Dimensional reduction in finite-temperature quantum chromodynamics

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The infrared behavior of four-dimensional quantum chromodynamics at finite temperature and chemical potential is examined within the context of perturbation theory. The reduction to an effective three-dimensional theory of the Yang-Mills field coupled to a massive adjoint scalar field is explicitly shown to occur at the one-loop level. A renormalization scheme especially appropriate for the reduction is exhibited. By working in a general Lorentz-covariant gauge, the (well-known) one-loop electrostatic mass is shown to be gauge invariant. Infrared divergences at the two-loop level indicate the need for a nonperturbative treatment of the effective theory; their gauge dependence implies that the naive method for computing the electrostatic mass in covariant gauges is invalid beyond one-loop. Further analysis is carried out in a class of gauges ("static gauges") that are particularly well suited for finite-temperature calculations. The systematic construction of the effective theory is outlined, and performed in a static gauge. At distance scales beyond the electrostatic screening length, pertinent to an investigation of possible magnetostatic screening, the effective theory simplifies further to pure three-dimensional Yang-Mills theory with coupling $T^{1/2}g(T)$. This implies that the leading-order magnetostatic mass gap must be proportional to g^2T.

I. INTRODUCTION

Hadronic matter, when subjected to sufficiently high temperatures and/or densities (such as might occur in astrophysical processes or in heavy-ion collision experiments), goes into a phase where its colored constituents (quarks and gluons) are no longer confined. While the nature and details of this deconfining phase transition are not yet completely understood, the existence of the quark-gluon plasma phase is widely accepted. We shall, therefore, assume the qualitative phase diagram shown in Fig. 1. ($T_c \sim 200$ MeV $\sim 2\times10^{12}$ °K, μ_c corresponds to several times nuclear density.) We choose to work

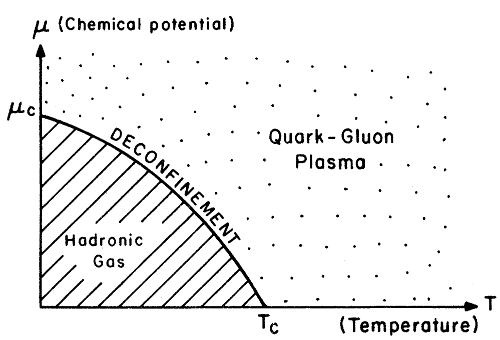

FIG. 1. Phase diagram for hadronic matter.

well above the phase transition (so that its details are irrelevant) and shall concern ourselves with the properties of the quark-gluon plasma in thermochemical equilibrium with its environment. Such a system is described statistically by the grand canonical ensemble, and we shall adopt quantum chromodynamics as the underlying field theory. Thus we are led to examine the properties of four-dimensional QCD at finite temperature and chemical potential ($QCD_{4T\mu}$).[1]

It is interesting to compare the QCD plasma with the QED plasma. The case of $QED_{4T\mu}$ is well known[2]: Only magnetostatic correlations persist over long distances; fermionic and nonstatic electromagnetic fluctuations are thermally damped out, while electrostatic fields undergo Debye screening. Since photons have no self-interaction, the static, long-range sector of the theory is trivial. Perturbation theory is completely applicable to $QED_{4T\mu}$.

Unlike its Abelian counterpart, however, $QCD_{4T\mu}$ is plagued by severe infrared divergences in the perturbation expansion. To some extent these are cured by the chromodynamic analog of the Debye effect: screening of static (chromo-) electric fields. Also, fermions and nonstatic (chromo-) electromagnetic fields are thermally damped out over large distances. The problem is with the static infrared (chromo-) magnetic sector of the theory: it is highly nontrivial and nonperturbative, on account of the selfinteractions between massless gluons. It seems likely that this nonperturbativity implies the generation of a mass.[3] It is unclear whether this mass is a Debye-type screening mass or whether it corresponds to a bound-state (glueball) mass gap, or even whether these are distinct, mutually exclusive choices. What *is* clear is that the singular properties of $QCD_{4T\mu}$ warrant further investigation.

Indeed, there has been much recent interest in this topic: analytical studies[1,3,4] as well as numerical Monte Carlo computations[5] have been carried out. Common to most of the analytical investigations of $QCD_{4T\mu}$ is the working hypothesis that the leading infrared behavior of the theory is completely described by an effective Lagrangian, which is essentially that of Yang-Mills theory (pure, quarkless QCD) in three Euclidean dimensions (QCD_3).[4] This phenomenon, which we call "dimensional reduction",[6] can be viewed in the context of the decoupling theorem[7]: The nonstatic modes of $QCD_{4T\mu}$ are suppressed in the infrared, their only effect being to renormalize the input parameters (coupling constants, masses) of the effective theory. Note that QCD_3 is itself ultraviolet finite. (Being superrenormalizable, it can have at most ultraviolet mass renormalization; however, gauge invariance prohibits even that from occurring.)

This paper has two main objectives: firstly, to explicitly demonstrate dimensional reduction in $QCD_{4T\mu}$ at the one-loop level of perturbation theory. This is done in Sec. II, where we compute the one-loop covariant-gauge gluon propagator at finite temperature and finite (quark) chemical potential, in the static limit. We show that maximal decoupling of the nonstatic modes can be achieved through a proper choice of renormalization constants. As a byproduct of the calculation, we establish the gauge invariance of the one-loop electrostatic mass. The infrared-effective one-loop theory that emerges is QCD_3 coupled to a massive adjoint scalar field (the electrostatic field) which we call extended QCD_3 ($EQCD_3$).

Our second goal is to see just how far we can go with a perturbative treatment. This is the subject of Secs. III and IV. In Sec. III, we examine the two-loop covariant-gauge gluon propagator (which also contains information about one-loop coupling-constant corrections) and find gauge-dependent infrared divergences. These indicate the breakdown of naive perturbation theory, due to the perturbative expansion of the effective theory. They also indicate the gauge dependence of the naive covariant-gauge electrostatic mass beyond one-loop, making covariant gauges inconvenient for further calculations. In Sec. IV, we abandon covariant gauges in favor of a more convenient class of gauges and propose a scheme for constructing the effective theory wherein the renormalization effects of the nonstatic sector are computed perturbatively, free of infrared divergences. The definition of $EQCD_3$ is broadened to include a quartic self-interaction for the adjoint scalars. In the far infrared these massive scalars decouple from $EQCD_3$, which then simplifies to yield QCD_3 as the effective theory.

Our main results and conclusions are recapitulated in Sec. V. Notation, Feynman rules, and some computational details are presented in Appendix A. In Appendix B, we consider the effects of colored scalars on dimensional reduction. We find that a finite chemical potential for scalars leads to antiscreening effects but, provided these are not too large, dimensional reduction goes through as before.

II. THE ONE-LOOP CALCULATION

Consider $QCD_{4T\mu}$ in the plasma phase. We shall assume the validity of perturbation theory until it breaks down at some loop order (by the occurrence of infrared divergences). Thus we are not too concerned with the size of the coupling. However, for the sake of rigor, we may take (T,μ) to be sufficiently large. Then the coupling is small and we can also neglect quark masses. (Their inclusion would, in any case, only *improve* our results.) Since we are interested in the collective behavior of the plasma, which involves the long-range properties of the theory, we shall focus our attention on infrared (momentum transfer $|\vec{k}|$ small compared to temperature T) correlation functions (propagators).

At the classical (tree-graph) level, only zero-energy bosonic modes ("static modes") propagate in the infrared; fermionic and finite-energy bosonic modes ("nonstatic modes") are suppressed by factors of $\sim |\vec{k}|^2/T^2$. The surviving propagators are thus those of the static A_4 field ("electrostatic potential") and the static A_i field ("magnetostatic potential"); thus the effective infrared theory is $EQCD_3$ (where the A_i's play the role of the Yang-Mills potential in three dimensions and the A_4 appears as a scalar, transforming under the adjoint representation of the gauge group, which is massless at the classical level).

Quantum corrections to the classical theory involve an infinite sum over nonstatic modes. Does dimensional reduction continue to work? To answer this question, we shall calculate the one-loop corrections to the covariant-gauge static gluon propagators. We shall find that, in a suitable renormalization scheme, dimensional reduction works even better at this level. The only effect of the nonstatic sector is to renormalize the input parameters of the static, three-dimensional sector. Thus, at this order in perturbation theory we see finite-mass generation for the A_4 field ("electrostatic or Debye mass") and wave-function renormalization. The A_i field remains massless.

We define the parameters of our calculations as follows (details of the calculation will be found in Ref. 8; for notation, Feynman rules, etc., see Appendix A):

SU(N): Gauge group ,
N_f: Number of quark flavors ,
μ_f: Chemical potential for flavor $f=1,\ldots,N_f$,
T: Temperature ,
D: Dimension of spacetime , $D\to 4$,
Λ: Scale of dimensional regularization ,
ξ: Covariant-gauge parameter .

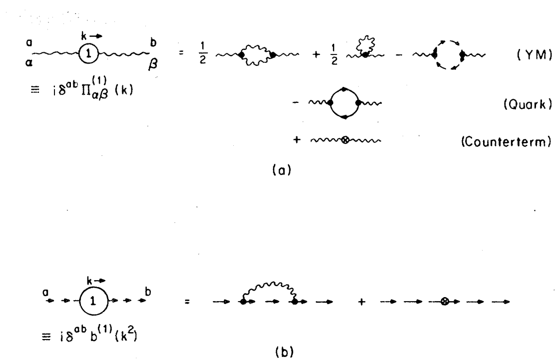

FIG. 2. One-particle irreducible (1PI) self-energy diagrams at the one-loop level for (a) gluon and (b) ghost.

A. Gluon self-energy

The renormalized one-particle-irreducible (1PI) gluon self-energy (polarization tensor) $\Pi_{\alpha\beta}(k)$ is shown to one-loop in Fig. 2(a). At finite T and μ we find, in the static infrared limit ($k_4=0,\vec{k}\to 0$), the following.

(i) Electrostatic mass and self-energy.

$$\Pi^{(1)}_{44}(\vec{k}^2)=\underbrace{-m_E{}^2+\frac{Ng^2T\,|\vec{k}\,|}{4}}_{\text{EQCD}_3}+\underbrace{g^2T\,|\vec{k}\,|\,O(\,|\vec{k}\,|^3/T^3)}_{\text{Nonstatic}}\ ,$$

where the electrostatic mass (squared) is given by

$$m_E{}^2=(N+\tfrac{1}{2}N_f)\frac{g^2T^2}{3}+\frac{g^2}{2\pi^2}\sum_{f=1}^{N_f}\mu_f{}^2\ .$$

(ii) Magnetostatic self-energy.

$$\Pi^{(1)}_{ij}(\vec{k})=(\delta_{ij}-k_ik_j/\vec{k}^2)\Pi^{(1)}(\vec{k}^2)\ ,$$

$$\Pi^{(1)}(\vec{k}^2)=\underbrace{[(\xi+1)^2+10-2]\frac{Ng^2T\,|\vec{k}\,|}{64}}_{\text{EQCD}_3}+\underbrace{g^2T\,|\vec{k}|\,O(|\vec{k}\,|^3/T^3)}_{\text{Nonstatic}}\ .$$

(iii) Gluon wave-function renormalization constant.

$$\underbrace{Z^{(1)}_3-1=\tfrac{1}{3}[(3\xi-13)N+4N_f]\frac{g^2}{16\pi^2}\left[\frac{1}{D-4}+\ln\frac{T}{\Lambda}+C_3\right]}_{\text{Nonstatic}}\ ,$$

where C_3 is a constant containing ξ,N,N_f and is *different* for Π_{44} and Π_{ij}.

B. Ghost self-energy

The ghost is part of the Yang-Mills field and, for dimensional reduction to work, should behave similarly to A_i. Calculating the renormalized 1PI ghost self-energy $b(k^2)$ to one-loop [Fig. 2(b)] at finite T and μ, in the static limit, we find that this is indeed the case,

$$b(\vec{k}^2)=\underbrace{\frac{Ng^2T\,|\vec{k}\,|}{16}}_{\text{EQCD}_3}+\underbrace{g^2T\,|\vec{k}\,|\,O(\,|\vec{k}\,|^3/T^3)}_{\text{Nonstatic}}\ ,$$

$$\tilde{Z}_3^{(1)}-1=\underbrace{\left[\frac{\xi-3}{2}\right]N\frac{g^2}{16\pi^2}\left[\frac{1}{D-4}+\ln\frac{T}{\Lambda}+\tilde{C}_3\right]}_{\text{Nonstatic}}\ ,$$

where $\tilde{C}_3$ is another constant, which depends on the parameter ξ.

C. Comments

(i) The result of a typical self-energy computation is of the form

$$\pi(\vec{k}^2)=\bar{\pi}(\vec{k}^2)-\vec{k}^2(Z-1)\ ,$$

where π is generic for Π_{44}, Π, or b and where $\bar{\pi}$, the result of the loop integrations, contains ultraviolet infinities that are removed via Z. At $T=\mu=0$, $\bar{\pi}(\vec{k}^2)$ typically contains a $\ln\vec{k}^2$ term; at $T/\mu\neq0$, the logarithmic dependence is cut off by the T/μ scale. (We have assumed $\mu\lesssim T$; for $\mu\gg T$, μ replaces T in the $\ln T/\Lambda$ term for the fermionic loop.) This absence of $\ln\vec{k}^2$ is crucial for dimensional reduction to work effectively. $\bar{\pi}(\vec{k}^2)$ in the infrared approximation is then of the form

$$\bar{\pi}(\vec{k}^2)=\bar{\pi}_0T^2+\bar{\pi}_1T\,|\vec{k}\,|+\bar{\pi}_2\vec{k}^2+O(\vec{k}^4/T^2)\ .$$

We renormalize by setting $(Z-1)=\bar{\pi}_2$, i.e., by subtracting off all terms proportional to $\vec{k}^2$ in the self-energy; we call this scheme "maximal decoupling" (MD), since it minimizes the effects of the nonstatic modes. The renormalization constant thus contains some pieces additional to the minimal-subtraction pole at $T=\mu=0$. In accordance with the breaking of manifest Lorentz invariance at finite T/μ, the MD renormalization constants for electrostatic and magnetostatic gluons are different. The scale parameter Λ is to be chosen to be appropriate to the renormalization of the coupling constant; we expect $\Lambda\sim T$ corresponding to a renormalized coupling $g(T)$ (see Sec. IV).

(ii) We have split up the contributions to the self-energies into EQCD_3 and nonstatic parts. In the EQCD_3 part we include renormalization effects of the nonstatic modes. We see that the only leading effect of the infinitely many nonstatic modes is a finite mass renormalization $(m_E{}^2)$ and infinite wave-function renormalization $(Z_3,\tilde{Z}_3)$; otherwise, their contribution is suppressed by a factor $\sim(\,|\vec{k}\,|/T)^3$. [In an arbitrary renormalization scheme, the suppression factor is $|\vec{k}\,|/T\gg(\,|\vec{k}\,|/T)^3$.] Thus the leading infrared behavior of $\text{QCD}_{4T\mu}$ is described by EQCD_3.

(iii) Electrostatic (A_4-A_4) correlations acquire a mass m_E which is $O(g)$. Both the mass and the electric self-energy are manifestly gauge invariant (i.e., ξ independent). The corresponding Debye screening length is $l_E\equiv m_E{}^{-1}$.

(iv) Magnetostatic (A_i-A_j and ghost) correlations remain massless to this order. Examine $\Pi^{(1)}(\vec{k}^2)$, the magnetic self-energy. The contribution $\sim[(\xi+1)^2+10]$ comes from pure QCD_3, while the electrostatic field (i.e., the "E" part of EQCD_3) contributes $\sim(-2)$. If the electrostatic mass were to be resummed, then in the *far infrared* $(\,|\vec{k}\,|\ll m_E)$ the electrostatic contribution to $\Pi^{(1)}(\vec{k}^2)$ would also be nonleading and the effective far-infrared theory would be pure QCD_3, as described in Ref. 4.

III. BEYOND ONE LOOP: INFRARED DIVERGENCES AND GAUGE DEPENDENCE

The one-loop calculation revealed the general scheme of dimensional reduction: Except for renormalization effects, the infrared contribution of nonstatic modes is suppressed relative to the leading contribution, which is given by an effective three-dimensional theory. In a suitable renormalization scheme, the amount of suppression can be maximized.

Naively, one might expect to be able to check this scenario to all orders in g^2. However, infrared divergences (IRD's) begin to appear, preventing any further analysis based on ordinary perturbation theory.

To understand the nature and origin of these two-loop IRD's, we shall examine the covariant-gauge gluon self-energy $\Pi_{\alpha\beta}$ to two-loops. More specifically, we shall focus on the IRD's in the electrostatic self-energy $\Pi_{44}^{(2)}$, the case of $\Pi_{ij}^{(2)}$ being similar. It will be seen that the IRD's are a result of the perturbative expansion of the static sector.

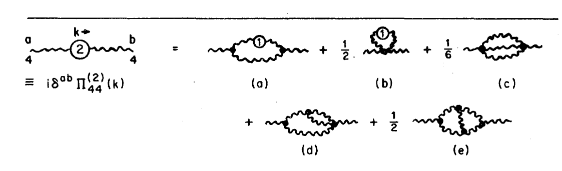

FIG. 3. Two-loop contribution to $\Pi_{44}(k)$, showing infrared-divergent graphs.

A. Infrared divergences in $\Pi_{44}^{(2)}$

The two-loop contributions to Π_{44} are shown in Fig. 3. To obtain the low-momentum behavior, we examine them for IRD's, first at $\vec{k}\neq 0$ and then at $\vec{k}=0$.

At $k_4=0$, $\vec{k}\neq 0$ we find that graphs 3(a) and 3(b) have scaleless logarithmic divergences which vanish under dimensional regularization. Thus,

$$\Pi_{44}^{(2)}(\vec{k})\Big|_{\rm IRD}=(Ng^2T)\left[\frac{\frac{4}{3}\vec{k}^2}{\vec{k}^2+m_E{}^2}-1\right]\times\int\frac{d_3\vec{q}}{\vec{q}^4}\Pi_{ii}^{(1)}(\vec{q}^2)=0$$

$$[d_3\vec{q}\equiv d^3\vec{q}/(2\pi)^3]\,,$$

where $\Pi_{ii}^{(1)}(\vec{q}^2)=2\Pi^{(1)}(\vec{q}^2)\propto|\vec{q}|$, as determined by the one-loop calculation of Sec. II. We have put in an $m_E{}^2$ in one of the denominators to indicate what would result if the Debye mass had been resummed [this also trivially modifies the low-momentum behavior of $\Pi^{(1)}(\vec{q}^2)$ according to comment (iv) in Sec. II C]—there is no significant difference. Thus the low-momentum behavior of $\Pi_{44}^{(2)}(\vec{k})$ is governed by the IRD's at $\vec{k}=0$.

It is more convenient to do the $\vec{k}=0$ calculation in the Landau and Feynman gauges rather than for arbitrary ξ. The results are as follows.

(i) Landau gauge ($\xi=0$). Graphs 3(a), 3(d), and 3(e) have no IRD. Graphs 3(b) and 3(c) have logarithmic divergences, but cancel each other out. However, the divergence in graph 3(b) is the same scaleless one as in the $\vec{k}\neq 0$ calculation, and must actually be set equal to zero. Thus we are left with

$$\Pi_{44}^{(2)}(0)\Big|_{\rm IRD}^{\xi=0}=\tfrac{9}{32}(Ng^2T)^2\int\frac{d_3\vec{q}}{|\vec{q}|^3}\,.$$

(ii) Feynman gauge ($\xi=1$). Graphs 3(a)–3(e) all have logarithmic divergences which cancel out. Again, on setting $(b)=0$ we have a surviving logarithmic divergence. But in addition graph 3(a) has a *linear* IRD which also survives. Thus,

$$\Pi_{44}^{(2)}(0)\Big|_{\rm IRD}^{\xi=1}=-m_E{}^2Ng^2T\int\frac{d_3\vec{q}}{\vec{q}^4}+\tfrac{3}{8}(Ng^2T)^2\int\frac{d_3\vec{q}}{|\vec{q}|^3}\,.$$

These divergences arise from a one-loop Debye mass insertion in the A_4—A_4 propagator and can be removed by resumming such insertions, as is evident from the following expansion of the resummed propagator:

$$\frac{1}{\vec{q}^2+m_E{}^2}=\frac{1}{\vec{q}^2}(1+m_E{}^2/\vec{q}^2)^{-1}=\frac{1}{\vec{q}^2}-\frac{m_E{}^2}{\vec{q}^4}+\cdots\,.$$

The first term in the expansion creates the logarithmic IRD, the second term is responsible for the linear IRD, and higher-order terms would make the IRD's even more severe.

At $\vec{k}\neq 0$, the IRD's at $\vec{k}=0$ translate into terms of the form $1/|\vec{k}|$ and $\ln|\vec{k}|$. In addition, one has terms $\sim T^2$ and $\sim T|\vec{k}|$ which come from the nonsingular static contribution and the leading nonstatic contribution; in the MD scheme, though, the $\sim T|\vec{k}|$ term vanishes. These terms then constitute the EQCD$_3$ contribution. The rest we label, in accordance with the one-loop calculation, the nonstatic part. This nonleading piece must vanish as $\vec{k}\to 0$. A power-counting analysis of the graphs defining $\Pi_{44}^{(2)}$ indicates that after performing the MD subtraction of terms proportional to $\vec{k}^2$, the coefficients being

$$(Z_3^{(2)}-1)\sim N^2g^4\left[\frac{1}{D-4}+\ln\frac{T}{\Lambda}+\text{const}\right],$$

we are left with

$$[\text{Nonstatic}]\sim N^2g^4O(|\vec{k}|^3/T)\,.$$

Purely nonstatic graphs contribute terms $O(\vec{k}^4/T^2)$ to [Nonstatic] as in the one-loop calculation; some of the graphs with both static and nonstatic internal propagators yield the nonanalytic $O(|\vec{k}|^3/T)$ terms.

To summarize, the infrared behavior of the two-loop electrostatic self-energy is of the form

$$\Pi_{44}^{(2)}(k_4=0,\vec{k})=\underbrace{N^2g^4\left[\frac{AT^3}{|\vec{k}|}+BT^2\ln\frac{|\vec{k}|}{T}+CT^2\right]}_{\text{EQCD}_3}+\underbrace{N^2g^4T|\vec{k}|O(|\vec{k}|^2/T^2)}_{\text{Nonstatic}}\,,$$

where the coefficients A and B, and perhaps also C, are gauge dependent. The divergence of $\Pi_{44}^{(2)}$ as $\vec{k}\to 0$ can be avoided by resumming the one-loop electrostatic mass; however, the $\vec{k}=0$ IRD's are then converted into nonanalytic *gauge-dependent* pieces, implying that Π_{44} in covariant gauges does not directly yield the electrostatic mass beyond one-loop.

B. Infrared divergences in $\Pi_{ij}^{(2)}$

The IR behavior of the two-loop magnetostatic self-energy has been extensively studied[1,3,4]; here, again, one finds gauge-dependent IRD's similar to those above. These IRD's are only partly removed by electrostatic mass resummation; a residual logarithmic term remains. Perhaps this residual IRD is the signal for, and is cured by, the generation of a magnetic mass m_M,[9] in analogy to the role played by m_E in Sec. III A above. In perturbation theory, $m_M=0$ at one-loop, and so $m_M \lesssim g^2T$. If $m_M \sim g^2T$, an infinite class of diagrams (of arbitrarily high loop order) contributes to a given order in the coupling constant. If $m_M < g^2T$ the situation is even worse: the size of the contribution *increases* with loop order. An equivalent manifestation of the breakdown of perturbation theory is the blowing-up at small momenta of the effective IR expansion parameter $g^2T/|\vec{k}|$. The only salvation of an expansion in powers of g would thus be the nonperturbative generation of a magnetic mass $m_M \sim 0(g^n)T$, where $1<n<2$. However, as we shall argue later, $n=2$ is required on dimensional grounds. Thus attempts to modify perturbation theory by some sort of magnetic-mass resummation are bound to fail, and only a completely nonperturbative treatment can work.

Attempts have been made recently to estimate the magnetic mass numerically in finite-temperature lattice gauge theory via Monte Carlo simulations.[5] While these studies indicate the presence of magnetic screening, the statistics are as yet somewhat crude. The reported value for the magnetic mass is $m_M=0.24g^2(T)T$. The computations are performed by using twisted boundary conditions on the lattice to simulate the effects of a monopole-antimonopole pair. It is not completely clear to us how this magnetic screening mass manifests itself in the context of the gauge-dependent static gluon propagator.

C. Discussion

The lesson learned from this section is twofold. First, we find that covariant gauges are inconvenient for direct computation of physical quantities such as the electrostatic mass. Therefore, if a more suitable class of gauges can be found, we should abandon covariant gauges.

Second, we find the breakdown of perturbation theory manifesting itself in the infrared blow-up of $\Pi_{\alpha\beta}^{(2)}$. The IRD's in $\Pi_{44}^{(2)}$ are completely removed by resumming the electrostatic mass, but this only partially cures the IRD's in $\Pi_{ij}^{(2)}$. Of course, these IRD's are gauge dependent and one may consider the possibility that a gauge-invariant computation would be free of IRD's. We find it hard to imagine that the IRD's would continue to be gauge artifacts to all orders of perturbation theory. Indeed, we expect gauge-invariant IRD's at higher levels due to the appearance of gauge-invariant operators in the operator-product expansion. Therefore, we shall take the broader point of view and suppose that we have genuine physical IRD's corresponding to possible nonanalyticities in the g^2 expansion for physical quantities such as the screening masses. Thus we conclude that the necessity for a nonperturbative treatment of the effective theory prevents any further analysis of $\mathrm{QCD}_{4T\mu}$ in ordinary perturbation theory.

Nevertheless, the same dimensional arguments that led to the infrared suppression of nonstatic modes at the classical and one-loop levels seem to work at two-loops also, and should continue to apply to all orders, since there are no IRD's in the purely nonstatic sector. Thus we fully expect dimensional reduction to be valid to all orders, and shall continue our analysis on the basis of this expectation.

IV. PERTURBATIVE CONSTRUCTION OF THE EFFECTIVE LAGRANGIAN

The leading infrared behavior of $\mathrm{QCD}_{4T\mu}$ is described, as we have argued in the previous sections, by an effective theory in three Euclidean dimensions. This effective theory is defined to be the static sector of $\mathrm{QCD}_{4T\mu}$ modulated by the renormalization effects of the nonstatic sector. The static sector must be treated nonperturbatively; we do not attempt such a treatment here. But since there are no infrared divergences in the nonstatic sector, it should be possible to perturbatively compute its renormalization effects. In the present section, we sketch a systematic way of doing this.

The effects of nonstatic modes are of two kinds: *First,* nonstatic modes induce effective interaction vertices in the static sector. The electrostatic mass generation at the one-loop level is the lowest-order example of this. There is an infinite set of induced vertices, corresponding to operators of arbitrarily high order. However, higher-order operators are heavily suppressed in the infrared and consequently do not occur in the effective Lagrangian. (Note that some lower-order induced vertices are just the nonstatic corrections to existing static-sector vertices.) *Second,* nonstatic modes produce wave-function and vertex renormalizations in the original four-dimensional theory. These give rise to coupling-constant renormalization.

The temperature T and the renormalized coupling $g(T)$ are the two parameters in terms of which all bare interaction vertices of the effective theory can be expressed. For example, the three-dimensional coupling constant G and the electrostatic bare mass parameter $m_0{}^2$ can be written

$$G=T^{1/2}g(1+A_1g^2+A_2g^4+\cdots),$$

$$m_0{}^2=T^2(B_1g^2+B_2g^4+\cdots),$$

where A_i, B_i are constants that can be evaluated perturbatively in a manner to be described below. In practice it will only be necessary to compute the first few orders. (Note that at lowest order, $m_0{}^2=m_E{}^2$ and $B_1=N/3$. For simplicity, we have neglected the quark chemical potentials here.)

We now describe the general strategy for evaluating the infrared effects of nonstatic modes, based on dimensional reduction. We shall temporarily include, in the effective theory, the entire infinite set of induced vertices. Later on we shall decide which of these we actually wish to retain.

Consider a one-particle irreducible n-point Green's function with *static* external legs. Within the effective theory, such a quantity can be split up into two parts: An n-point bare vertex and an n-point "blob." The latter, if perturbatively expanded in the bare propagators and vertices of the effective theory, would contain graphs of at least one-loop order. Such an expansion would not in general be infrared finite. Now consider the perturbative expansion of the same Green's function in $\mathrm{QCD}_{4T\mu}$. One finds three kinds of graphs, depending on whether the internal propagators are (i) purely static ("static graphs"), (ii) static as well as nonstatic ("mixed graphs"), (iii) purely nonstatic ("nonstatic graphs").

It is easily seen that in the infrared limit, the sum of all static and mixed graphs is precisely the n-point blob of the effective theory. (Dimensional reduction allows us to ascertain the infrared effects of nonstatic modes by shrinking nonstatic internal propagators down to points. Thus it is seen that mixed graphs are just static graphs built up from the induced vertices of the effective theory. Together with the static graphs they then constitute the perturbative expansion, within the effective theory, of the n-point blob.) The remaining nonstatic graphs define the n-point bare vertex of the effective theory after including a factor of $T^{n/2-1}$ to make the dimensions come out right.

Thus, the interaction vertices necessary to define the effective theory may be obtained by retaining only the purely nonstatic graphs in the perturbative expansion of static Green's functions in $\mathrm{QCD}_{4T\mu}$. Furthermore, the nonstatic graphs for two- and three-point vertices also provide the $\mathrm{QCD}_{4T\mu}$ renormalization constants necessary for computing the β function and the running coupling constant.

We must now decide which vertices to retain in the effective Lagrangian. We do this by comparing the n-point vertex with the corresponding n-point blob, using naive power counting to determine their relative strengths. A vertex with n static external legs is first induced at $O(g^n)$. If there are p momentum factors in the vertex (i.e., p derivatives in the operator corresponding to the vertex) each of scale $|\vec{k}|$, then the strength of the vertex is

$$V\sim g^nT^{3-n/2-p}|\vec{k}|^p .$$

The corresponding blob on the other hand arises, at one loop, from a three-dimensional integration over n powers of inverse momentum; the strength of the blob is then

$$B\sim(T^{1/2}g)^n|\vec{k}|^{3-n} .$$

Thus the "suppression factor" for the vertex, defined to be its strength relative to the blob, is

$$R\equiv V/B\sim(|\vec{k}|/T)^{n+p-3} .$$

The actual suppression factor may be better than R if the vertex is induced at a level higher than $O(g^n)$. In particular, one can have $R=0$ due to some symmetry.

At the classical and one-loop levels, we saw that in a suitable renormalization scheme the suppression of nonstatic modes was $O(\vec{k}^2/T^2)$ or better. We shall use this as our criterion for including induced vertices, rejecting those with a suppression factor smaller than or of the order of $(|\vec{k}|/T)^2$. Thus we only retain vertices satisfying

$$n+p<5 .$$

(Note that for renormalizability of the effective theory, one needs $n/2+p\leq 3$.)

Based on this criterion, we reject vertices with five or more legs. Consider vertices with $n=4$. We must have $p=0$. Designating the electrostatic propagator by E and the magnetostatic propagator by M, the only vertices with $p=0$ are $4M$, $2M$-$2E$, and $4E$. The first two already exist at the classical level and are related by gauge symmetry to three-point vertices through the Ward identities for the effective theory. The $4E$ vertex, on the other hand, is new and must be computed. Next consider $n=3$ vertices. $3M$ and $1M$-$2E$ vertices already exist at the classical level; the nonstatic graphs represent corrections to these. Because of gauge symmetry, only one, e.g., the $1M$-$2E$ vertex, need be computed. The other three-point vertices, viz., $2M$-$1E$ and $3E$, vanish by symmetry. Finally consider the two-point vertices. The $2M$ vertex vanishes by gauge symmetry. The $2E$ vertex yields the bare electrostatic mass. The $1M$-$1E$ vertex vanishes by symmetry.

We had previously defined $EQCD_3$ to be QCD_3 coupled to an adjoint scalar field with mass. Let us now broaden this definition to include a quartic self-interaction for the scalar field. From the previous paragraph we then see that the effective theory is $EQCD_3$, which is superrenormalizable, needing only mass renormalization for the scalar field. We stress that the quartic term in $EQCD_3$ represents higher-order corrections to the one-loop effective theory of Sec. II, and is not needed for its renormalizability. Note that since there is no ultraviolet mass renormalization in $QCD_{4T\mu}$, the nonstatic graphs defining the scalar mass yield the *bare* mass $m_0{}^2 = m^2 + \delta m^2$, where δm^2 is the mass counterterm. At one-loop in the dimensional regularization scheme, $\delta m^2 = 0$ and $m^2 = m_E{}^2$.

We summarize this section so far as follows. The infrared properties of $QCD_{4T\mu}$ are represented, up to $O(\vec{k}^2/T^2)$ corrections, by $EQCD_3$. The bare vertices in the $EQCD_3$ Lagrangian may be computed by perturbatively expanding the corresponding static Green's functions of $QCD_{4T\mu}$, keeping only nonstatic graphs. If $\vec{k}^2 \sim g^n T^2$, then the expansion need only be carried out up to corrections $O(g^n)$.

The separation of static and nonstatic sectors is very clean and elegant in a class of gauges known as "static gauges",[10] which are, therefore, particularly suitable for our purposes. We shall briefly describe the salient features of these gauges below, and then outline some sample computations of the $EQCD_3$ bare vertices. We end the section by concluding that for distance scales beyond the electrostatic screening length the effective theory simplifies further to the three-dimensional theory of the pure Yang-Mills field, QCD_3.

A. Static gauges

Static gauges are finite-temperature analogs of the temporal ($A_4 \equiv -iA_0 = 0$) gauge. At $T \neq 0$, the condition $A_4 = 0$ conflicts with the periodic boundary conditions and discrete energies required by the standard finite-temperature formalism (see Ref. 1, pp. 47—48). Instead, one demands that A_4 be time independent (static), i.e., that $\partial_4 A_4 = 0$. Additional gauge fixing in the static sector is required to completely specify the gauge. The resulting Feynman rules[10] are given in Appendix A 3. We note that there is an electrostatic ($A_4 - A_4$) propagator and a magnetostatic ($A_i - A_j$) propagator with a possible static ghost depending on the nature of gauge fixing in the static sector. The static Feynman rules are thus those of $EQCD_3$ minus nonstatic renormalization effects. The only nonstatic propagator is the $A_i - A_j$ gluon, which is completely unaffected by the details of gauge fixing in the static sector. There is no nonstatic $A_4 - A_4$ gluon, and the nonstatic ghost effectively decouples from the theory. There is no mixed ($A_i - A_4$) gluon in either the static or the nonstatic case.

Static gauges are particularly well suited for studying the infrared properties of finite-temperature QCD. Since $T \neq 0$ breaks Lorentz covariance, nothing further is lost by choosing a gauge which also breaks it in exactly the same manner. Another such gauge is the more familiar Coulomb gauge. However, static gauges are superior to the Coulomb gauge for three reasons: first, the separation of static and nonstatic propagators is much cleaner in static gauges. In the Coulomb gauge, the presence of a nonstatic $A_4 - A_4$ gluon is an added complication. Secondly, the additional gauge degree of freedom in static gauges makes it possible to study the gauge dependence of various quantities. This is not possible in the usual Coulomb gauge. If one tries to generalize the Coulomb gauge via a ξ parameter, one introduces a mixed $A_j - A_4$ propagator which is physically undesirable. Thirdly, static gauges afford an elegant means of resumming electrostatic mass insertions in a gauge-invariant way[10]: There exists a gauge-invariant operator which, in static gauges, reduces to the electrostatic mass term $\sim \mathrm{Tr}(A_4{}^2)$. By adding such a mass term to the quadratic part of the Lagrangian and subtracting it from the interaction part, one can perform a gauge-invariant resummation of the electrostatic mass in perturbation theory. This cures the infrared divergences arising in the electrostatic sector. The mass is determined self-consistently by relating it to the decay of gauge-invariant electrostatic correlations.

B. Computation of the $EQCD_3$ bare vertices

$EQCD_3$ is the non-Abelian gauge theory in three space dimensions of vectors (the magnetostatic potentials) coupled to massive adjoint scalars (the electrostatic potentials) with a quartic self-interaction. Consequently, the $EQCD_3$ Lagrangian contains three parameters: the gauge coupling G, the scalar bare mass $m_0{}^2$, and the scalar self-interaction κ. Since $EQCD_3$ is really the limit of $QCD_{4T\mu}$, one can express G, $m_0{}^2$, and κ in terms of the $QCD_{4T\mu}$ parameters $g(T)$, T and μ (we shall usually omit explicit references to μ for simplicity). This is done by computing the relevant nonstatic graphs through some perturbative order determined by the infrared momentum scale, at small, static external momenta.

We illustrate the procedure by outlining the computations of G, $m_0{}^2$, and κ to one-loop order in any static gauge. The relevant nonstatic graphs are shown in Fig. 4. For our purposes it is not neces-

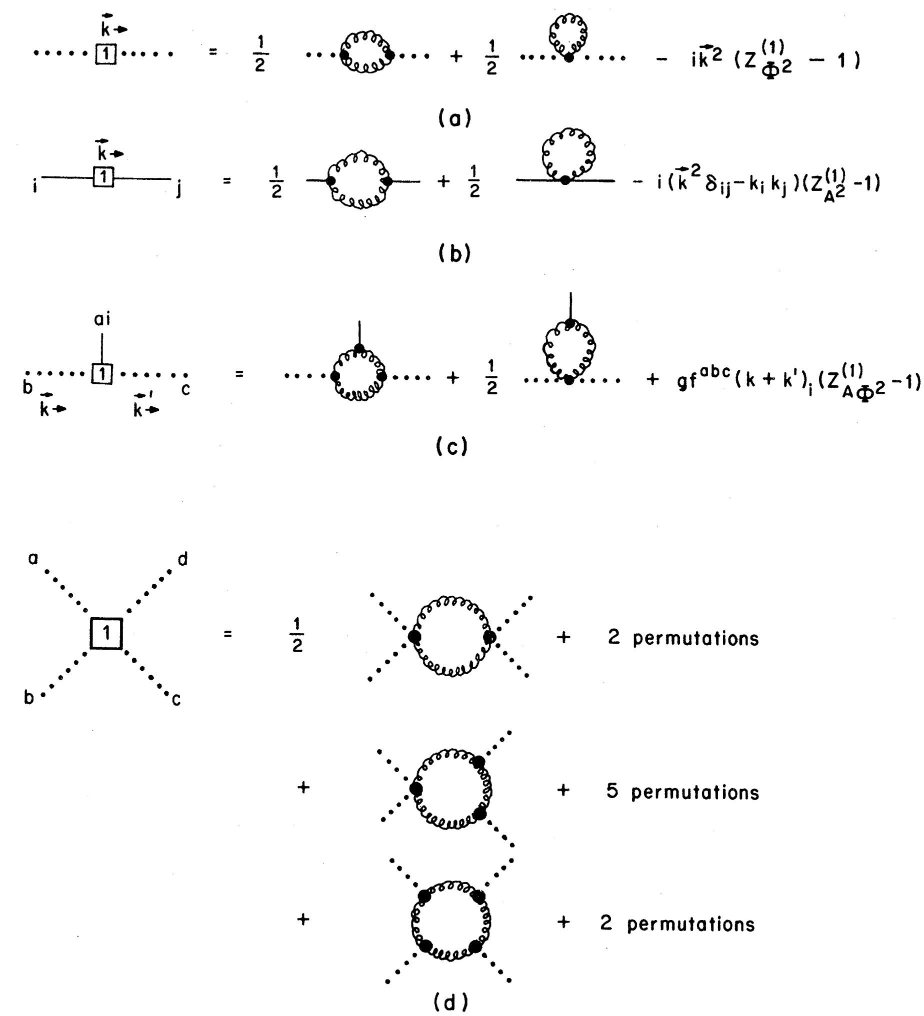

FIG. 4. One-loop nonstatic graphs for (a) electrostatic self-energy, (b) magnetostatic self-energy, (c) $A_iA_4{}^2$ triple vertex, and (d) $A_4{}^4$ quadruple vertex, yielding induced bare vertices of EQCD$_3$ and renormalization constants of QCD$_{4T\mu}$. (Quark loops have been omitted for simplicity.)

sary to actually perform any of the computations, since we can rely instead on previous results where needed.

Graphs 4(a) give us the induced electrostatic mass (already known from the covariant-gauge calculation of Sec. II) and also the electrostatic wave-function renormalization constant Z_{Φ^2}. Graphs 4(b) tell us that the induced magnetostatic mass is zero at this level, and also yield Z_{A^2}. Graphs 4(c) give us the coefficient A_1 in the expansion of G in powers of g, and also the vertex renormalization constant $Z_{A\Phi^2}$. Finally, graphs 4(d) yield the leading value of the quartic coupling κ.

We now show that it is appropriate to use the coupling $g(T)$, defined at scale T in a suitable renormalization scheme such as MD, all the way down to arbitrarily small momentum scales. Consider the relation of the bare coupling constant to the coupling constant $g(\Lambda)$ defined at scale Λ using dimensional regularization,

$$g_0 = Z_g \Lambda^{(4-D)/2} g(\Lambda) ,$$

where Z_g is the gauge-invariant combination

$$Z_g \equiv Z_{A\Phi^2} Z_{A^2}{}^{-1/2} Z_{\Phi^2}{}^{-1} .$$

Using the invariance properties of the lowest-order β function and our knowledge of Z_g at $T=0$, we may write (without actually performing any computations) in the MD scheme

$$Z_g = 1 + \frac{g^2}{24\pi^2}\left(\tfrac{11}{2}N - N_f\right)\left[\frac{1}{D-4} + \ln\frac{T}{\Lambda} + \text{const}\right] + O(g^4) .$$

If we identify Λ with T we obtain the standard one-loop β function, corresponding to the running coupling constant

$$g^2(T)=\frac{g^2(T_0)}{1+\frac{g^2(T_0)}{12\pi^2}(\frac{11}{2}N-N_f)\ln\frac{T}{T_0}},$$

where $g(T_0)$ is the experimentally determined coupling constant at scale T_0. But if we vary Λ keeping T fixed, the "β function" so obtained vanishes to $O(g^3)$, i.e., the coupling constant does not run any farther to this order. Thus $g(\Lambda)=g(T)$, and the value $g(T)$ can be used even at vanishingly small momentum scales.

In the MD scheme we have $A_1=0$ (and, in fact, expect $A_i=0$ to all orders) since the renormalization constant absorbs all constant pieces. Then

$$G=T^{1/2}g(T).$$

Finally, consider the quartic coupling κ. Ultraviolet divergences in graphs 4(d) must cancel, leaving a one-loop value for the vertex of the form $\sim -iN^2g^4T(\delta_{ad}\delta_{bc}+\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd})$. From this we extract the value of κ,

$$\kappa\sim N^2g^4T,$$

which on physical grounds we expect to be positive.

C. Distance scales and effective theories

We have found that up to the one-loop level the parameters of EQCD_3 are $m_0{}^2\equiv m^2+\delta m^2=m_E{}^2$, $G=T^{1/2}g(T)$, $\kappa\sim N^2g^4T$. Depending on the infrared momentum scale at which the theory is to be employed, it may be necessary to compute higher-order nonstatic corrections to these parameters. Typically, one would use EQCD_3 as the effective theory at distance scales on the order of the Debye screening length, corresponding to $|\vec{k}|\sim m_E\sim gT$. This means that in dimensionally reducing we neglect corrections $O(|\vec{k}|^2/T^2\sim g^2)$. For G and κ, the suppression factor is $R\sim|\vec{k}|/T\sim g$; two-loop corrections would therefore be $O(g^3)$ and need not be computed. For $m_0{}^2$, $R\sim T/|\vec{k}|\sim g^{-1}$ and so the two-loop correction, which is $O(g)$, must be computed.

At distance scales beyond the Debye screening length, $|\vec{k}|\ll gT$, EQCD_3 becomes an even better approximation to $\mathrm{QCD}_{4T\mu}$, and to fully exploit this circumstance one would have to compute its parameters even more accurately. At such scales the scalar of EQCD_3 acts as a heavy particle which decouples and so one also has the option of approximating EQCD_3 itself by a simpler theory, QCD_3. Being ultraviolet finite, QCD_3 has only one dimensional parameter, its coupling constant G. Consequently, the magnetostatic mass, if it exists, must be proportional to G^2 to leading order, i.e., $m_M\sim g^2T\ll m_E$. One is thus justified in using pure QCD_3 as the effective theory appropriate to distance scales on the order of the magnetic screening length. The corrections to QCD_3 from EQCD_3 would then be $O(m_M{}^2/m_E{}^2)\sim O(g^2)$, which is the size of one-loop corrections. Thus in the leading approximation it is unnecessary to compute the QCD_3 coupling constant beyond the renormalization-group-improved classical value of $G=T^{1/2}g(T)$.

V. CONCLUSION

The equilibrium behavior of the quark-gluon plasma at high temperatures is characterized by at least two distance scales. The first distance scale is the inverse temperature β, which requires the use of the full theory $\mathrm{QCD}_{4T\mu}$. The second scale is the Debye screening length $l_E\equiv m_E{}^{-1}\gg\beta$. At distances on the order of l_E or beyond, $\mathrm{QCD}_{4T\mu}$ may be approximated up to corrections $O(m_E{}^2/T^2)$ by a dimensionally reduced effective theory, EQCD_3. The parameters entering the EQCD_3 Lagrangian can be computed perturbatively through any required order.

At distance scales $l\gg l_E$, one can approximate EQCD_3 itself, up to $O(l_E{}^2/l^2)$ corrections, by pure QCD_3 with gauge coupling $T^{1/2}g(T)$. The nonperturbativity of QCD_3 is responsible for that of EQCD_3 and ultimately for the infrared divergences that plague $\mathrm{QCD}_{4T\mu}$. It is of considerable importance to determine the implications of these divergences—for example, whether they cure themselves by generating a magnetostatic mass m_M, corresponding to a possible third distance scale, the magnetic screening length $l_M\equiv m_M{}^{-1}\gg l_E$. As the leading approximation to $\mathrm{QCD}_{4T\mu}$ at this third distance scale, QCD_3 is the natural starting point for such nonperturbative investigations. Any mass gap dynamically generated within QCD_3 must be proportional to the square of its coupling constant. Thus dimensional reduction predicts that in the leading approximation the magnetostatic mass, if it exists, is of the order $m_M\sim Tg^2(T)$. To determine the coefficient one must go beyond perturbation theory.

QCD_3 has been studied on the lattice via Monte Carlo simulation[11]; again, there are indications of mass generation but we feel the data are too crude to be decisive. A computation with better statistics, focusing on the question of magnetic screening, is strongly urged. If a magnetic mass *is* obtained, it would be interesting to compare it with the value $0.24\,g^2T$ obtained in Ref. 5.

We conclude by noting that the program of systematic dimensional reduction in static gauges, outlined in Sec. IV, is equally applicable to the range of theories mentioned in Ref. 6. (See Ref. 12 for a recent calculation along these lines.)

ACKNOWLEDGMENTS

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APPENDIX A: FEYNMAN RULES, ETC.

1. General formalism (Ref. 13)

To obtain $QCD_{4T\mu}$ from QCD_{1+3}: (i) Perform a Wick rotation to Euclidean space and impose periodic (antiperiodic) boundary conditions in the Euclidean-time direction for bosonic (fermionic) fields, with period $\beta\equiv T^{-1}$; this gives QCD_{4T}, with discrete Euclidean energies. (ii) Give these discrete energies an imaginary component $-i\mu$; the result is $QCD_{4T\mu}$.

Thus, to go from the Minkowski version of the theory to the grand canonical version make the following replacements:

$$-ik_0\to k_4=\omega_n-i\mu ,$$
$$k^i\to k_i \quad (i=1,2,3) ,$$

where

$$\omega_n=\begin{cases} n(2\pi T)\text{: bosons and ghosts} , \\ (n+\tfrac{1}{2})(2\pi T)\text{: fermions} . \end{cases}$$

Euclidean γ matrices are defined via

$$\gamma_0\to-\gamma_4 ,$$
$$\gamma^j\to i\gamma_j ,$$

and satisfy the relation

$$\{\gamma_\alpha,\gamma_\beta\}=2\delta_{\alpha\beta} \quad (\alpha,\beta=1,2,3,4) .$$

Covariant-gauge Feynman rules are shown in Fig. 5. Factors of i have been included to facilitate comparison with the corresponding Minkowski-space rules. Group-theoretical factors are the same as in the Minkowski version; in the usual notation,

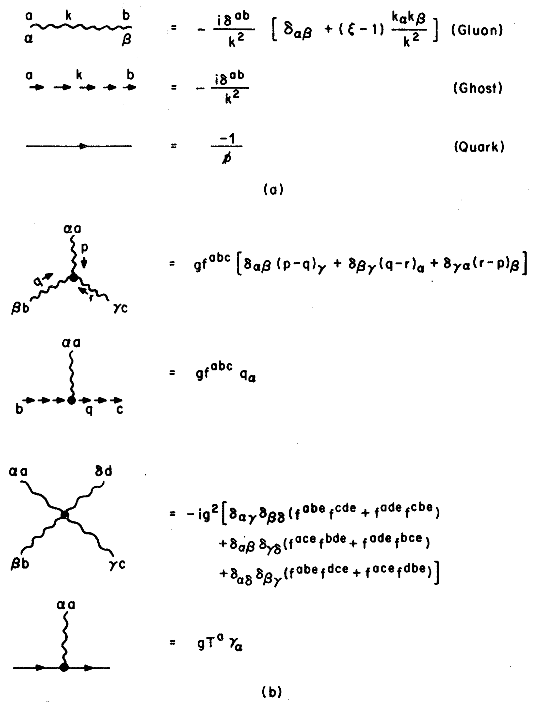

FIG. 5. Feynman rules for $QCD_{4T\mu}$ in the covariant gauge. (a) Propagators. (b) Vertices. (c) Rules for loops are (i) integrate and sum, $iT\sum_n\int d_3\vec{q}$, (ii) ghost and fermion loops get factor -1, (iii) single (double) gluon loop gets factor $\frac{1}{2}(\frac{1}{6})$.

$$(T^aT^a)_{ij}=C_F\delta_{ij} , \quad f^{acd}f^{bcd}=C_A\delta^{ab} .$$

For SU(N),

$$i,j,\ldots=1,\ldots,N ,$$
$$a,b,\ldots=1,\ldots,(N^2-1) ,$$
$$C_F=(N^2-1)/2N , \quad C_A=N .$$

To perform the sum over discrete energies, the standard trick of converting the sum into a contour integral is used. If $f=f(q)$ is a function of the Euclidean momentum $q\equiv(\vec{q},q_4)$, the Feynman integral of f at finite T and μ is defined as

$$[f]_{T\mu}\equiv T\sum_n\int d_3\vec{q}\,f(\vec{q},\omega_n-i\mu) ,$$

where $d_3\vec{q}\equiv d^3\vec{q}/(2\pi)^3$.

One can split up $[f]_{T\mu}$ into two parts:

$$[f]_{T\mu}\equiv[f]_{00}+\Delta_{T\mu}f .$$

The "vacuum contribution"

$$[f]_{00}=\int d_4q\,f(q) , \quad d_4q\equiv d^4q/(2\pi)^4$$

is just the Euclidean version of the standard Minkowski-space result. All ultraviolet infinities are contained in $[f]_{00}$. The "matter contribution" contains no ultraviolet infinities, and needs no regularization. It can be represented as a contour integral,

$$\Delta_{T\mu} f = \int \frac{d_3\vec{q}}{2\pi i}\left[\oint_{\Gamma_\mu} d\alpha\, f(\vec{q},-i\alpha) + \oint_{\Gamma} d\alpha \frac{[f(\vec{q},i\alpha-i\mu)+f(\vec{q},-i\alpha-i\mu)]}{1\mp e^{\alpha/T}}\right],$$

where the contours Γ and Γ_μ are shown in Fig. 6. In this formula, as elsewhere in the Appendices, the upper (lower) value—in this case the sign in the factor $1\mp e^{\alpha/T}$—is for bosons (fermions).

2. Certain Feynman integrals at finite T and μ (Ref. 8)

To perform the one-loop calculation, we need to compute $\Delta_{T\mu} f$ for certain functions $f(q)$ parametrized by a static IR external momentum $k=(\vec{k},k_4=0)$, where $|\vec{k}| \ll T$. To do this, we first perform the energy sum via the contour integration formula of Appendix A 1. The angular integration over $\vec{q}$ is then carried out. This typically leaves us with an integral over $|\vec{q}|$ of a product of powers of $|\vec{q}|$, the Bose (Fermi) distribution function, and a logarithmic factor. The following series expansions (and their derivatives) are useful:

(i) $\frac{-1}{1\mp e^{x}} = \sum_{n=0}^{\infty} \lambda_n x^{n-1}$,

where

$$\lambda_n = \begin{cases} B_n/n! \text{ and } |x| < 2\pi \text{ (bosons)}, \\ (2^n-1)B_n/n! \text{ and } |x| < \pi \text{ (fermions)}, \end{cases}$$

the B_n being Bernoulli numbers

$$B_0 = 1,\quad B_1 = -\tfrac{1}{2},$$

$$B_{2m+1} = 0 \text{ for } m = 1,2,\ldots,$$

$$\lambda_0 = \begin{cases} 1, \\ 0, \end{cases} \quad \lambda_1 = \tfrac{1}{2}.$$

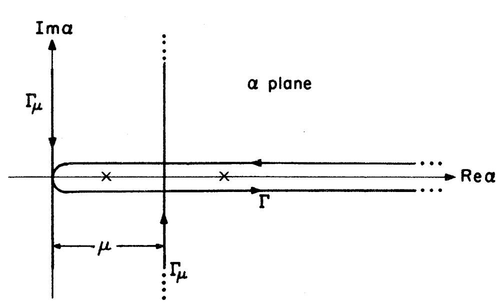

FIG. 6. Contours for computing $\Delta_{T\mu f}$.

(ii) $\frac{1}{2}\ln\left|\frac{1+\epsilon}{1-\epsilon}\right| = \sum_{r=1}^{\infty} \frac{\epsilon^{2r-1}}{2r-1} \quad (|\epsilon| < 1)$.

The integral over $|\vec{q}|$ is obtained as an expansion in $|\vec{k}| < T$ by dividing the integration region into three parts: $0 \to |\vec{k}|$, $|\vec{k}| \to T$, and $T \to \infty$. The appropriate power expansions are used in each region.

The following specific functions $f(q)$ arise in the one-loop calculation:

$$f_1 \equiv \frac{1}{q^2},\quad f_2 \equiv \frac{\vec{k}^2}{q^2(q+\vec{k})^2},\quad f_3 \equiv \frac{\vec{q}^2}{q^2(q+\vec{k})^2},$$

$$f_4 \equiv \frac{\vec{k}\cdot\vec{q}}{q^2(q+\vec{k})^2},\quad f_5 \equiv \frac{(\hat{k}\cdot\vec{q})^2}{q^2(q+\vec{k})^2},$$

$$g_i \equiv \frac{\vec{k}^2}{q^2} f_i \quad (i=1,2,3,4,5),$$

$$h_j \equiv \frac{\vec{k}^2}{(q+\vec{k})^2} g_j \quad (j=2,3,4,5).$$

Integrals over f_i are done for $\mu=0$ bosons (gluons) and for fermions (quarks). Integrals over g_i and h_i are needed only for gluons, and only for specific combinations of g_i and h_i. The results are (upper value is for gluons, lower value for quarks)

$$\Delta_{T\mu} f_1 = \begin{cases} T^2/12, \\ -(T^2/24+\mu^2/8\pi^2), \end{cases}$$

$$\Delta_{T\mu} f_2 = \lambda_0 \frac{T|\vec{k}|}{8} + \frac{\vec{k}^2}{8\pi^2}\ln\frac{|\vec{k}|}{T} + O(\vec{k}^2),\quad \lambda_0 = \begin{cases} 1, \\ 0, \end{cases}$$

$$\Delta_{T\mu} f_3 = \tfrac{3}{2}\Delta_{T\mu} f_1 + \frac{\vec{k}^2}{96\pi^2}\ln\frac{|\vec{k}|}{T} + O(\vec{k}^2),$$

$$\Delta_{T\mu} f_4 = -\tfrac{1}{2}\Delta_{T\mu} f_2,$$

$$\Delta_{T\mu} f_5 = \tfrac{1}{2}\Delta_{T\mu} f_1 + \tfrac{1}{4}\Delta_{T\mu} f_2,$$

$$\Delta_T(g_1-g_2) = \frac{\vec{k}^2}{8\pi^2}\ln\frac{|\vec{k}|}{T} + O(\vec{k}^2),$$

$$\Delta_T g_3 = \Delta_T f_2 - \tfrac{1}{4}\Delta_T(g_1-g_2),$$

$$\Delta_T g_4 = -\tfrac{1}{2}\Delta_T f_2 + \tfrac{1}{2}\Delta_T(g_1-g_2),$$

$$\Delta_T g_5=\tfrac{1}{2}\Delta_T f_2-\tfrac{1}{4}\Delta_T(g_1-g_2)\ ,$$
$$\Delta_T(g_2-h_3)=O(\vec{k}^2)\ ,$$
$$\Delta_T(g_2-\tfrac{1}{2}h_2)=2\Delta_T(g_2-h_3)\ ,$$
$$\Delta_T h_4=-\tfrac{1}{2}\Delta_T h_2\ ,$$
$$\Delta_T h_5=-\tfrac{1}{2}\Delta_T f_2+\tfrac{1}{2}\Delta_T g_1+\tfrac{1}{4}\Delta_T h_2\ .$$

(Note: For fermions, we have assumed $\mu \lesssim T$. For $\mu \gg T$, T is replaced by μ in the $\ln|\vec{k}|/T$ factors.)

3. Feynman rules for static gauges

Static-gauge Feynman rules are derived in Ref. 10; we rederive them here as a special case of the general formalism of Ref. 14. We work with a Euclidean metric, and $A_0=iA_4$.

Consider the four vector

$$\Lambda\equiv(-i\vec{\Lambda},\lambda\partial_4)\underset{k\text{ space}}{\longrightarrow}-i(\vec{\Lambda},\lambda k_4)$$

in terms of which the gauge-fixing and ghost terms in the Lagrangian are

$$\Delta\mathscr{L}_{\rm gf}=-\tfrac{1}{2}(\Lambda_\mu A_\mu^a)^2\ ,$$
$$\Delta\mathscr{L}_{\rm ghost}=\eta^{a\dagger}\Lambda_\mu(\partial_\mu\eta^a+gf^{abc}A_\mu^b\eta^c)\ ,$$

where A_μ^a is the gluon field and η^a the ghost field. By specializing Λ one obtains specific gauges. ($\lambda=1/\sqrt{\xi}$, $\vec{\Lambda}=\vec{k}/\sqrt{\xi}$ give covariant gauges; $\lambda=0$, $\vec{\Lambda}=\vec{k}/\sqrt{\xi}$, $\xi\to 0$ gives the Coulomb gauge.)

Static gauges are obtained by letting $\lambda\to\infty$. This corresponds to setting $\partial_4 A_4^a=0$. We are left with an additional gauge degree of freedom in the magnetostatic sector which is fixed on specifying $\vec{\Lambda}$ in the propagators,

$$iD_{ij}^{ab}(\vec{k})=-\frac{i\delta_{ab}}{\vec{k}^2}\left[\delta_{ij}-\frac{k_i\Lambda_j+\Lambda_i k_j}{\vec{k}\cdot\vec{\Lambda}}+\frac{k_ik_j(\vec{\Lambda}^2+\vec{k}^2)}{(\vec{k}\cdot\vec{\Lambda})^2}\right]$$

(static, spacelike gluon) ,

$$iG^{ab}(\vec{k})=-i\delta_{ab}/\vec{k}\cdot\vec{\Lambda}$$

(ghost, couples only to above) .

The other propagators, like the vertices, are completely unaffected by the details of $\vec{\Lambda}$. Thus it is possible to do a certain class of calculations (e.g., nonstatic graphs) without even specifying $\vec{\Lambda}$. However, in general we must specify $\vec{\Lambda}$, which is entirely analogous to fixing the gauge in EQCD_3. A convenient choice is the O(3)-covariant gauge, $\vec{\Lambda}=\vec{k}/\sqrt{\xi}$.

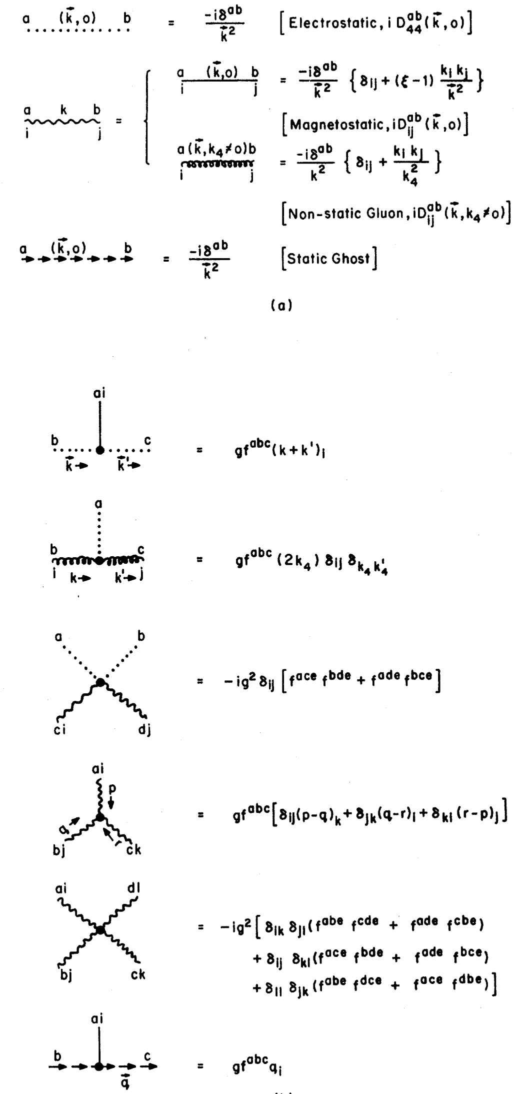

FIG. 7. Feynman rules for $\text{QCD}_{4T\mu}$ in the O(3)-covariant static gauge. (Rules for quarks are essentially the same as for covariant gauges and have been omitted.) (a) Propagators. (b) Vertices. (c) Rules for loops are (i) integrate static loops: $iT\int d_3\vec{q}$, integrate and sum nonstatic loops: $iT\sum_{n\neq 0}\int d_3\vec{q}$; (ii) ghost loops get factor (-1); (iii) gluon loops get appropriate symmetry factors.

The Feynman rules for the O(3)-covariant static gauge are given in Fig. 7. The nonstatic ghost effectively decouples from the theory so it has not been included. [If desired, the electrostatic mass may be treated self-consistently in a gauge-invariant manner by replacing $\vec{k}^2\to(\vec{k}^2+m^2)$ in the electrostatic propagator; this creates an additional mass insertion vertex $im^2\delta_{ab}$; see Ref. 10 for details.]

APPENDIX B: DIMENSIONAL REDUCTION IN THE PRESENCE OF SCALARS

We consider here the effects on dimensional reduction if, in addition to quarks, there are colored scalars coupled to the gauge field. Unlike quarks, scalars obey Bose-Einstein statistics and their propagators contain static modes, so we expect qualitatively different behavior. Calculations bear out this expectation.

Consider first the case where the chemical potential of a particular flavor of scalar vanishes. (This would occur, for example, for an adjoint scalar Yukawa-coupled to the quarks with a possible quartic self-coupling in addition.) The behavior of such a scalar is clearly similar to that of the A_4 field, in the infrared limit. At the classical level, only the static mode propagates; at the one-loop level it acquires a screening mass. In addition, the electrostatic mass is enhanced by the presence of scalar loops. The effective infrared theory is thus EQCD_3 with additional massive scalars.

Next, examine the case where $\mu \neq 0$. For concreteness, consider scalars of a single flavor transforming under the fundamental representation, with quartic self-interaction. The additional Feynman rules generated by such scalars are shown in Fig. 8.

We compute $\Delta_{T\mu} f_i$ (see Appendix A 2) in the infrared, $|\vec{k}| \ll \mu \lesssim T$, and obtain

$$\Delta_{T\mu} f_1 = \left[\frac{T^2}{12} - \frac{\mu^2}{8\pi^2} \right],$$

$$\Delta_{T\mu} f_2 = \frac{\vec{k}^2}{8\pi^2} \ln \frac{|\vec{k}|}{T} + O(\vec{k}^2),$$

$\Delta_{T\mu} f_i$, $i = 3,4,5$, are given by the same formulas as in Appendix A 2 .

Comparing these values with the corresponding ones for fermions, we notice only one difference: the sign of the μ^2 term is *opposite* to that of the T^2 term in $\Delta_{T\mu} f_1$ for bosons, while both terms have the same sign for fermions,

$$\Delta_{T\mu} f_1 \Big|_{\text{fermions}} = - \left[\frac{T^2}{24} + \frac{\mu^2}{8\pi^2} \right].$$

Remembering that fermion loops occur with an overall (−) sign, it is easily seen that mass terms generated by a finite bosonic chemical potential have the *wrong* sign. This "antiscreening" effect is due to the *attractive* nature of Bose-Einstein statistics.

How does dimensional reduction work in the presence of $\mu \neq 0$ scalars? In the infrared ($|\vec{k}| \ll \mu \lesssim T$), the scalars themselves are screened out. At the classical level, the static scalar propagator is $(\vec{k}^2 - \mu^2)^{-1}$; the mass term generated by including one-loop corrections is of the form $m_S{}^2 = [O(g^2,\lambda)T^2 - (1 + O(g^2,\lambda))\mu^2]$. (Note that this mass term could be negative.) The computation of interest is that of the one-loop scalar contribution to the gluon propagator. The relevant graphs are shown in Fig. 9.

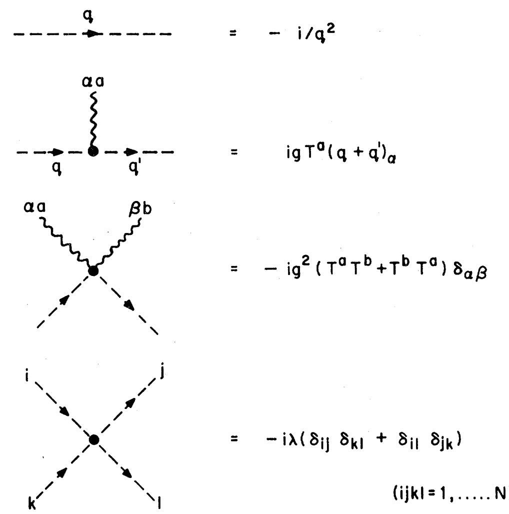

FIG. 8. Additional Feynman rules for fundamental scalars with quartic self-coupling.

The results are as follows (these are to be added to the results of Sec. II):

(i) Electrostatic self-energy.

$$\Pi_{44}^{(s)}(\vec{k}^2) = -\tfrac{1}{2} g^2 \left[\frac{T^2}{3} - \frac{\mu^2}{2\pi^2} \right] + g^2 T\, |\vec{k}|\, O(|\vec{k}|^3/T^3) .$$

(ii) Magnetostatic self-energy.

$$\Pi^{(s)}(\vec{k}^2) = g^2 T\, |\vec{k}|\, O(|\vec{k}|^3/T^3) .$$

(iii) Gluon wave-function renormalization constant.

$$(Z_3 - 1)^{(s)} = \frac{1}{3} \frac{g^2}{16\pi^2} \left[\frac{1}{D-4} + \ln \frac{T}{\Lambda} + \text{const} \right] .$$

As expected, the $\mu \neq 0$ scalar contribution to the gluon propagator is similar to that of fermions, except for the antiscreening effect.

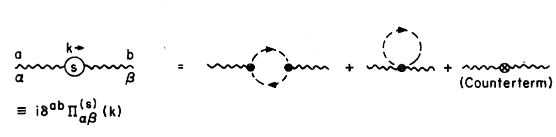

FIG. 9. Scalar contribution to 1PI gluon self-energy.

To summarize, in the presence of scalars with vanishing or sufficiently small chemical potentials, dimensional reduction goes through with trivial modifications: The effective infrared theory is QCD_3 coupled to scalars (one of the flavors being the electrostatic field) which become massive at the one-loop level and screen out over very long distances.

If sufficiently large scalar chemical potentials are present, some of the scalars coupled to QCD_3 could acquire mass terms with the wrong sign, leading to a Higgs effect in the three-dimensional sector. Such a situation is more appropriately discussed in the context of spontaneously broken gauge theories and their phase transitions, and we shall not pursue it any further here. The interested reader is referred to recent work on this subject.[15]

[1]For a review of $QCD_{4T\mu}$ with earlier references, see D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[2]E. S. Fradkin, Proc. Lebedev Phys. Inst. 29, 7 (1965), English translation by Consultants Bureau, New York, 1967.

[3]A. D. Linde, Rep. Prog. Phys. 42, 389 (1979); Phys. Lett. 96B, 289 (1980).

[4]T. Appelquist and R. D. Pisarski, Phys. Rev. D 23, 2305 (1981).

[5]A. Billoire, G. Lazarides, and Q. Shafi, Phys. Lett. 103B, 450 (1981); T. A. DeGrand and D. Toussaint, Phys. Rev. D 25, 526 (1982).

[6]The phrase "dimensional reduction" is already in wide usage, e.g., in the contexts of quantum gravity and supergravity.

[7]T. Appelquist and J. Carazzone, Phys. Rev. D 11, 2856 (1975).

[8]S. Nadkarni, manuscript in preparation.

[9]However, mass generation is not necessarily implied. For discussions of a counterexample (three-dimensional massless scalar electrodynamics in the $1/N$ expansion), see Ref. 8, as well as T. Appelquist and U. Heinz, Phys. Rev. D 24, 2169 (1981). For additional insight see R. Jackiw and S. Templeton, *ibid.* 23, 2291 (1981).

[10]E. D'Hoker, Nucl. Phys. B201, 401 (1982).

[11]E. D'Hoker, Nucl. Phys. B180 [FS2], 341 (1981).

[12]T. Appelquist and A. Chodos, Phys. Rev. Lett. 50, 141 (1983).

[13]C. W. Bernard, Phys. Rev. D 9, 3312 (1974); P. D. Morley and M. B. Kislinger, Phys. Rep. 51, 63 (1979).

[14]J. Frenkel and J. C. Taylor, Nucl. Phys. B109, 439 (1976).

[15]J. I. Kapusta, Phys. Rev. D 24, 426 (1981); H. E. Haber and H. A. Weldon, *ibid.* 25, 502 (1982).