

$$G^a_{\mu\nu}G^{a,\mu\nu} = \frac{1}{2}(B^a_iB^a_i - E^a_iE^a_i) \tag{3.5}$$

we can represent a vacuum with gluon condensation in terms of real fields. We postpone, for the present, the interpretation of the continuation back into Minkowksi space. A final simplification will be to look for solutions which are time independent and display the gauge-theory version of spherical symmetry. By this we mean that vectorial objects either point in a fixed direction and depend on the magnitude of the radius or they exhibit an angular dependence in such a way that the radial asymmetry can be compensated by a local gauge transformation.

Incorporating the simplifications mentioned above, we follow closely the formalism of Mathelitsch, Mitter, and Widder[25] and of Jacobs and Wudka.[26] Working in the class of gauges with $r_iA^a_i = 0$, we parametrize the vector potential in the form

$$\begin{aligned} grA^a_4 &= \hat{r}_a f(x)\,, \\ grA^a_i &= \phi_{ia}\beta(x) - \psi_{ia}[1-\alpha(x)]\,, \end{aligned} \tag{3.6}$$

where $\hat{r}_i$ is the unit vector in the radial direction and $x = r/r_0$. The tensors in (3.6),

$$\phi_{ia} = (\hat{r}_i\hat{r}_a - \delta_{ia})\,, \tag{3.7a}$$

$$\psi_{ia} = \epsilon_{ail}\hat{r}_l \tag{3.7b}$$

have both group indices ($a=1,2,3$) and three-space indices ($i=1,2,3$). The gauge condition guarantees that the component of A_i^a proportional to

$$\rho_{ia}=\hat{r}_i\hat{r}_a \tag{3.7c}$$

vanishes. The current induced in the medium will be parametrized,

$$\begin{aligned} gr_0{}^3 j_4^a &= \hat{r}_a q(x) , \\ gr_0{}^3 j_i^a &= \rho_{ia} v(x)+\phi_{ia}\nu(x)+\psi_{ia}\mu(x) . \end{aligned} \tag{3.8}$$

With these definitions, the Yang-Mills field equations

$$(D_\mu G_{\mu\nu})^a = j_\nu^a , \tag{3.9a}$$

where

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c \tag{3.9b}$$

and

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g\epsilon^{abc}A_\mu^c \tag{3.10}$$

can be reduced to the system of ordinary differential equations

$$(f-xf')'+\frac{2}{x}f(\alpha^2+\beta^2)=x^2 q , \tag{3.11a}$$

$$2(\alpha\beta'-\beta\alpha')=x^2 v , \tag{3.11b}$$

$$\alpha''+\frac{\alpha}{x^2}(1-\alpha^2-\beta^2-f^2)=-x\nu , \tag{3.11c}$$

$$\beta''+\frac{\beta}{x^2}(1-\alpha^2-\beta^2-f^2)=-x\mu , \tag{3.11d}$$

where the primes denote derivatives with respect to x. Because of the antisymmetry of the field tensor $G_{\mu\nu}^a$, the current must satisfy the covariant divergence relation

$$(D_\mu j_\mu)^a = 0 . \tag{3.12}$$

With the parametrization above, this gives the constraint

$$xv'+2v+2(\nu\alpha-\mu\beta)=0 . \tag{3.13}$$

In order to understand the properties of a possible gluon condensate, it is useful to have explicit expressions for the color electric and magnetic fields. Recalling the assumption of time independence, it is straightforward to show

$$gr^2 E_a^i = \rho_{ia}(f-xf')+\phi_{ia}(\alpha f)-\psi_{ia}(\beta f) , \tag{3.14a}$$

$$gr^2 B_a^i = \rho_{ia}(1-\alpha^2-\beta^2)+\phi_{ia}(x\alpha') -\psi_{ia}(x\beta') . \tag{3.14b}$$

Solutions to Eqs. (3.11) have been studied in some detail in Refs. 25 and 26. For the special case of vanishing magnetic sources, $v=\mu=\nu=0$, they have also been studied in Refs. 27 and 28. However, the analysis in these papers concentrates on "particle-like" solutions. These are localized solutions with finite

$$E_c = \frac{1}{2}\int d^3x (E_i^a E_i^a + B_i^a B_i^a) . \tag{3.15}$$

This is a completely natural constraint within the framework of classical dynamics. However, it is not a constraint which would be satisfied by a model of the vacuum involving gluon condensation. As mentioned earlier, if we are to take seriously the arguments of Savvidy,[14] Fukuda and Kazama,[15] or the phenomenology of Shifman, Vainshtein, and Zakharov[12] as discussed in Sec. II we must relax this requirement. Since the equations of motion are nonlinear, this involves more than merely superposing localized solutions. Our approach to the equations will not be dynamical. That is, we will not be using the equations to see how localized sources interact. Instead, we will try to construct global solutions with certain desirable properties.

Transverse solutions

It is not immediately obvious what set of classical charges leads to a physically "interesting" model for the vacuum. To find out, we employ some extra constraints. One possible constraint which can lead to some interesting features is the requirement that E_i^a and B_i^a be transverse to ρ_{ia}. That is,

$$\begin{aligned} f-xf' &= 0 , \\ \alpha^2+\beta^2 &= 1 . \end{aligned} \tag{3.16}$$

This gives $f=cx$ and (3.11a) becomes

$$\frac{2c}{x^2}=q(x) . \tag{3.17}$$

To keep correspondence between electric and magnetic charges we can also assume $v(x)=-2\omega/x^2$ so that

$$\begin{aligned} \alpha(x) &= \sin\omega x , \\ \beta(x) &= \cos\omega x \end{aligned} \tag{3.18}$$

and Eqs. (3.11c) and (3.11d) become

$$\begin{aligned} (\omega^2+c^2)\alpha &= x\mu , \\ (\omega^2+c^2)\beta &= x\nu \end{aligned} \tag{3.19}$$

so that (in this solution) the "induced" currents proportional to the transverse tensors are linearly related to the fields. Replacing $x=r/r_0$ and introducing $\Omega=\omega/r_0$, $C=c/r_0$, we can write the electric and magnetic fields in the simple form

$$E_a^i = \frac{C}{gr}[\phi_{ia}\sin(\Omega r) - \psi_{ia}\cos(\Omega r)] , \tag{3.20a}$$

$$B_a^i = \frac{\Omega}{gr}[\phi_{ia}\cos(\Omega r) + \psi_{ia}\sin(\Omega r)] . \tag{3.20b}$$

Euclidean invariance requires $C=\Omega$. In this simple solution, both E_i^a and B_i^a have magnitudes which fall like $1/r$ and both rotate transversely in three-space and color space with a period $\Omega = |grB/\sqrt{2}|$ proceeding radially from the origin. Because

$$E_i^a B_i^a = 0 \tag{3.21}$$

there is no special topological charge in the field configuration. The gauge-invariant combination

$$G_{\mu\nu}^a G_{\mu\nu}^a = \tfrac{1}{2}(E_i^a E_i^a + B_i^a B_i^a) = \frac{2\Omega^2}{g^2 r^2} . \tag{3.22}$$

The amount of condensate contained in a sphere of radius R

$$\sigma_G(R) = \frac{\alpha_s}{\pi}\int 4\pi r^2 dr (G_{\mu\nu}^a G_{\mu\nu}^a)(r) , \quad \sigma_G(R) = \frac{2\Omega^2}{\pi} R . \tag{3.23}$$

Since the value of G^2 depends on the choice of origin, this trial solution does not immediately provide an acceptable model of the vacuum. However, we can achieve translational invariance with a "dilute" gas of such fields. Since the trivial vanishing field configuration also provides an acceptable solution to (3.11) we can terminate this solution at radius R (at the expense of introducing δ-function terms into the sources). A volume of three-space filled with a density $\rho = n/L^3$ of these condensation centers would have

$$\sigma_G = \frac{2\rho}{\pi}\Omega^2(RL^3) . \tag{3.24}$$

This corresponds to a dilute gas if the total volume filled by the spheres is small,

$$\rho\frac{4\pi}{3}R^3 \ll 1 . \tag{3.25}$$

The equations of motion provide a nontrivial constraint if there are many oscillations of the fields within each sphere,

$$\Omega R \gg 2\pi . \tag{3.26}$$

These two requirements are consistent if

$$\Omega \gg 2\pi \sigma_G^{1/4} , \tag{3.27}$$

and a dilute ensemble of such field configurations would interact very weakly so that the directions in group space and three-space of each "condensation" site are approximately independent.

Constant sources

In the solution discussed we were forced into the artifice of putting together solutions with different origins. Since the condensate density followed the $1/r^2$ density of $q(r/r_0)$ and $v(r/r_0)$, this suggests that we choose constant sources,

$$q(x) = 2cr_0^2 , \quad v(x) = -2wr_0^2 . \tag{3.28}$$

Equations (3.11a) and (3.11b),

$$(f - xf')' + \frac{2}{x} f(\alpha^2+\beta^2) = 2cr_0^2 x^2 , \quad 2(\alpha\beta' - \beta\alpha') = -2wr_0^2 x^2 , \tag{3.29}$$

permit the ansatz

$$f(x) = cx , \quad \alpha(x) = r_0 x \sin\omega x , \quad \beta(x) = r_0 x \cos\omega x , \tag{3.30}$$

so that the color electric and magnetic fields can be written

$$E_i^a = \frac{C}{g}(\phi_{ia}\sin\Omega r - \psi_{ia}\cos\Omega r) ,$$
$$B_i^a = \frac{\Omega}{g}\left[\left[\frac{1}{r^2} - 1\right]\frac{1}{\Omega}\rho_{ia} + \phi_{ia}\left[\cos\Omega r + \frac{1}{\Omega r}\sin\Omega r\right] + \psi_{ia}\left[\sin\Omega r - \frac{1}{\Omega r}\cos\Omega r\right]\right] , \tag{3.31}$$

where, as before $c = c/r_0$ and $\Omega = \omega/r_0$. This gives

$$\tfrac{1}{2}[(E_i^a E_i^a) + (B_i^a B_i^a)] = \tfrac{1}{2}\frac{1}{g^2}\left[\frac{1}{r^4} - 1\right] + \frac{\Omega^2 + C^2}{g^2} . \tag{3.32}$$

Because of the $1/r^4$ singularity, this density is not integrable over $4\pi r^2 dr$ and cannot be used directly as a model for the vacuum in a region containing the origin. It is also important to observe that, unlike the transverse solutions discussed earlier, there is a value

$$E_i^a B_i^a = \frac{2c}{g^2 r} \tag{3.33}$$

associated with these fields. We therefore seem to have topological charges imbedded among the electric and magnetic charges in this solution.

The patched solution

We can combine the features of the two solutions above by patching them together at $x=1$ ($r=r_0$) to give

$$\begin{aligned} f(x)&=cx\ ,\\ \alpha(x)&=\sin\omega x,\quad x<1\\ &=x\sin\omega x,\quad x\geq 1\ ,\\ \beta(x)&=\cos\omega x,\quad x<1\\ &=x\cos\omega x,\quad x\geq 1\ . \end{aligned} \tag{3.34}$$

This can be generated by

$$\begin{aligned} q(x)&=\frac{2c}{x^2},\quad x<1\\ &=2c,\quad x>1\ ,\\ v(x)&=-\frac{2\omega}{x^2},\quad x<1\\ &=-2\omega,\quad x>1\ ,\\ x\mu(x)&=(\omega^2+c^2)\sin\omega x,\quad x<1\\ &=(\omega^2+c^2+1-1/x^2)x\sin\omega x\\ &\quad -2\omega\cos\omega x,\quad x>1\ ,\\ x\nu(x)&=(\omega^2+c^2)\cos\omega x,\quad x<1\\ x\nu(x)&=(\omega^2+c^2+1-1/x^2)x\cos\omega x\\ &\quad +2\omega\sin\omega x,\quad x>1\ . \end{aligned} \tag{3.35}$$

In addition, the sources defined at $x=1$ contain δ functions not explicitly shown. This matchup gives the field strengths of (3.20) ($x<1$) and (3.31) ($x>1$) and the condensate density

$$\begin{aligned} \frac{g^2}{4\pi^2}\frac{1}{2}(E_i^aE_i^a+B_i^aB_i^a)&=\frac{\Omega^2}{2\pi^2r^2},\quad r<r_0\\ &=\frac{\Omega^2}{2\pi^2r_0^2}+\frac{1}{4\pi^2}\left[\frac{1}{r^4}-1\right],\\ &\qquad r>r_0 \end{aligned} \tag{3.36}$$

so that we can arrange in this manner to have an "almost" constant value of the gluon condensate.

Wilson loops and classical confinement

The above preliminary exercise was instructive in showing how the equations of motion can be used to construct a "medium" with specific chromodynamic properties. We are not going to pursue these particular solutions further because they lack one property believed to be crucial in understanding the QCD vacuum; they do not confine.

To understand the concept of confinement in this type of classical approach where there are no "quanta" to play the roles of quarks and gluons we can borrow from the work on other nonperturbative approaches and simply form a closed loop from the classical fields

$$W^{\rm cl}(s)=\rho^a g\oint_s A_a^\mu dl_\mu\ . \tag{3.37}$$

This is a gauge-invariant but path-dependent function of the classical fields. If the function W grows linearly (or faster) with the area of the surface, then we may say we have a "classical" version of confinement where two hypothetical test charges whose own fields can be neglected would experience a linear confining potential. If the fields give a behavior which depends more slowly than linearly with the area, then we have the classical version of a nonconfining medium. With the gauge choice $r_iA_i=0$ and the assumption of time independence, it is particularly convenient to evaluate the loop with two radial legs and two time-directed legs,

$$\begin{aligned} W^{\rm cl}(r)&=g[A_4^a(r_2)-A_4^a(r_1)]\hat{r}_aT\\ &=\left[\frac{1}{r_2}f(r_2/r_0)-\frac{1}{r_1}f(r_1/r_0)\right]T\ . \end{aligned} \tag{3.38}$$

For the solutions discussed above (3.16), (3.30), and (3.34), $f(x)=cx$ and the classical loop vanishes. This means that a medium constructed from a set of classical charges leading to these field configurations would have little in common with a "confining" vacuum. In order for the radial loop to have an area-law behavior, we must have $f(x)=\sigma x^2$ so that

$$\begin{aligned} W^{\rm cl}(r)&=\frac{\sigma}{r_0{}^2}(r_2-r_1)T\\ &=\frac{\sigma}{r_0{}^2}\ (\text{area})\ . \end{aligned} \tag{3.39}$$

We can use this requirement to construct model solutions to the equations (3.11) which display classical confinement.

Classical confinement by static magnetic charges

One of the most persistent attempts to understand the dynamics of confinement involves the "dual Meissner effect."[4] The idea is that the QCD vacuum is full of mobile magnetic charges which repel electric flux in much the same way as the mobile Cooper pairs in a superconductor repel magnetic flux. We will not go into any detail concerning the origin of this idea but it does suggest the possibility of "constructing" a vacuum from source configurations with $q(x)=0$. If we insert the "confining" ansatz

$$f(x)=\sigma x^2 \tag{3.40}$$

into (3.11a), we get

$$2\sigma x(\alpha^2+\beta^2-1)=x^2 q(x) . \tag{3.41}$$

For the static electric charge distribution to vanish, we must have $\alpha^2+\beta^2=1$, which can be parametrized in the form

$$\alpha=\sin[\omega(x)] ,\quad \beta=\cos[\omega(x)] . \tag{3.42}$$

With this parametrization, Eq. (3.11b) becomes

$$\omega(x)=-\int \frac{x^2}{2}v(x) , \tag{3.43}$$

where the choice of an integration constant involves a boundary condition. The two equations for the transverse sources can be written in the form

$$\begin{bmatrix} \sigma^2x^2+\omega'^2 & -\omega'' \\ \omega'' & \sigma^2x^2+\omega'^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x\nu \\ x\mu \end{bmatrix} \tag{3.44}$$

which shows that everything is determined in terms of $v(x)$ and σ. Inserting this solution into (3.14) gives

$$E_i^a=\frac{\sigma}{gr_0^2}[-\rho_{ia}+\phi_{ia}\sin\omega(x)-\psi_{ia}\cos\omega(x)] ,$$
$$B_i^a=-\frac{xv(x)}{2gr_0^2}[\phi_{ia}\cos\omega(x)+\psi_{ia}\sin\omega(x)] . \tag{3.45}$$

An interesting feature of this solution is that the magnitude of E_i^a does not depend on position. The color electric field merely rotates in position and isospin space so that

$$E_i^aE_i^a=\frac{\sigma^2}{g^2r_0^4}\{1+2[\sin^2\omega(x)+\cos^2\omega(x)]\} \tag{3.46}$$

is independent of position. We can also have $B_i^aB_i^a$ independent of position if we choose

$$-xv(x)=2l ,\quad \omega(x)=\tfrac{1}{2}lx^2 \tag{3.47}$$

so that

$$E_i^a=\frac{\sigma}{gr_0^2}[-\rho_{ia}+\phi_{ia}\sin(\tfrac{1}{2}lx^2) -\psi_{ia}\cos(\tfrac{1}{2}lx^2)] ,$$
$$B_i^a=\frac{l}{gr_0^2}[\phi_{ia}\cos(\tfrac{1}{2}lx^2)+\psi_{ia}\sin(\tfrac{1}{2}lx^2)] . \tag{3.48}$$

The stronger requirement of generalized Euclidean invariance

$$E_i^aE_i^a=B_i^aB_i^a \tag{3.49}$$

gives $3\sigma^2=2l^2$ so that everything is specified in terms of the parameter σ which gives the coefficient of the area-law behavior for Wilson loops in Eq. (3.38),

$$\sigma_G=\langle \text{vac} \,|\, (\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu} \,|\, \text{vac}\rangle = \frac{3}{4\pi^2}\frac{\sigma^2}{r_0^4}=\frac{3}{4\pi^2}(W_{\text{cl}}/A)^2 . \tag{3.50}$$

This equation can be used to test whether we have constructed a reasonable model for a confining vacuum since it relates two, otherwise independent, observables. Inserting the value

$$\sigma/r_0^2=0.45\ \text{GeV}^2 \tag{3.51}$$

such as is found in SU_2 lattice calculations[8] gives a value

$$\sigma_G=0.015\ \text{GeV}^4 \tag{3.52}$$

which is consistent with the value (1.3) found phenomenologically by Shifman, Vainshtein, and Zakharov.[12] However, since we are neglecting fermions and dealing with SU_2 rather than SU_3 it is not clear that this numerical agreement is significant.

It is important to observe that the solution (3.48) has $E_i^aB_i^a=0$ so that we are not introducing any topological charge. As is obvious from comparison with earlier solutions, it is not trivial that this naive construction should result in an expression for $(E_i^aE_i^a+B_i^aB_i^a)$ which is independent of position. In fact, the fields (3.48) have three properties: (1) area law for classical Wilson loops; (2) no static color electric charges; (3) $E_i^aE_i^a+B_i^aB_i^a=$const, widely attributed to the QCD vacuum. The fact that a nonlinear set of equations admits a solution with this much in common with our naive expectations is highly encouraging.

IV. DISCUSSION

We found, in Sec. III, that the classical equations of motion for SU_2 gluon dynamics admit solutions which have the characteristics of an extended medium. Within the limitations imposed by our simplifying assumptions, it is important to ask if these solutions can be useful in constructing a quantum-field-theoretical understanding of the physical vacuum. In this regard, it is interesting to examine the algorithm proposed by Nielsen and Olesen[5] for constructing the Copenhagen vacuum.

As mentioned in the Introduction the starting point for the construction of the Nielsen-Olesen vacuum is the Savvidy ansatz,[14] a state with

$$G^3_{12}=H \tag{4.1}$$

and all other components vanishing. Direct substitution shows that this satisfies

$$(D^\mu G^a_{\mu\nu})=0 \tag{4.2}$$

since both the commutator term and the derivative vanish separately. As pointed out by Ambjørn and Olesen,[5] however, this solution is not stable against the addition of small fluctuations. The construction of Nielsen and Olesen[5] systematically "improves" the approximation in several steps while retaining the property $G^a_{\mu\nu}G^{a,\mu\nu}\neq 0$. The enumeration of these steps proves to be instructive.

Step 1. The linear instability in the classical equations is used to give the field some spatial dependence,

$$G^3_{12}=H\{1-2\exp[-gH(x_1{}^2+x_2{}^2)/2]\} \tag{4.3}$$

so that the field configuration corresponds to a flux tube of radius $r=(2\ln 2/gH)^{1/2}$ surrounded by an infinite return yoke. It is at this point that the model starts to develop domain structure.

Step 2. The construction of the model proceeds by filling the x_1-x_2 plane by a lattice of flux tubes separated a distance

$$\Delta=\left[\frac{2\pi}{gH}\right]^{1/2}. \tag{4.4}$$

At this stage the color field is still strictly magnetic.

Step 3. Nielsen and Olesen then formulate an estimate of the fluctuations of the centers of the flux tubes. They find

$$\theta=\frac{(x_1-\langle x_1\rangle)^2}{\Delta^2}\gtrsim 0.2 \tag{4.5}$$

which strongly suggests that the medium constructed from these flux tubes is more akin to a classical liquid than to a solid. They conclude that the lattice constructed in step 2 does not remain fixed in place but undulates. The time fluctuations of the magnetic fields at this stage of the construction generate color electric fields.

Step 4. The bending of the flux tubes in the direction orthogonal to the x_1-x_2 plane is taken into account.

Step 5. Tunneling and fluctuations of the domain walls are taken into account. More electric fields are produced at this stage.

Step 6. Statistical averaging over directions in three-space and in color space are used to finally produce a state which is a Lorentz and color scalar.

These steps in this construction are outlined here to demonstrate how far away one must go from the original starting point of a constant magnetic field in order to generate an acceptable model for the QCD vacuum. It is notable that one requirement for Lorentz invariance, $E^a_iE^a_i=-B^a_iB^a_i$, is not directly addressed in the construction since electric fields are only generated as quantum fluctuations while the magnetic fields are treated classically. Equally important, once past step 2 in the algorithm, there are no longer explicit expressions for the field strengths. This means, for example, that the question of whether the vacuum constructed in this manner actually confines color charge has to be dealt with using indirect arguments about the random nature of the flux.

We will not be able to go into detail here about the steps which must be taken to generate a comparable quantum model of the vacuum starting from one of the classical solutions generated in Sec. III. It is to be hoped that the construction is less involved since the starting point has more in common with the physical state we are trying to describe. The formalism for quantizing a non-Abelian gauge theory using fluctuations around an arbitrary background field has been discussed recently by Ambjørn and Hughes[29] and by Horibe and Hosoya.[30] It appears that the methods developed can be applied to the solutions to the field equations represented in Sec. III.

In analogy to the construction of Nielsen and Olesen,[5] one of the most important things to do is to investigate the classical stability of the solutions to the equations under the addition of small fluctuations. This can be done in several different ways. A first approximation might involve leaving the classical sources fixed and varying only the fields around these classical values. It is not clear, however, that this is the most appropriate approach to the stability problem so we will not go further into the question.

One problem which merits some study involves the interpretation of these solutions once the continuation is made from Euclidean space to Minkowski space. The appearance of imaginary values for the vacuum expectation of "classical" fields is certainly disconcerting. It is not clear that the constraint

$$\langle \text{vac} \mid B_i^a B_i^a \mid \text{vac} \rangle = -\langle \text{vac} \mid E_i^a E_i^a \mid \text{vac} \rangle \qquad (4.6)$$

should necessarily be imposed at the classical level. The solution found by the Copenhagen vacuum which ignores electric fields except those generated by fluctuations in the magnetic fields provides room for simpler classical interpretation but makes it difficult to deal with the issue of confinement through the introduction of classical Wilson loops.

Finally, it is obviously important at some point to go back and study the properties of a system which includes fermions in a proper way. Sources which yield $E_i^a B_i^a \neq 0$ must be examined in a more careful manner in conjunction with the problem of chiral-symmetry breaking. The general question of the vacuum structure of non-Abelian gauge theories provides an interesting starting point for many other problems.

ACKNOWLEDGMENTS

This work was begun during a visit to the University of Colorado. I would like to thank Professor K. T. Mahanthappa and Professor Thomas DeGrand for their interest and support. The advice and criticism of David Maloof and David Calloway have also been influential. This work was performed under the auspices of the United States Department of Energy.

---

[1]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[2]For a review, see, for example, P. Hasenfratz and J. Kuti, Phys. Rep. 40C, 75 (1978).

[3]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Rev. D 19, 1826 (1979).

[4]S. Mandelstam, Phys. Rep. 23C, 245 (1976).

[5]H. B. Nielsen and P. Olesen, Nucl. Phys. B160, 380 (1979).

[6]T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. B129, 493 (1977); G. 't Hooft, Nucl. Phys. B138, 1 (1978).

[7]T. H. Hansson, K. Johnson, and C. Peterson, Phys. Rev. D 26, 2069 (1982).

[8]M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rev. Lett. 42, 1390 (1979).

[9]See, for example, C. Rebbi, at the lectures presented at the 1981 G.I.F.T. summer school, Trieste Report No. IC/81/151 (unpublished).

[10]H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).

[11]T. D. Lee, Phys. Rev. D 19, 1802 (1979).

[12]M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147, 385 (1979); B147, 448 (1979); B147, 519 (1979).

[13]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[14]G. K. Savvidy, Phys. Lett. 71B, 133 (1977).

[15]R. Fukuda and Y. Kazama, Phys. Rev. Lett. 45, 1142 (1980).

[16]D. H. Miller, Report No. SLAC Pub-2841 (unpublished).

[17]R. Fukuda and Y. Kazama, report, 1982 (unpublished).

[18]S. J. Chang and G.-J. Ni, Phys. Rev. D 26, 864 (1982).

[19]H. B. Nielsen and P. Olesen, Report No. NBI-HE-79-45 (unpublished); P. Olesen, Nucl. Phys. B200 [FS4], 381 (1982).

[20]For a review, see A. Actor, Rev. Mod. Phys. 51, 461 (1979).

[21]H. Luetwyler, Bern report, 1982 (unpublished).

[22]S. Coleman, Phys. Lett. 70B, 59 (1977).

[23]P. Sikivie and N. Weiss, Phys. Rev. Lett. 40, 1411 (1978); Phys. Rev. D 18, 3809 (1978).

[24]C. H. Oh, Phys. Rev. D 25, 2194 (1982), and references therein.

[25]L. Mathelitsch, H. Mitter, and F. Widder, Phys. Rev. D 25, 1123 (1982).

[26]L. Jacobs and J. Wudka, Phys. Rev. D 25, 1114 (1982).

[27]R. Jackiw, L. Jacobs, and C. Rebbi, Phys. Rev. D 20, 474 (1979).

[28]R. Jackiw and P. Rossi, Phys. Rev. D 21, 426 (1980).

[29]J. Ambjorn and R. J. Hughes, Report No. CALT-68-903 (unpublished).

[30]M. Horibe and A. Hosoya, Prog. Theor. Phys. 64, 1814 (1980).