Parametric formula for the sixth-order vacuum polarization contribution in quantum electrodynamics

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We have obtained a Feynman-parametric integral for the complete vacuum polarization contribution of sixth order in quantum electrodynamics in a form convenient for numerical integration.

I. INTRODUCTION

As measurements of various electromagnetic processes become more and more precise, the need for pushing the theoretical calculations in QED to higher orders becomes more acute. In this article we present a formula which expresses the complete sixth-order photon self-energy (vacuum polarization) effect in a form which is convenient for numerical integration. Knowledge of this expression is necessary in calculating the anomalous magnetic moments of leptons beyond order α^3. It also contributes to the hyperfine structures of positronium and muonium.

We present our result as an integral over Feynman parameters. Although it can be expressed in the form of the Källén-Lehmann spectral representation, this does not prove particularly useful insofar as the spectral function obtained is not in a convenient analytic form. However, for the above-mentioned applications, and possibly for all practical purposes (in particular for numerical evaluation), we find our formula entirely satisfactory.

We express the photon self-energy amplitude of $2n$th order, which consists of $2n$ vertices, $2n$ electron lines forming a single closed loop, and $n-1$ photon lines, as an integral over $3n-1$ Feynman parameters z_i (Ref. 1):

$$\Pi_{\mu\nu}^{(2n)}(q)=\left[\frac{-\alpha}{4\pi}\right]^n(n-2)!\,F_{\mu\nu}\int\frac{(dz)}{U^2(V-i\epsilon)^{n-1}}\ , \tag{1.1}$$

where q is the external momentum entering at the vertex ν and leaving at the vertex μ and

$$(dz)=\delta\left[1-\sum_i z_i\right]\prod_i dz_i,\quad z_i\geq 0,\ i\ \text{running over all internal lines}\ , \tag{1.2}$$

$$F_{\mu\nu}=\mathrm{Tr}[\gamma_\mu(\not{D}_1+m_1)\gamma^\alpha(\not{D}_2+m_2)\gamma^\beta\cdots\gamma_\nu\cdots\gamma_\lambda(\not{D}_{2n}+m_{2n})]\ , \tag{1.3}$$

$$D_i^\mu=\frac{1}{2}\int_{m_i^2}^\infty dm_i^2\frac{\partial}{\partial q_{i\mu}}\ . \tag{1.4}$$

We have included in (1.1) a factor $(-i)(2\pi)^{-4}$ coming from one of the external photon lines. In (1.4) q_i is that part of the momentum carried by the line i which is independent of loop momenta.[1]

The function U is a homogeneous form of degree n in the z_i and is completely determined by the topological structure of the diagram. Examples are given in Sec. II and in the Appendix. The function V is of the form

$$V=V_0-q^2G,\quad V_0=\sum_{i=1}^{2n}m_i^2z_i\ ,$$
$$G=\sum_i^{P(\mu,\nu)}z_iA_i\ , \tag{1.5}$$

where the sum in V_0 is over all electron lines and that in G is over the continuous path $P(\mu,\nu)$ of electron lines connecting the vertices μ and ν whose fixed momenta q_i flow in the same direction as that of the external momentum q. The function A_i is the "Kirchhoff current" associated with the line i:

$$A_i=\sum_j^{P(\mu,\nu)}(\delta_{ij}-z_jB_{ij}/U)\ . \tag{1.6}$$

Finally the function B_{ij} is defined by

$$B_{ij}=\sum_c\eta_{ic}\eta_{jc}U_c\ , \tag{1.7}$$

where the sum goes over all (not just independent) self-nonintersecting loops c that contain both lines i and j, η_{ic} is the projection (± 1, 0) of line i along the loop c, and U_c is the U function for the reduced diagram obtained from the original diagram by shrinking the loop c to a point. Note that U can in turn be expressed as a linear combination of B_{ij}'s:

$$\eta_{js} U = \sum_i \eta_{is} z_i B_{ij} \ . \tag{1.8}$$

Thus all functions can be calculated once the B_{ij} are known.

Of course the integral (1.1) is not well defined unless the overall and subdiagram ultraviolet (UV) divergences are regularized. Although subdiagram UV divergences can be handled adequately by simple Feynman cutoffs, the overall divergence of (1.1) requires more careful cutoff procedures such as that of Pauli and Villars[2] to maintain gauge invariance of the theory. However, if we extract the coefficient of $q_\mu q_\nu$ in (1.1) and write it as

$$\Pi_{\mu\nu}^{(2n)}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\widetilde{\Pi}^{(2n)}(q^2) + \text{gauge-dependent terms} \ , \tag{1.9}$$

the function $\widetilde{\Pi}^{(2n)}$ has only a logarithmic overall UV divergence and the number of auxiliary masses in Pauli-Villars regularization can be reduced to one. Furthermore, this regularization and the renormalization of subvertex UV divergences can be carried out independent of each other. The charge renormalization can be achieved by

$$\Pi^{(2n)}(q^2) = \widetilde{\Pi}^{(2n)}(q^2) - \widetilde{\Pi}^{(2n)}(0) \ . \tag{1.10}$$

We carry out subvertex renormalizations by the method of the intermediate renormalization developed previously[3,4] which is particularly convenient for numerical integration. The remaining finite renormalization can be carried out without difficulty as it involves only lower-order diagrams.

In Sec. II we explain how the parametric formulation is applied to the second- and fourth-order vacuum polarization loops. Section III is devoted to a discussion of the sixth-order vacuum polarization diagrams. Functions necessary to define the sixth-order integrals are listed in the Appendix.

II. SECOND- AND FOURTH-ORDER VACUUM POLARIZATION LOOPS

For simplicity we shall omit the factor α/π and put the electron mass m equal to unity in the remainder of this article.

Although the vacuum polarization tensors of second and fourth orders are known analytically,[5] we shall begin by putting them in the Feynman-parametric form to illustrate our general approach. Applying formula (1.1) to the second-order photon self-energy diagram of Fig. 1(a), we obtain

$$\Pi_{\mu\nu}^{(2)}(q) = -\tfrac{1}{4}\,\mathrm{Tr}[\gamma_\mu(\not{D}_1+1)\gamma_\nu(\not{D}_2+1)] \times \int (dz) \int_1^{M^2} dm^2 \frac{z_{12}}{U^2 V_m} \ , \tag{2.1}$$

where M is the Pauli-Villars cutoff mass and[6]

$$\begin{aligned} &(dz) = dz_1 dz_2 \delta(1-z_{12}) \ , \\ &U = z_{12}, \quad V_m = z_{12} m^2 - q^2 G \ , \\ &G = z_1 A_1, \quad A_1 = z_2/z_{12} \ . \end{aligned} \tag{2.2}$$

Carrying out the D operation according to (1.4), and extracting the coefficient of $q_\mu q_\nu$ we find

$$\begin{aligned} &\widetilde{\Pi}(q^2) = \int (dz) \frac{D_0}{U^2} \ln\left[\frac{z_{12} M^2}{z_{12} - q^2 G}\right] \ , \\ &D_0 = 2A_1(1-A_1) \ . \end{aligned} \tag{2.3}$$

Renormalization at $q=0$ according to (1.10) leads to

$$\Pi^{(2)}(q^2) = \int (dz) \frac{D_0}{U^2} \ln\left[\frac{V_0}{V}\right] \ , \tag{2.4}$$

where $V_0 = z_{12}$, $V = V_0 - q^2 G$. By a simple change of variable $[z_1 = (1+t)/2]$ and integration by parts, (2.4) can be readily rewritten as

$$\begin{aligned} &\Pi^{(2)}(q^2) = -q^2 \int_0^1 dt \frac{\rho_2(t)}{q^2 - 4/(1-t^2)} \ , \\ &\rho_2(t) = \frac{t^2(1-t^2/3)}{1-t^2} \ , \end{aligned} \tag{2.5}$$

which is essentially the Källén-Lehmann spectral representation of $\Pi^{(2)}$.

The fourth-order vacuum polarization tensor consists of contributions from three diagrams, one of the form Fig. 1(b) and two of the form Fig. 1(c). Using (1.1) one can express the contribution of Fig. 1(b) as

$$\Pi_{\mu\nu}^{(4a)} = \tfrac{1}{16}\,\mathrm{Tr}[\gamma_\mu(\not{D}_1+1)\gamma^\alpha(\not{D}_2+1)\gamma_\nu(\not{D}_3+1)\gamma_\alpha(\not{D}_4+1)] \int \frac{(dz)}{U^2 V} \ , \tag{2.6}$$

where

$$(dz)=dz_1 \cdots dz_5\delta(1-z_{12345}),\quad U=z_{1234}z_5+z_{14}z_{23}\ ,$$
$$B_{11}=z_{235},\quad B_{12}=z_5,\quad B_{22}=z_{145},\quad A_1=(z_3B_{12}+z_4B_{11})/U,\quad A_2=(z_3B_{22}+z_4B_{12})/U\ , \tag{2.7}$$
$$V_0=z_{1234},\quad G=z_1A_1+z_2A_2,\quad V=V_0-q^2G\ .$$

Carrying out the D operations and extracting the coefficient of $q_\mu q_\nu$, we can write $\tilde{\Pi}^{(4a)}$ in the form[7]

$$\tilde{\Pi}^{(4a)}(q^2)=\int (dz)\left[\frac{D_0}{U^2V}+\frac{D_1}{U^3}\ln\left[\frac{V_0M^2}{V}\right]\right], \tag{2.8}$$

where

$$D_0=(A_1+A_4)(A_2+A_3)-A_1A_4-A_2A_3\ ,$$
$$D_1=(A_1A_2+A_3A_4)B_{12}-A_1A_4B_{22}-A_2A_3B_{11}\ , \tag{2.9}$$

with

$$A_4=A_1-1,\quad A_3=A_2-1\ . \tag{2.10}$$

Charge renormalization at $q=0$ leads to

$$\Pi^{(4a)}(q^2)=\int (dz)\left[\frac{D_0}{U^2}\left[\frac{1}{V}-\frac{1}{V_0}\right]+\frac{D_1}{U^3}\ln\left[\frac{V_0}{V}\right]\right], \tag{2.11}$$

where V_0 is defined in (2.7).

This expression has UV divergences arising from the subvertices $S'=\{2,3,5\}$ and $S''=\{1,4,5\}$. The divergent terms can be isolated from the integrand of (2.11) by the K-renormalization method.[3,4] For the $\{2,3,5\}$ vertex we find it to be

$$K_{S'}\Pi^{(4a)}=\int (dz)\frac{D_1'}{U'^3}\ln\left[\frac{V_0'}{V'}\right], \tag{2.12}$$

where

$$D_1'=-A_1'A_4'B_{22}',\quad A_1'=z_4/z_{14},\quad A_4'=A_1'-1\ ,$$
$$B_{22}'=z_{14},\quad U'=z_{14}z_{235},\quad G'=z_1A_1'\ , \tag{2.13}$$
$$V_0'=z_{14}+z_{23}{}^2/z_{235},\quad V'=V_0'-q^2G'\ .$$

A similar result is obtained for $K_{S''}\Pi^{(4a)}$. It is readily seen that (2.12) factorizes as

$$K_{S'}\Pi^{(4a)}=\hat{L}_2\Pi^{(2)}\quad (\text{similarly } K_{S''}\Pi^{(4a)}=\hat{L}_2\Pi^{(2)})\ , \tag{2.14}$$

where $\hat{L}_2$ is the UV-divergent part of the second-order vertex renormalization constant L_2 (Ref. 8):

$$L_2=\hat{L}_2+\Delta'L_2\ ,$$
$$\hat{L}_2=\tfrac{1}{2}\ln\Lambda-\tfrac{1}{8}\ , \tag{2.15}$$
$$\Delta'L_2=\ln\lambda+\tfrac{5}{4}\ ,$$

Λ and λ being the UV and IR (infrared) cutoffs, respectively. Thus we can relate $\Pi^{(4a)}$ to $\Pi^{(4a)}_{\text{ren}}$ (which is obtained by a standard renormalization of the subvertices) by

$$\Pi^{(4a)}_{\text{ren}}=\Pi^{(4a)}-2L_2\Pi^{(2)}$$
$$=\Delta\Pi^{(4a)}-2\Delta'L_2\Pi^{(2)}\ , \tag{2.16}$$

where

$$\Delta\Pi^{(4a)}=(1-K_{S'}-K_{S''})\Pi^{(4a)} \tag{2.17}$$

is free from UV divergences.

The contribution of the diagram of Fig. 1(c) has a form similar to (2.8):

$$\tilde{\Pi}^{(4b)}(q^2)=\int (dz)\left[\frac{D_0+q^2C_0}{U^2V}+\frac{D_1}{U^3}\ln\left[\frac{V_0M^2}{V}\right]\right], \tag{2.18}$$

where

$$U=z_{134}z_{25}+z_2z_5,\quad G=z_{13}A_1+z_2A_2\ ,$$
$$V_0=z_{1234},\quad V=V_0-q^2G\ ,$$
$$B_{11}=z_{25},\quad B_{12}=z_5,\quad B_{22}=z_{1345}\ ,$$
$$A_1=z_4B_{11}/U,\quad A_2=z_4B_{12}/U\ , \tag{2.19}$$
$$A_3=A_1,\quad A_4=A_1-1\ ,$$
$$C_0=-A_1{}^2A_2A_4\ ,$$
$$D_0=(4A_1-A_2)A_4\ ,$$
$$D_1=B_{12}A_1(A_1+3A_4)\ .$$

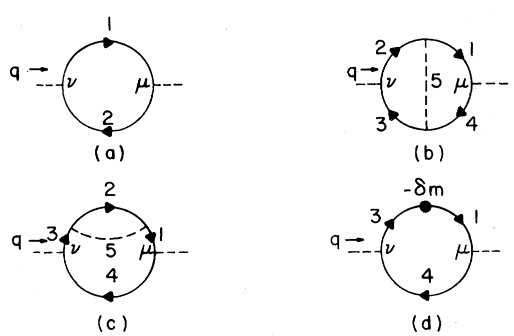

FIG. 1. Second- and fourth-order photon self-energy (or vacuum polarization) diagrams.

Renormalization of the overall divergence of $\tilde{\Pi}^{(4b)}$ at $q=0$ leads to

$$\Pi^{(4b)}(q^2)=\int (dz)\left[\frac{D_0}{U^2}\left[\frac{1}{V}-\frac{1}{V_0}\right]+\frac{q^2C_0}{U^2V}+\frac{D_1}{U^3}\ln\left[\frac{V_0}{V}\right]\right]. \quad (2.20)$$

In this case a UV divergence arises from the subdiagram $S=\{2,5\}$. The K_S operation on $\Pi^{(4b)}$ gives an integral of the same form as (2.20) with the integrand modified as follows:

$$\begin{aligned}&U=z_{134}z_{25},\quad G=z_{13}A_1\,,\\&A_1=z_4/z_{134},\quad A_2=(z_5/z_{25})A_1\,,\\&V_0=z_{134}+{z_2}^2/z_{25},\quad V=V_0-q^2G\,,\\&B_{22}=z_{134}\,,\end{aligned} \quad (2.21)$$

other definitions being unchanged. By the construction of the K_S operation we obtain[3,4]

$$K_S\Pi^{(4b)}=\delta m_2\Pi^{(2^*)}+\hat{B}_2\Pi^{(2)}\,, \quad (2.22)$$

where $\Pi^{(2^*)}$ is the contribution from the diagram of Fig. 1(d), δm_2 is the second-order electron self-mass, and $\hat{B}_2$ is the UV-divergent part of the second-order wave-function renormalization constant B_2 (Ref. 9):

$$\begin{aligned}&B_2=\hat{B}_2+\Delta' B_2\,,\\&\hat{B}_2=-\tfrac{1}{2}\ln\Lambda-\tfrac{5}{8}\,,\\&\Delta' B_2=\Delta B_2-\Delta' L_2,\quad \Delta B_2=\tfrac{3}{4}\,.\end{aligned} \quad (2.23)$$

It is important to note here that $\Pi^{(2)}$ in (2.22) is not of the form (2.4) but is given by

$$\int (dz)z_1\left[\frac{2A_1(4A_1-3)}{U^3}\ln\left[\frac{V_0}{V}\right]-\frac{D_0}{U^2}\left[\frac{1}{V_0}-\frac{1-q^2{A_1}^2}{V}\right]\right]=-\int (dz)z_1\frac{\partial}{\partial z_1}\left[\frac{D_0}{U^2}\ln\left[\frac{V_0}{V}\right]\right], \quad (2.24)$$

where all functions are defined in (2.2) with z_2 replaced by z_4. This can be turned into (2.4) by making use of an identity due to Nakanishi.[10]

We can therefore write the renormalized quantity

$$\Pi_{\text{ren}}^{(4b)}=\Pi^{(4b)}-\delta m_2\Pi^{(2^*)}-B_2\Pi^{(2)}$$

as

$$\Pi_{\text{ren}}^{(4b)}=\Delta\Pi^{(4b)}-\Delta' B_2\Pi^{(2)}\,, \quad (2.25)$$

where

$$\Delta\Pi^{(4b)}=(1-K_S)\Pi^{(4b)} \quad (2.26)$$

is free from UV divergences. Collecting (2.16) and (2.25) we can finally express the vacuum polarization of fourth order in the form

$$\begin{aligned}\Pi_{\text{ren}}^{(4)}&=\Pi_{\text{ren}}^{(4a)}+2\Pi_{\text{ren}}^{(4b)}\\&=\Delta\Pi^{(4a)}+2\Delta\Pi^{(4b)}-2\Delta B_2\Pi^{(2)}\,,\end{aligned} \quad (2.27)$$

where ΔB_2 is defined in (2.23).

III. SIXTH-ORDER VACUUM POLARIZATION DIAGRAMS

Of the 18 vacuum polarization diagrams of sixth order, the three represented by Fig. 2 are reducible to fourth-order diagrams when the second-order photon self-energy subdiagram is shrunk to a point. Thus, these can be calculated readily once the fourth-order vacuum polarization expression is known. Noting that insertion of a second-order vacuum polarization diagram is equivalent to the substitution of the photon propagator

$$\frac{1}{q^2}\to\int_0^1 dt\frac{\rho_2(t)}{q^2-4/(1-t^2)}\,, \quad (3.1)$$

where

$$\rho_2(t)=\frac{t^2(1-t^2/3)}{(1-t^2)}\,, \quad (3.2)$$

we find from (2.11) that the contribution of Fig. 2(a) can be written as

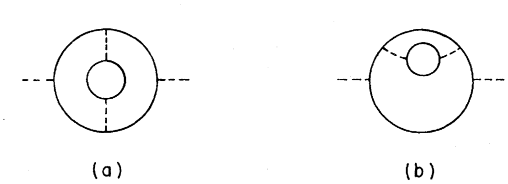

FIG. 2. Sixth-order vacuum polarization diagrams obtained by inserting a second-order vacuum polarization loop in the fourth-order vacuum polarization diagrams.

$$\Pi^{(4a,P2)}(q^2)=\int_0^1 dt\,\rho_2(t)\int (dz)\left[\frac{D_0}{U^2}\left[\frac{1}{V}-\frac{1}{V_0}\right]+\frac{D_1}{U^3}\ln\left[\frac{V_0}{V}\right]\right], \tag{3.3}$$

where

$$V_0=z_{1234}+4z_5/(1-t^2)\,, \tag{3.4}$$

all other quantities being defined by (2.7) through (2.10).

It is readily seen, in parallel with (2.12) and (2.14), that the $K_{S'}$ and $K_{S''}$ operations yield

$$K_{S'}\Pi^{(4a,P2)}=K_{S''}\Pi^{(4a,P2)}=\hat{L}_{2,P2}\Pi^{(2)}\,, \tag{3.5}$$

where $\hat{L}_{2,P2}$ is the UV-divergent part of the vertex renormalization constant $L_{2,P2}$ for the diagram shown in Fig. 3(a). The integral

$$\Delta\Pi^{(4a,P2)}=(1-K_{S'}-K_{S''})\Pi^{(4a,P2)} \tag{3.6}$$

is now free from UV divergences, and

$$\Pi^{(4a,P2)}_{\text{ren}}=\Delta\Pi^{(4a,P2)}-2\Delta' L_{2,P2}\Pi^{(2)} \tag{3.7}$$

with[11]

$$\Delta' L_{2,P2}=L_{2,P2}-\hat{L}_{2,P2}\,. \tag{3.8}$$

Similarly, the contribution of Fig. 2(b) can be written as

$$\Pi^{(4b,P2)}(q^2)=\int_0^1 dt\,\rho_2(t)\int (dz)\left[\frac{D_0}{U^2}\left[\frac{1}{V}-\frac{1}{V_0}\right]+\frac{q^2C_0}{U^2V}+\frac{D_1}{U^3}\ln\left[\frac{V_0}{V}\right]\right], \tag{3.9}$$

where V_0 is given by (3.4) and the rest by (2.19).

The K_S operation on $\Pi^{(4b,P2)}$ now leads to

$$K_S\Pi^{(4b,P2)}=\delta m_{2,P2}\Pi^{(2^*)}+\hat{B}_{2,P2}\Pi^{(2)}\,, \tag{3.10}$$

where $\Pi^{(2^*)}$ is the contribution of the diagram of Fig. 1(d), $\delta m_{2,P2}$ is the electron self-mass, and $\hat{B}_{2,P2}$ is the UV-divergent part of the wave-function renormalization constant $B_{2,P2}$ for the diagram of Fig. 3(b). The comment (2.24) on $\Pi^{(2)}$ applies here as well.

The sixth-order vacuum polarization effect arising from the gauge-invariant set of diagrams of Fig. 2 is thus given by

$$\Pi^{(4,P2)}_{\text{ren}}=\Pi^{(4a,P2)}_{\text{ren}}+2\Pi^{(4b,P2)}_{\text{ren}}=\Delta\Pi^{(4a,P2)}+2\Delta\Pi^{(4b,P2)}-2\Delta B_{2,P2}\Pi^{(2)}\,, \tag{3.11}$$

where[11]

$$\Delta\Pi^{(4b,P2)}=(1-K_S)\Pi^{(4b,P2)},\quad \Delta B_{2,P2}=\Delta' B_{2,P2}+\Delta' L_{2,P2},\quad \Delta' B_{2,P2}=B_{2,P2}-\hat{B}_{2,P2}\,. \tag{3.12}$$

We find

$$\Delta B_{2,P2}=\int_0^1 dt\,\rho_2(t)\int_0^1 dy\frac{1-y/2}{1+[4/(1-t^2)](1-y)/y^2]}=\tfrac{41}{24}-\pi^2/6=0.063\,399\cdots\,. \tag{3.13}$$

The remaining 15 vacuum polarization diagrams are shown in Fig. 4. Applying formula (1.1) to the diagrams $6A,6B,\ldots,6H$ of Fig. 4, we find

$$\Pi^{(6i)}_{\mu\nu}(q)=-\tfrac{1}{64}\,\text{Tr}[\gamma_\mu(\not{D}_1+1)\gamma^\alpha(\not{D}_2+1)\gamma^\beta\cdots\gamma_\nu\cdots(\not{D}_6+1)]\int (dz)\frac{1}{U^2V^2},\quad i=A,B,\ldots,H\,, \tag{3.14}$$

where U, V, and other necessary functions are listed in the Appendix and

$$(dz)=\prod_{i=1}^{8}dz_i\,\delta\left[1-\sum_{j=1}^{8}z_j\right]. \tag{3.15}$$

Carrying out the D operations and extracting the coefficient of $q_\mu q_\nu$ we can write $\tilde{\Pi}^{(6i)}$ in the form

$$\tilde{\Pi}^{(6i)}(q^2)=\tfrac{1}{2}\int (dz)\left[\frac{D_0+q^2B_0+q^4C_0}{U^2V^2}+\frac{D_1+q^2B_1}{U^3V}+\frac{D_2}{U^4}\ln\left[\frac{V_0M^2}{V}\right]\right], \tag{3.16}$$

where

$$V_0=z_{123456}\ . \tag{3.17}$$

The functions D_0, B_0, C_0, D_1, B_1, D_2 have been obtained using the algebraic computation program SCHOONSCHIP[12] and are listed in the Appendix. Overall renormalization at $q=0$ according to (1.10) leads to[13]

$$\Pi^{(6i)}(q^2)=\tfrac{1}{2}\int (dz)\left[\frac{D_0}{U^2}\left[\frac{1}{V^2}-\frac{1}{V_0^2}\right]+\frac{q^2B_0+q^4C_0}{U^2V^2}+\frac{D_1}{U^3}\left[\frac{1}{V}-\frac{1}{V_0}\right]+\frac{q^2B_1}{U^3V}+\frac{D_2}{U^4}\ln\left[\frac{V_0}{V}\right]\right]. \tag{3.18}$$

The integral (3.18) has UV divergences arising from various subdiagrams which must be extracted before it is evaluated. We achieve this by constructing subtraction terms which are obtained by applying K_S operations[3,4] on the integrand of (3.18). In the following we shall carry out this procedure explicitly for each diagram of Fig. 4.

Diagram 6A. In this case UV divergences arise from two subdiagrams $S'=\{2,7\}$ and $S''=\{4,8\}$. The $K_{S'}$ operation on the integrand of (3.18) extracts the leading term in the neighborhood of the singular point $z_{27}=0$. The integrand of $K_{S'}\Pi^{(6A)}$ is of the form (3.18) except that the various functions are replaced by their limits at $z_{27}=0$. By construction we get

$$K_{S'}\Pi^{(6A)}=\delta m_2\Pi^{(4b,1^*)}+\hat{B}_2\Pi^{(4b)}\ , \tag{3.19}$$

where δm_2 and $\hat{B}_2$ are as in (2.22), $\Pi^{(4b,1^*)}$ is the contribution from the diagram of Fig. 5(a), and $\Pi^{(4b)}$ here can be equated to that of (2.20) after a transformation similar to (2.24). Clearly $K_{S''}\Pi^{(6A)}$ is of the form (3.19) as well. Finally we have

$$K_{S'}K_{S''}\Pi^{(6A)}=(\delta m_2)^2\Pi^{(2^{**})}+2\delta m_2\hat{B}_2\Pi^{(2^*)}+(\hat{B}_2)^2\Pi^{(2)}\ , \tag{3.20}$$

where $\Pi^{(2^*)}$ is defined in (2.22) and $\Pi^{(2^{**})}$ is the contribution from the diagram of Fig. 5(b).

Thus the properly renormalized $\Pi^{(6A)}_{\rm ren}$ is given by

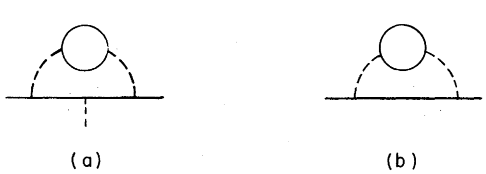

FIG. 3. Fourth-order electron vertex and self-energy diagrams containing a second-order vacuum polarization loop.

$$\begin{aligned}\Pi^{(6A)}_{\rm ren}&=\Pi^{(6A)}-2\delta m_2\Pi^{(4b,1^*)}-2B_2\Pi^{(4b)}\\&\quad+(\delta m_2)^2\Pi^{(2^{**})}+2\delta m_2B_2\Pi^{(2^*)}+B_2{}^2\Pi^{(2)}\\&=\Delta\Pi^{(6A)}-2\Delta' B_2\Delta\Pi^{(4b)}+(\Delta' B_2)^2\Pi^{(2)}\ ,\end{aligned} \tag{3.21}$$

where $\Delta' B_2$ and $\Delta\Pi^{(4b)}$ are given by (2.23) and (2.26) and

$$\Delta\Pi^{(6A)}=(1-K_{S'})(1-K_{S''})\Pi^{(6A)} \tag{3.22}$$

is free from UV divergences and can be evaluated numerically.

Diagram 6B. The results (3.21) and (3.22) hold for this diagram with a trivial adjustment.

Diagram 6C. The UV divergences arise from the subdiagrams $S'=\{3,8\}$ and $S''=\{3,8,2,4,7\}$. The $K_{S'}$ operation on $\Pi^{(6C)}$ produces a result similar to that on $\Pi^{(6A)}$:

$$K_{S'}\Pi^{(6C)}=\delta m_2\Pi^{(4b,2^*)}+\hat{B}_2\Pi^{(4b)}\ , \tag{3.23}$$

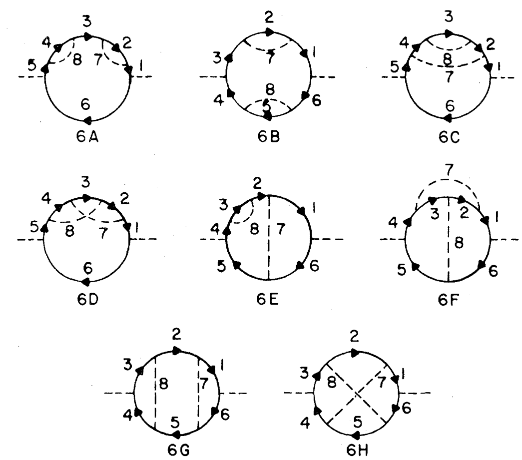

FIG. 4. Sixth-order vacuum polarization diagrams containing only one closed electron loop.

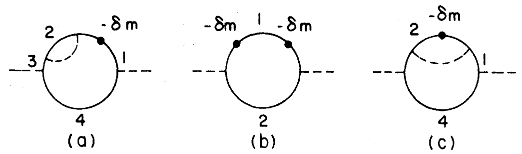

FIG. 5. Vacuum polarization diagrams containing electron self-mass counterterms.

where $\Pi^{(4b,2^*)}$ is the contribution from the diagram of Fig. 5(c). The $K_{S''}$ operation yields

$$K_{S''}\Pi^{(6C)}=\delta\hat{m}_{4b}\Pi^{(2^*)}+\hat{B}_{4b}\Pi^{(2)} , \quad (3.24)$$

where $\delta\hat{m}_{4b}$ and $\hat{B}_{4b}$ are UV-divergent parts of the electron self-mass δm_{4b} and the wave-function renormalization constant B_{4b} for the diagram of Fig. 6(a) [see Ref. 4, formulas (4.47) and (4.54)]. Since $S'\subset S''$, the $K_{S'}$ operation on $K_{S''}\Pi^{(6C)}$ reduces to the calculation of $K_{S'}\delta\hat{m}_{4b}$ and $K_{S'}\hat{B}_{4b}$. We find[14]

$$\begin{aligned} &K_{S'}\delta\hat{m}_{4b}=\delta m_2\delta\hat{m}_{2*}+\hat{B}_2\delta m_2' , \\ &K_{S'}\hat{B}_{4b}=\hat{B}_2\hat{B}_2' . \end{aligned} \quad (3.25)$$

We also have

$$\begin{aligned} &K_{S'}\delta m_{4b}=\delta m_2\delta m_{2*}+\hat{B}_2\delta m_2 , \\ &K_{S'}B_{4b}=\delta m_2 B_{2*}+\hat{B}_2 B_2 , \end{aligned} \quad (3.26)$$

where δm_{2*} and B_{2*} are the electron self-mass and the wave-function renormalization constant for the diagram of Fig. 6(b):

$$\delta m_{2*}=\int (dz)\int dm_{1'}{}^2 z_{1'}\left[-\frac{1-A_1+A_1{}^2}{U^2V^2}+\frac{2B_{11}}{U^3V}\right], \quad (3.27)$$

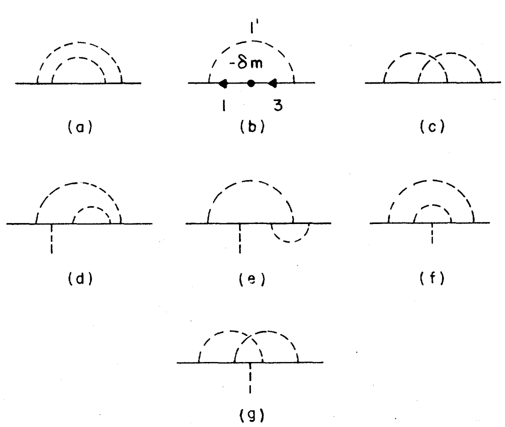

FIG. 6. Various fourth-order electron self-energy and vertex diagrams.

$$B_{2*}=\int (dz)\left[\frac{A_1(1-2A_1)}{U^2V}-\frac{2G(1-A_1+A_1{}^2)}{U^2V^2}+\frac{4GB_{11}}{U^3V}\right], \quad (3.28)$$

with $G=z_{13}A_1$. $\delta\hat{m}_{2*}$ is the last term of (3.27) whereas

$$\delta\hat{m}_2'=\tfrac{1}{2}\int (dz)\int dm_{1'}{}^2 z_{1'}\frac{(4-3A_1)B_{11}}{U^3V} \quad (3.29)$$

and

$$\hat{B}_2'=\tfrac{1}{2}\int (dz)\int dm_{1'}{}^2 z_{1'}\frac{-3A_1B_{11}}{U^3V} \quad (3.30)$$

are the UV-divergent parts of $\delta m_2'$ and B_2', the latter being equal to δm_2 and B_2 by Nakanishi's identity.[15] In terms of these relations we can express the renormalized amplitude

$$\begin{aligned} \Pi_{\rm ren}^{(6C)}=&\Pi^{(6C)}-\delta m_2\Pi^{(4b,2^*)}-B_2\Pi^{(4b)} \\ &-\delta m_{4b}\Pi^{(2^*)}-B_{4b}\Pi^{(2)}+\delta m_2\delta m_{2*}\Pi^{(2^*)} \\ &+\delta m_2 B_{2*}\Pi^{(2)}+B_2\delta m_2\Pi^{(2^*)}+B_2{}^2\Pi^{(2)} \end{aligned} \quad (3.31)$$

as

$$\begin{aligned} \Pi_{\rm ren}^{(6C)}=&\Delta\Pi^{(6C)}-\Delta'B_2\Delta\Pi^{(4b)}-\Delta'B_{4b}\Pi^{(2)} \\ &-\Delta\delta m_{4b}\Pi^{(2^*)}+(\Delta'B_2)^2\Pi^{(2)} , \end{aligned} \quad (3.32)$$

where

$$\Delta\Pi^{(6C)}=(1-K_{S'})(1-K_{S''})\Pi^{(6C)} \quad (3.33)$$

and[15]

$$\begin{aligned} &\Delta'B_{4b}=B_{4b}-\hat{B}_{4b}-\delta m_2\Delta'B_{2*}-B_2\Delta'B_2' , \\ &\Delta'B_{2*}=B_{2*}-\hat{B}_{2*},\quad \Delta'B_2'=B_2'-\hat{B}_2' , \\ &\Delta\delta m_{4b}=\delta m_{4b}-\delta\hat{m}_{4b}-\delta m_2\Delta'\delta m_{2*}+\hat{B}_2\Delta'\delta m_2' , \\ &\Delta'\delta m_{2*}=\delta m_{2*}-\delta\hat{m}_{2*},\quad \Delta'\delta m_2'=\delta m_2'-\delta\hat{m}_2' . \end{aligned} \quad (3.34)$$

Diagram 6D. This diagram contains three divergent subdiagrams: $S=\{2,3,4,7,8\}$, $S'=\{2,3,7\}$, $S''=\{3,4,8\}$. We find

$$K_S\Pi^{(6D)}=\delta\hat{m}_{4a}\Pi^{(2^*)}+\hat{B}_{4a}\Pi^{(2)} , \quad (3.35)$$

where $\delta\hat{m}_{4a}$ and $\hat{B}_{4a}$ are the UV-divergent parts of the electron self-mass δm_{4a} and the wave-function renormalization constant B_{4a} for the diagram of Fig. 6(c) [see Ref. 4, formulas (4.5) and (4.9)]. The $K_{S'}$ and $K_{S''}$ operations give

$$K_{S'}\Pi^{(6D)} = K_{S''}\Pi^{(6D)} = \hat{L}_2\Pi^{(4b)} , \quad (3.36)$$

where $\hat{L}_2$ is defined in (2.15). Using these results we can rewrite

$$\Pi^{(6D)}_{\text{ren}} = \Pi^{(6D)} - 2L_2\Pi^{(4b)} - \delta m_{4a}\Pi^{(2*)} - B_{4a}\Pi^{(2)} + 2L_2(\delta m_2\Pi^{(2*)} + B_2\Pi^{(2)}) \quad (3.37)$$

as

$$\Pi^{(6D)}_{\text{ren}} = \Delta\Pi^{(6D)} - 2\Delta' L_2\Delta\Pi^{(4b)} - \Delta\delta m_{4a}\Pi^{(2*)} - \Delta' B_{4a}\Pi^{(2)} + 2\Delta' L_2\Delta' B_2\Pi^{(2)} , \quad (3.38)$$

where

$$\Delta\Pi^{(6D)} = (1-K_S)(1-K_{S'}-K_{S''})\Pi^{(6D)} \quad (3.39)$$

and

$$\Delta\delta m_{4a} = \Delta'\delta m_{4a} = \delta m_{4a} - \delta\hat{m}_{4a} , \quad \Delta' B_{4a} = B_{4a} - \hat{B}_{4a} - 2\hat{L}_2\Delta' B_2 . \quad (3.40)$$

The remaining diagrams are much easier to renormalize. We list only the final results.

Diagram 6E. In this case we have

$$\Pi^{(6E)}_{\text{ren}} = \Delta\Pi^{(6E)} - \Delta' B_2\Delta\Pi^{(4a)} - \Delta' L_2\Delta\Pi^{(4b)} - \Delta' L_{4s}\Pi^{(2)} + 2\Delta' L_2\Delta' B_2\Pi^{(2)} , \quad (3.41)$$

where $\Delta' L_{4s}$ is the UV-finite part of the vertex renormalization constant L_{4s} for the diagram of Fig. 6(d) [see Ref. 4, formula (4.56)], $\Delta\Pi^{(4a)}$ and $\Delta\Pi^{(4b)}$ are given by (2.17) and (2.26), and

$$\Delta\Pi^{(6E)} = (1-K_S)(1-K_{S'}-K_{S''})\Pi^{(6E)} , \quad (3.42)$$

with $S=\{3,8\}$, $S'=\{2,3,4,5,7,8\}$, $S''=\{1,6,7\}$.

Diagram 6F. Here we get

$$\Pi^{(6F)}_{\text{ren}} = \Delta\Pi^{(6F)} - \Delta' L_2\Delta\Pi^{(4a)} - 2\Delta' L_{4c}\Pi^{(2)} + 2(\Delta' L_2)^2\Pi^{(2)} , \quad (3.43)$$

where $\Delta' L_{4c}$ is the UV-finite part of the vertex renormalization constant L_{4c} for the diagram of Fig. 6(e) [see Ref. 4, formula (4.32)], and

$$\Delta\Pi^{(6F)} = (1-K_S)(1-K_{S'}-K_{S''})\Pi^{(6F)} , \quad (3.44)$$

with $S=\{2,3,7\}$, $S'=\{1,2,3,6,7,8\}$, $S''=\{2,3,4,5,7,8\}$.

Diagram 6G. We have

$$\Pi^{(6G)}_{\text{ren}} = \Delta\Pi^{(6G)} - 2\Delta' L_2\Delta\Pi^{(4a)} - 2\Delta' L_{4l}\Pi^{(2)} + 3(\Delta' L_2)^2\Pi^{(2)} , \quad (3.45)$$

where $\Delta' L_{4l}$ is the UV-finite part of the vertex renormalization constant L_{4l} for the diagram of Fig. 6(f) [see Ref. 4, formula (4.61)], and

$$\Delta\Pi^{(6G)} = (1-K_{S_1})(1-K_{S_2})(1-K_{S_3}-K_{S_4})\Pi^{(6G)} , \quad (3.46)$$

with $S_1=\{1,6,7\}$, $S_2=\{3,4,8\}$, $S_3=\{1,2,5,6,7,8\}$, and $S_4=\{2,3,4,5,7,8\}$.

Diagram 6H. In this case we find

$$\Pi^{(6H)}_{\text{ren}} = \Delta\Pi^{(6H)} - 2\Delta' L_{4x}\Pi^{(2)} , \quad (3.47)$$

where $\Delta' L_{4x}$ is the UV-finite part of the vertex renormalization constant L_{4x} for the diagram of Fig. 6(g) [see Ref. 4, formula (4.29)], and

$$\Delta\Pi^{(6H)} = (1-K_{S'}-K_{S''})\Pi^{(6H)} , \quad (3.48)$$

with $S'=\{1,2,5,6,7,8\}$ and $S''=\{2,3,4,5,7,8\}$.

Finally, collecting the results for the diagrams 6*A* to 6*H* of Fig. 4 and including appropriate multiplicity factors, we can write the vacuum polarization term of sixth order as

$$\Pi^{(6)}_{\text{ren}} = 2(\Delta\Pi^{(6A)} + \Delta\Pi^{(6C)} + \Delta\Pi^{(6D)} + \Delta\Pi^{(6F)}) + \Delta\Pi^{(6B)} + 4\Delta\Pi^{(6E)} + \Delta\Pi^{(6G)} + \Delta\Pi^{(6H)} - 4\Delta B_2\Pi^{(4)}_{\text{ren}} - 2[\Delta B_{4a} + \Delta L_{4x} + 2\Delta L_{4c} + \Delta B_{4b} + \Delta L_{4l} + 2\Delta L_{4s} + \tfrac{3}{2}(\Delta B_2)^2]\Pi^{(2)}_{\text{ren}} - 2(\Delta\delta m_{4a} + \Delta\delta m_{4b})\Pi^{(2*)}_{\text{ren}} , \quad (3.49)$$

where ΔB_{4a}, ΔL_{4x}, ΔL_{4c}, ΔB_{4b}, ΔL_{4l}, ΔL_{4s} are the IR-finite parts of $\Delta' B_{4a}, \ldots, \Delta' L_{4s}$ defined by (4.43), (4.29), (4.33), (4.67), (4.61), and (4.58) of Ref. 4. The numerical values of the coefficients of $\Pi^{(4)}_{\text{ren}}$, $\Pi^{(2)}_{\text{ren}}(=\Pi^{(2)})$, and $\Pi^{(2*)}_{\text{ren}}(=\Pi^{(2*)})$ are as follows:

$$-4\Delta B_2 = -3 , \quad (3.50)$$

$$-2[\Delta B_{4a} + \cdots + \tfrac{3}{2}(\Delta B_2)^2] = -1.7447(36) , \quad (3.51)$$

$$-2(\Delta\delta m_{4a} + \Delta\delta m_{4b}) = -3.8132(22) . \quad (3.52)$$

The complete vacuum polarization effect of sixth order is the sum of $\Pi^{(4,P2)}_{\text{ren}}$ of (3.11) and $\Pi^{(6)}_{\text{ren}}$ of (3.49). This result has been applied to the calculation of the vacuum polarization effect on the eighth-order electron anomalous magnetic moment.[16] The contributions of $\Pi^{(4,P2)}_{\text{ren}}$ and $\Pi^{(6)}_{\text{ren}}$ to the hyperfine structures of muonium and positronium are not difficult to calculate either. However, they are too small to be of immediate interest.

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APPENDIX

In this appendix we list all functions that are needed to construct integrals (except for renormalization terms) for the sixth-order vacuum polarization tensor corresponding to the 15 single-electron-loop diagrams represented by Fig. 4. The functions are referenced by diagram as in Fig. 4. The assignment of Feynman parameters $z_1, z_2, \ldots, z_8$ to respective lines is also found in Fig. 4. Note that for all diagrams of Fig. 4

$$V = V(q) = V_0 - q^2 G \ , \qquad \text{(A1)}$$

where[6]

$$V_0 = z_{123456} \qquad \text{(A2)}$$

and G is given below for each diagram.

Functions needed for construction of UV-divergence subtraction terms are not shown explicitly. But they can be constructed readily by applying the K_S operations[3,4] on the functions given below. For the readers' convenience we describe briefly the K_S operation. Let M be the parametric integral corresponding to the Feynman diagram G which has a UV divergence arising from the subdiagram S. Suppose S consists of the lines $1,2,\ldots,m$. Denote by G/S the diagram which is obtained by shrinking all lines of S to a point. Then the K_S operation proceeds as follows.

(1) Determine for each term of the integrand of M whether it is UV divergent or not by examining its behavior in the limit

$$z_1 + z_2 + \cdots + z_m \rightarrow 0$$

(power counting). Retain only the divergent terms.

(2) Simplify the divergent terms by removing all nonleading terms of B_{ij}, A_i, and U in the same limit. This leads to

$$B_{ij} \equiv B_{ij}^{G} \rightarrow U^S B_{ij}^{G/S} \ \text{ for } i,j \in S \ ,$$
$$U \equiv U^G \rightarrow U^S U^{G/S} \ . \qquad \text{(A3)}$$

The limit of A_i is slightly more complicated, but is readily found using (A3).

(3) Replace $V(q) \equiv V^G(q)$ by $V^{G/S}(q) + V^S(\text{m.s.})$ and $V_0 \equiv V^G(q=0)$ by $V^{G/S}(q=0) + V^S(\text{m.s.})$, respectively. The function $V^S(\text{m.s.})$ is the V function for the vertex or self-energy subdiagram S in which the momenta of electron (photon) lines external to S are put on the mass shell (equal to zero). Note that V^S is not necessarily given by (A1) which is specific to the vacuum polarization diagram. Note also that we maintain the restriction

$$\sum_{i \in G} z_i = 1$$

in this replacement.

As an example, for diagram A and $S = \{2,7\}$, we have

$$V^{G/S}(q) = z_{13456} + z_6 A_6^{G/S} q^2 \ ,$$
$$V^S(\text{m.s.}) = z_2 - z_2 A_2^S = z_2{}^2 / z_{27} \qquad \text{(A4)}$$

with

$$A_6^{G/S} = z_6 \frac{B_{11}^{G/S}}{U^{G/S}} - 1 \ ,$$
$$B_{11}^{G/S} = z_{48}, \quad U^{G/S} = z_{1356} z_{48} + z_4 z_8 \ . \qquad \text{(A5)}$$

The function $V^S(\text{m.s.})$ is added to $V^{G/S}$ to avoid introducing a spurious singularity in the integrand. It also enables us to factorize the integral $K_S M$ thus constructed.

We are now ready to list the functions.

Diagram A:

$$B_{11} = z_{27} z_{48}, \quad B_{12} = z_7 z_{48}, \quad B_{14} = z_8 z_{27}, \quad B_{24} = z_7 z_8 \ ,$$

$$U = z_{1356} B_{11} + z_2 B_{12} + z_4 B_{14} \ ,$$

$$A_1 = z_6 B_{11}/U, \quad A_2 = z_6 B_{12}/U, \quad A_4 = z_6 B_{14}/U, \quad A_6 = A_1 - 1 \ ,$$

$$G = -z_6 A_6 \ ,$$

$$D_0 = 2A_6(A_2 + A_4 - 6A_1) \ ,$$

$$B_0 = A_1 A_6 [-3A_2 A_4 + 6A_1(A_2 + A_4) - 4A_1{}^2] \ ,$$

$$C_0=-A_1{}^3A_2A_4A_6\ ,$$

$$D_1=B_{11}[3A_2A_4-6(A_2+A_4)(A_1+3A_6)+4A_1(A_1+3A_6)]+9B_{24}A_1A_6\ ,$$

$$B_1=2B_{24}A_1{}^3(A_1+4A_6)\ ,$$

$$D_2=-4B_{11}B_{24}A_1(2A_1+3A_6)\ .$$

Diagram B:

$$B_{11}=z_{58}z_{27},\ \ B_{12}=z_7z_{58},\ \ B_{15}=z_8z_{27},\ \ B_{25}=z_7z_8\ ,$$

$$U=z_{1346}B_{11}+z_2B_{12}+z_5B_{15}\ ,$$

$$A_1=(z_{46}B_{11}+z_5B_{15})/U,\ \ A_4=A_1-1,\ \ A_2=(z_{46}B_{12}+z_5B_{25})/U\ ,$$

$$A_5=-(z_{13}B_{15}+z_2B_{25})/U\ ,$$

$$G=z_{13}A_1+z_2A_2\ ,$$

$$D_0=-(4A_1-A_2)(4A_4-A_5)\ ,$$

$$B_0=(4A_1-A_2)A_4{}^2A_5+(4A_4-A_5)A_1{}^2A_2\ ,$$

$$C_0=-A_1{}^2A_2A_4{}^2A_5\ ,$$

$$D_1=2B_{12}A_1(-2A_1-6A_4+A_5)+4B_{15}A_4(-3A_1+A_2-A_4)+B_{25}(A_1{}^2+A_4{}^2)\ ,$$

$$B_1=B_{11}A_2A_5(3A_1{}^2+4A_1A_4+3A_4{}^2)\ ,$$

$$D_2=-B_{11}B_{25}(3A_1{}^2+14A_1A_4+3A_4{}^2)\ .$$

Diagram C:

$$B_{12}=z_7z_{38},\ \ B_{13}=z_7z_8,\ \ B_{23}=z_8z_{1567}\ ,$$

$$B_{11}=z_{247}z_{38}+z_3z_8,\ \ B_{22}=z_{1567}z_{38}\ ,$$

$$U=z_{156}B_{11}+z_{24}B_{12}+z_3B_{13}\ ,$$

$$A_1=z_6B_{11}/U,\ \ A_2=z_6B_{12}/U,\ \ A_3=z_6B_{13}/U,\ \ A_6=A_1-1\ ,$$

$$G=-z_6A_6\ ,$$

$$D_0=A_6(-8A_1+4A_2-A_3)\ ,$$

$$B_0=A_6[8A_1A_2(A_3-A_2)+4A_1{}^2A_2-A_3(A_1{}^2+A_2{}^2)]\ ,$$

$$C_0=-A_1{}^2A_2{}^2A_3A_6\ ,$$

$$D_1=B_{12}[8(A_1+A_6)A_2-4A_1{}^2-12A_1A_6]+B_{13}[-8(A_1+A_6)A_2+A_1{}^2+A_2{}^2+3A_1A_6]$$
$$\qquad +B_{23}(-16A_1A_6+3A_2A_6)+16A_1A_6B_{22}\ ,$$

$$B_1=B_{12}A_1A_2A_3(2A_1+5A_6)+3B_{23}A_1{}^2A_2A_6\ ,$$

$$D_2=-B_{13}[B_{12}A_2(5A_1+3A_6)+B_{22}A_1(3A_1+9A_6)]\ .$$

Diagram D:

$$B_{12}=z_{348}z_7+z_3z_4,\ \ B_{13}=z_7z_8-z_2z_4,\ \ B_{14}=z_{237}z_8+z_2z_3\ ,$$

$$B_{23}=z_{156}z_{48}+z_8z_{47},\ \ B_{24}=z_7z_8-z_{156}z_3,\ \ B_{11}=z_{348}z_{27}+z_3z_{48}\ ,$$

$$U=z_{156}B_{11}+z_2B_{12}+z_3B_{13}+z_4B_{14}\ ,$$

$$A_1=z_6B_{11}/U,\ \ A_2=z_6B_{12}/U,\ \ A_3=z_6B_{13}/U,\ \ A_4=z_6B_{14}/U,\ \ A_6=A_1-1\ ,$$

$$G=-z_6A_6\ ,$$

$A_B=A_2B_{34}+4A_3B_{24}+A_4B_{23},\quad S_B=B_{23}+B_{24}+B_{34}$,

$A_A=A_2A_3+A_2A_4+A_3A_4,\quad S_A=A_2+A_3+A_4$,

$D_0=A_6(4A_1-S_A)$,

$B_0=A_6(-2A_1A_A-A_1{}^2S_A+2A_2A_3A_4)$,

$C_0=2A_1{}^2A_2A_3A_4A_6$,

$D_1=B_{11}S_A(A_1+3A_6)+2B_{11}A_A-2A_3B_{12}(A_4-A_6)+2(B_{13}+B_{12})A_4A_6+A_6(4A_1S_B-A_B)$,

$B_1=-A_1{}^2A_6A_B-2B_{11}A_2A_3A_4(2A_1+5A_6)$,

$D_2=B_{11}A_B(A_1+3A_6)+2B_{12}A_4(5B_{11}A_3+3B_{13}A_6)$.

Diagram E:

$B_{12}=z_7z_{38},\quad B_{13}=z_7z_8,\quad B_{23}=z_{167}z_8$,

$B_{11}=z_{2457}z_{38}+z_3z_8,\quad B_{22}=z_{167}z_{38}$,

$U=z_{16}B_{11}+z_{245}B_{12}+z_3B_{13}$,

$A_1=(z_5B_{12}+z_6B_{11})/U,\quad A_2=(z_5B_{22}+z_6B_{12})/U$,

$A_3=(z_5B_{23}+z_6B_{13})/U,\quad A_5=A_2-1,\quad A_6=A_1-1$,

$G=-z_5A_5-z_6A_6$,

$D_0=(A_3-4A_2)(A_1-A_5+A_6)-2[A_1(A_5-A_6)+A_5A_6]$,

$B_0=2A_2(A_3-A_2)[A_1(A_5-A_6)+A_5A_6]+A_2{}^2A_3(A_1-A_5+A_6)$,

$C_0=0$,

$$D_1=B_{11}A_5(4A_2-A_3)+2B_{22}(A_1+A_6)(A_2+2A_5)+B_{12}[-2A_2(3A_1+A_6)+4A_5(A_2-A_6)+2A_3(A_1-A_5)+A_3A_6]$$
$$+B_{13}\{A_2[A_1+A_6-2(A_2+A_5)]+A_5A_6\}+B_{23}[3(A_1A_6+A_2A_5)+A_2{}^2-(A_1+A_6)(5A_2+4A_5)]\ ,$$

$B_1=A_2{}^2[A_1(B_{12}A_3-B_{23}A_6)-A_5(B_{11}A_3+B_{13}A_6)]+2B_{12}A_2A_3A_5A_6$,

$D_2=B_{11}B_{23}A_2(A_2+3A_5)-B_{12}B_{13}A_2{}^2-3B_{13}B_{22}(A_1A_2+A_5A_6)+3B_{22}B_{23}A_1A_6$.

Diagram F:

$B_{12}=z_{45}Z_{37}+z_{38}z_7,\quad B_{13}=z_7z_8-z_2z_{45},\quad B_{14}=z_{237}z_8+z_2z_3$,

$B_{23}=z_{1456}z_8+z_{16}z_{45}+z_7z_8,\quad B_{24}=z_7z_8-z_3z_{16}$,

$B_{34}=z_{16}z_{27}+z_{28}z_7$,

$B_{11}=z_{458}z_{237}+z_3z_{27},\quad B_{44}=z_{168}z_{237}+z_2z_{37}$,

$U=z_{16}B_{11}+z_2B_{12}+z_3B_{13}+z_{45}B_{14}$,

$A_1=(z_5B_{14}+z_6B_{11})/U,\quad A_2=(z_5B_{24}+z_6B_{12})/U,\quad A_3=(z_5B_{34}+z_6B_{13})/U$,

$A_4=(z_5B_{44}+z_6B_{14})/U,\quad A_5=A_4-1,\quad A_6=A_1-1$,

$G=-z_5A_5-z_6A_6$,

$D_0=(A_1-A_4)(A_5-A_6)-(A_1+A_4)(A_2+A_3)+A_1A_4+A_2A_3+A_5A_6$,

$B_0=-(A_1-A_4)(A_5-A_6)A_2A_3-(A_1+A_4)(A_2+A_3)A_5A_6$,

$C_0=A_1A_2A_3A_4A_5A_6$,

$$D_1=B_{11}[A_2A_3-A_4(A_2+A_3+A_5)]+B_{44}[A_2A_3-A_1(A_2+A_3+A_6)]$$
$$+B_{14}(A_1A_4+A_5A_6)+B_{23}[(A_1-A_4)(A_5-A_6)+A_1A_4+A_5A_6]$$
$$+B_{12}[A_4(A_3+A_5+A_6)+A_5(2A_1+A_6)]+B_{34}[A_1(A_2+A_5+A_6)+A_6(2A_4+A_5)]$$
$$+B_{13}[2A_1(A_5-A_2)+A_5(A_2+A_4+A_6)+A_6(A_4-2A_2)]$$
$$+B_{24}[2A_4(A_6-A_3)+A_5(A_1-2A_3)+A_6(A_1+A_3+A_5)]\ ,$$
$$B_1=A_2A_3(B_{11}A_4A_5+B_{44}A_1A_6)-A_4A_5(A_1+A_6)(B_{12}A_3+2B_{13}A_2)-A_1A_6(A_4+A_5)(2B_{24}A_3+B_{34}A_2)\ ,$$
$$D_2=(A_1A_4+A_5A_6)(B_{12}B_{34}+4B_{13}B_{24}+B_{14}B_{23})+2(A_1A_5+A_4A_6)(B_{12}B_{34}+2B_{13}B_{24})$$
$$-A_2(A_1+A_6)(2B_{13}B_{44}+B_{14}B_{34})-A_3(A_4+A_5)(2B_{11}B_{24}+B_{12}B_{14})+A_1A_6(4B_{24}B_{34}-B_{23}B_{44})$$
$$+A_4A_5(4B_{12}B_{13}-B_{11}B_{23})+A_2A_3(B_{11}B_{44}+B_{14}^2)\ .$$

Diagram G:

$$B_{12}=z_7z_{348},\quad B_{13}=z_7z_8,\quad B_{23}=z_8z_{167}\ ,$$
$$B_{11}=z_{348}z_{257}+z_{34}z_8,\quad B_{22}=z_{348}z_{167},\quad B_{33}=z_{167}z_{258}+z_{16}z_7\ ,$$
$$U=z_{16}B_{11}+z_{25}B_{12}+z_{34}B_{13}\ ,$$
$$A_1=(z_4B_{13}+z_5B_{12}+z_6B_{11})/U,\quad A_2=(z_4B_{23}+z_5B_{22}+z_6B_{12})/U\ ,$$
$$A_3=(z_4B_{33}+z_5B_{23}+z_6B_{13})/U,\quad A_4=A_3-1,\quad A_5=A_2-1\ ,$$
$$A_6=A_1-1\ ,$$
$$G=-z_4A_4-z_5A_5-z_6A_6\ ,$$
$$D_0=-2(A_1+A_6)(A_3+A_4)+(A_2+A_5)(A_1+A_3+A_4+A_6)-A_1A_6-A_2A_5-A_3A_4\ ,$$
$$B_0=(A_2+A_5)[A_1A_6(A_3+A_4)+A_3A_4(A_1+A_6)]-2A_2A_5(A_1+A_6)(A_3+A_4)\ ,$$
$$C_0=-A_1A_2A_3A_4A_5A_6\ ,$$
$$D_1=B_{11}[(A_2+A_5)(A_3+A_4)-A_2A_5-A_3A_4]+B_{22}[4(A_1+A_6)(A_3+A_4)-A_1A_6-A_3A_4]$$
$$+B_{33}[(A_1+A_6)(A_2+A_5)-A_1A_6-A_2A_5]$$
$$+B_{12}[A_1A_2-4A_3A_4+A_5A_6+2(A_2+A_5)(A_3+A_4)-4(A_1+A_6)(A_3+A_4)]+B_{13}(A_1A_4+A_3A_6)$$
$$+B_{23}[A_2A_3-4A_1A_6+A_4A_5+2(A_2+A_5)(A_1+A_6)-4(A_1+A_6)(A_3+A_4)]\ ,$$
$$B_1=B_{12}A_3A_4[A_1(A_2+2A_5)+A_6(2A_2+A_5)]-B_{13}A_2A_5(A_1A_3+A_4A_6)$$
$$+B_{23}A_1A_6[A_2(A_3+2A_4)+A_5(2A_3+A_4)]\ ,$$
$$D_2=(A_1A_2+A_5A_6)(B_{12}B_{33}+2B_{13}B_{23})+(A_1A_3+A_4A_6)(B_{13}B_{22}-4B_{12}B_{23})$$
$$+(A_2A_3+A_4A_5)(B_{11}B_{23}+2B_{12}B_{13})-A_3A_4(B_{11}B_{22}+5B_{12}^2)$$
$$-A_1A_6(B_{22}B_{33}+5B_{23}^2)-A_2A_5(B_{11}B_{33}+B_{13}^2)-6B_{12}B_{23}(A_1A_4+A_3A_6)\ .$$

Diagram H:

$$B_{12}=z_{34}z_{57}+z_7z_{58},\quad B_{13}=z_7z_8-z_2z_5,\quad B_{15}=z_{34}z_{28}+z_8z_{27}\ ,$$
$$B_{23}=z_{16}z_{58}+z_8z_{57},\quad B_{25}=z_7z_8-z_{16}z_{34},\quad B_{35}=z_{16}z_{27}+z_{28}z_7\ ,$$
$$B_{11}=B_{12}+z_{34}z_{28}+z_2z_{58},\quad B_{33}=B_{23}+z_{16}z_{27}+z_2z_{57}\ ,$$
$$U=z_{16}B_{11}+z_2B_{12}+z_{34}B_{13}+z_5B_{15}\ ,$$
$$A_1=(z_4B_{13}+z_5B_{15}+z_6B_{11})/U,\quad A_2=(z_4B_{23}+z_5B_{25}+z_6B_{12})/U\ ,$$

$$A_3=(z_4B_{33}+z_5B_{35}+z_6B_{13})/U ,$$

$$A_4=A_3-1,\ A_5=A_4-A_2+A_1,\ A_6=A_1-1 ,$$

$$G=z_1A_1+z_2A_2+z_3A_3 ,$$

$$D_0=2(1+A_5)-(A_1+A_3)(1+A_2+A_5)-(A_1{}^2+4A_1A_3+A_3{}^2) ,$$

$$B_0=A_2A_5[2(A_1{}^2-A_1A_3+A_3{}^2)-A_1-A_3-1]+(A_2+A_5)(2-A_1-A_3)A_1A_3+A_2(A_1{}^2+A_3{}^2-A_1-A_3) ,$$

$$C_0=0 ,$$

$$\begin{aligned}D_1=&B_{11}[A_3(1-A_2-A_3-A_5)+A_5]+B_{33}[A_1(1-A_1-A_2-A_5)+A_5]+B_{13}(2A_1A_3-A_1-A_3)\\&+B_{12}[4A_1A_3-3(A_1+A_3)+A_5(A_3-2A_1)+A_3{}^2+2]\\&+B_{23}[4A_1A_3-3(A_1+A_3)+A_5(A_1-2A_3)+A_1{}^2+2]\\&+B_{15}[4A_1A_3-A_1+A_2-3A_3+A_2(A_3-2A_1)+A_3{}^2+1]\\&+B_{35}[4A_1A_3-3A_1+A_2-A_3+A_2(A_1-2A_3)+A_1{}^2+1]\\&+B_{25}[4A_1A_3+2(A_1+A_3)-4(A_1{}^2+A_3{}^2)+1] ,\end{aligned}$$

$$B_1=2A_2A_5[B_{11}A_3(A_3-1)+B_{33}A_1(A_1-1)+B_{13}(-2A_1A_3+A_1+A_3-1)] ,$$

$$\begin{aligned}D_2=&(2A_1A_3-A_1-A_3+1)(B_{12}B_{35}+4B_{13}B_{25}+B_{15}B_{23})\\&+(2A_1-1)[B_{13}(B_{23}A_5+B_{35}A_2)-B_{33}(B_{12}A_5+B_{15}A_2)]\\&+(2A_3-1)[B_{13}(B_{12}A_5+B_{15}A_2)-B_{11}(B_{23}A_5+B_{35}A_2)]\\&-2A_1(A_1-1)(B_{23}B_{35}+2B_{25}B_{33})-2A_3(A_3-1)(2B_{11}B_{25}+B_{12}B_{15}) .\end{aligned}$$

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[1]P. Cvitanović and T. Kinoshita, Phys. Rev. D 10, 3978 (1974).

[2]W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[3]P. Cvitanović and T. Kinoshita, Phys. Rev. D 10, 3991 (1974). For the reader's convenience the essential features of the K_S operation are outlined in the Appendix of this article.

[4]P. Cvitanović and T. Kinoshita, Phys. Rev. D 10, 4007 (1974).

[5]G. Källén and A. Sabry, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. 29, No. 17 (1955); J. Schwinger, *Particles, Sources, and Fields* (Addison Wesley, Reading, Massachusetts, 1973), Vol. II, p. 397.

[6]For brevity we use the notation $z_{12\cdots n}=z_1+z_2+\cdots+z_n$.

[7]Although $\Pi^{(4a)}$ is not gauge invariant by itself, application of the procedure (1.9) to $\Pi^{(4a)}$ gives the correct result when it is combined with the contribution of $\Pi^{(4b)}$.

[8]See formulas (3.13) and (3.14) of Ref. 4.

[9]See formulas (3.22) and (3.23) of Ref. 4.

[10]N. Nakanishi, Prog. Theor. Phys. 17, 401 (1975).

[11]A comment on notation is in order here. In (2.15) and (2.23) we write the UV-finite (but IR-divergent) parts of L_2 and B_2 as $\Delta'L_2$ and $\Delta'B_2$, respectively. We find it useful to decompose $\Delta'B_2$ further into IR-divergent and IR-finite parts. Clearly this decomposition is not unique. Taking advantage of the Ward identity which shows that $\Delta'B_2$ and $-\Delta'L_2$ have the same IR divergence, we choose to write $\Delta'B_2$ as $-\Delta'L_2+\Delta B_2$ where ΔB_2 is free from any divergence. With the insertion of a vacuum polarization loop, $\Delta'L_{2,P2}$ and $\Delta'B_{2,P2}$, which are obtained from the Feynman integrals for $\Delta'L_2$ and $\Delta'B_2$ by integration over the spectrum of virtual photon mass, become IR finite. Nevertheless we maintain the notation Δ' in order to distinguish them from $\Delta L_{2,P2}(=0)$ and $\Delta B_{2,P2}=\Delta'B_{2,P2}+\Delta'L_{2,P2}$.

[12]H. Strubbe, Comput. Phys. Commun. 8, 1 (1974); 18, 1 (1979).

[13]A comment similar to that of Ref. 7 applies here, too.

[14]The formulas (4.50) and (4.54) of Ref.4 are incorrect in the sense that they do not distinguish between δm_2, $\hat{B}_2$ and $\delta\hat{m}_2'$, $\hat{B}_2'$. This oversight does not affect the final result of the calculation in Ref. 4.

[15]By Nakanishi's identity (Ref. 10), $\delta m_2=\delta m_2'$ where [see Ref. 4, (3.18)]

$$\delta m_2=\frac{1}{2}\int(dz)\int z_{1'}dm_{1'}^2\frac{(2-A_1)}{U^2V} ,$$

$$(dz)=dz_1dz_{1'}\delta(1-z_{11'}) ,$$

and

$$\delta m_2' = \frac{1}{2}\int (dz)\int z_{1'} dm_{1'}^2 \left[-z_1\frac{\partial}{\partial z_1}\right]\frac{(2-A_1)}{U^2V}$$

$$= \frac{1}{2}\int (dz)\int z_{1'} dm_{1'}^2 \left[\frac{(2-A_1)(1-A_1{}^2)}{U^2V^2} + \frac{(4-3A_1)B_{11}}{U^3V}\right].$$

δm_2 is UV divergent which, according to our intermediate renormalization scheme, is written also as $\delta\hat{m}_2$. In $\delta m_2'$ defined above, on the other hand, the UV divergence is confined to the second term. Thus we write $\delta m_2' = \Delta' \delta m_2' + \delta\hat{m}_2'$, where $\delta\hat{m}_2'$ is given in (3.29). A similar analysis for B_2 and B_2' leads to (3.30). In the last term of each line of (3.26) it is the parametric forms of $\delta m_2'$ and B_2' that actually appear. They are replaced by δm_2 and B_2 using Nakanishi's identity. This, however, cannot be done for $\delta\hat{m}_2'$ and $\hat{B}_2'$ in (3.25).

[16]T. Kinoshita and W. B. Lindquist, following paper, Phys. Rev. D 27, 867 (1983).