Light-cone pathology of theories with noncausal propagation

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Couplings of fields with spin values 0, $\frac{1}{2}$, and 1 are examined in light-front coordinates. It is found that all theories which have noncausal modes of propagation in ordinary spacetime suffer from loss of constraints.

I. INTRODUCTION

The first example of noncausal propagation in a relativistic field theory was discovered by Velo and Zwanziger[1] in the case of the minimally coupled spin-$\frac{3}{2}$ Rarita-Schwinger field. This theory, though consistent in the free-field case, was already known to be plagued with indefinite-metric trouble when second quantized with minimal electromagnetic coupling.[2] Subsequently, other couplings of the Rarita-Schwinger field have been considered[3-6] and in each case both inconsistencies, namely the existence of noncausal modes at the classical level and the appearance of indefinite metric in the second-quantized theory, have been encountered. Indeed, the appearance of both difficulties is indicated by the occurrence of the same expression involving external fields, and it is believed that the two problems go hand in hand. It is also believed that all Lagrangian theories of fields with spin greater than 1 suffer from these defects in the presence of interactions[7] as an example of which the minimally coupled spin-2 theory has been shown to have noncausal modes.[8] What is even more interesting perhaps is the fact that noncausal modes have also been shown to exist for various couplings involving fields of spin values 0, $\frac{1}{2}$, and 1.[9-12]

Recently, various couplings of the Rarita-Schwinger spin-$\frac{3}{2}$ field have been studied in terms of light-front coordinates.[13,14] It was found that unless the external fields are subjected to noncovariant conditions, the interacting field suffers from a loss of constraints, thereby exhibiting more degrees of freedom than are possessed by the free field. Subsequently, the same "pathology" was shown to exist for the minimally coupled massive spin-2 field.[15]

It is natural to ask whether this behavior on the light cone is another manifestation of the Velo-Zwanziger pathology. To this end we examine various couplings involving "lower-spin" fields (i.e., fields with spin values 0, $\frac{1}{2}$, and 1). It is found that all theories which have noncausal modes in ordinary spacetime indeed suffer from a loss of constraints on the light cone.

The next two sections deal with nonderivative, bilinear, external field couplings of the spin-0 and spin-1 fields, respectively. Section IV deals with couplings of a spin-1 field to the Dirac field.

II. THE SPIN-0 FIELD

The spin-0 field is described by a five-component object comprising a scalar ϕ and a vector ϕ_μ. The free Lagrangian

$$\mathscr{L}_0=\phi_\mu^*\phi^\mu-m^2\phi^*\phi+\phi^*\partial_\mu\phi^\mu-\phi^{\mu*}\partial_\mu\phi$$

leads to the equations[14]

$$\phi_\mu-\partial_\mu\phi=0\ , \tag{1a}$$

$$\partial^\mu\phi_\mu-m^2\phi\equiv\partial_i\phi_i-\partial_+\phi_--\partial_-\phi_+-m^2\phi=0\ . \tag{1b}$$

Equation (1a) for $\mu=+$ and Eq. (1b) are equations of motion for ϕ and ϕ_-, respectively, whereas Eqs. (1a) for $\mu=1,2,\ldots,$ are constraints which determine ϕ and ϕ_- in terms of ϕ. One of the constraints, namely,

$$\phi_--\partial_-\phi=0\ ,$$

connects only the dynamical components ϕ and ϕ_-, and hence, in conjunction with the equations of motion leads to a secondary constraint, namely,

$$2\partial_-\phi_++m^2\phi-\partial_i\phi_i=0\ . \tag{2}$$

Equation (2) serves to determine the remaining component ϕ_+. Thus the free scalar field on the light front has only one degree of freedom.

The most general Lorentz-invariant interaction with external fields, which is bilinear in ϕ and ϕ_μ, is given by

$$\mathscr{L}_i=\alpha\phi^*\phi+\beta\phi^*_\mu\phi^\mu+T^{\mu\nu}\phi^*_\mu\phi_\nu+iA_\mu(\phi^{*\mu}\phi-\phi^*\phi^\mu)+B_\mu(\phi^{\mu *}\phi+\phi^*\phi^\mu)\ .$$

Here α,β are scalar fields, A_μ,B_μ are vector fields, and $T^{\mu\nu}=G^{\mu\nu}+iH^{\mu\nu}$ is a traceless second-rank tensor field with symmetric and antisymmetric parts $G_{\mu\nu}$ and $H_{\mu\nu}$, respectively. The field equations are

$$(1+\beta)\phi_\mu+T_{\mu\nu}\phi^\nu-D^{(-)}_\mu\phi=0 \tag{3a}$$

and

$$D^{(+)}_\mu\phi^\mu-(m^2-\alpha)\phi=0\ , \tag{3b}$$

where

$$D^{(\pm)}_\mu=(\partial_\mu-iA_\mu\pm B_\mu)\ .$$

Once again ϕ and ϕ_- are seen to be the dynamical components. When the $T_{\mu\nu}$ term is present, however, the three equations of constraints, namely, Eqs. (3a) for $\mu\neq +$, are seen to contain ϕ_+ as well as ϕ_i. The constraints thus define these components in terms of ϕ and ϕ_-, the latter being left as an independent degree of freedom. It is thus seen that the tensor couplings lead to an increase in the degrees of freedom. In ordinary spacetime these same couplings lead to noncausal behavior.[9,11] The remaining couplings (which are known to be causal[9,11]) remain consistent on the light front as well.

III. THE SPIN-1 FIELD—INTERACTIONS WITH EXTERNAL FIELDS

The spin-1 field can be described by the first-order Proca formalism involving an antisymmetric tensor $\psi_{\mu\nu}$ and a vector ϕ_μ. The free Lagrangian

$$\mathscr{L}_0=\tfrac{1}{2}\psi^*_{\mu\nu}\psi^{\mu\nu}-m^2\phi^{*\alpha}\phi_\alpha-\psi^*_{\mu\nu}\partial^\mu\phi^\nu+\phi^{*\nu}\partial^\mu\psi_{\mu\nu}$$

leads to the equations

$$\psi_{\mu\nu}=\partial_\mu\phi_\nu-\partial_\nu\phi_\mu \tag{4}$$

and

$$\partial^\mu\psi_{\mu\alpha}-m^2\phi_\alpha=0\ . \tag{5}$$

It is useful to note at this point that (at least in the massive case with nonderivative couplings) the Takahashi-Palmer[16] spin-1 field is completely equivalent to the Proca field. The former is obtainable from the above Lagrangian via the substitution

$$\psi_{\mu\nu}=-\frac{m}{2}\epsilon_{\mu\nu\alpha\beta}\tilde{\psi}^{\alpha\beta}$$

and the subsequent elimination of ϕ_α.

Returning to the free-field equations (4) and (5), one notices that six of the ten components, namely, ϕ_i, ϕ_-, ψ_{-i}, and ψ_{-+} are dynamical, i.e., they satisfy equations of motion, these being Eq. (4) with $\mu=+$, and Eq. (5) with $\alpha\neq -$. The remaining choices of indices in Eqs. (4) and (5) yield constraints. These read

$$\psi_{12}=\partial_1\phi_2-\partial_2\phi_1\ , \tag{6}$$

$$\psi_{-i}=\partial_-\phi_i-\partial_i\phi_-\ , \tag{7}$$

and

$$\partial_i\psi_{-i}+\partial_-\psi_{+-}+m^2\phi_-=0\ . \tag{8}$$

Equation (6) serves to determine the dependent component ψ_{12}; Eqs. (7) and (8) on the other hand involve only dynamical variables, thereby reducing the number of independent components to three, as desired. Used in conjunction with the equations of motion, the x^+ derivatives of Eqs. (7) and (8) result in secondary constraints which determine the remaining components ψ_{+i} and ϕ_+.

Next let us consider the most general (nonderivative) bilinear coupling to external fields, via the Lagrangian

$$\mathscr{L}_I=\phi^{*\alpha}\phi^\beta T_{\alpha\beta}+i\psi^{*\mu\nu}\phi^\alpha C_{\alpha\mu\nu}-i\psi^{\mu\nu}\phi^{*\alpha}C^*_{\alpha\mu\nu}+\tfrac{1}{2}\psi^{*\mu\nu}\psi^{\alpha\beta}f_{\mu\nu\alpha\beta}\ ,$$

where the field $C_{\alpha\mu\nu}$ is antisymmetric in μ,ν, and $f_{\mu\nu\alpha\beta}$ is antisymmetric in each of the pairs of indices μ,ν and α,β. These external fields are decomposed into their SL(2,C) irreducible components as follows:

$$T_{\alpha\beta}=S^{(1)}_{\alpha\beta}+ia^{(1)}_{\alpha\beta}+g_{\alpha\beta}X\ , \tag{9}$$

$$C_{\alpha\mu\nu}=(\tilde{C}_{\alpha\mu\nu}+i\tilde{d}_{\alpha\mu\nu})+\epsilon_{\mu\nu\alpha\beta}(U^\beta+iV^\beta)+\tfrac{1}{2}g_{\mu\alpha}(A_\nu+iB_\nu)-\tfrac{1}{2}g_{\nu\alpha}(A_\mu+iB_\mu)\ , \tag{10}$$

$$f_{\mu\nu\alpha\beta}=\tilde{f}_{\mu\nu\alpha\beta}+\tfrac{1}{2}(g_{\mu\alpha}g_{\nu\beta}-g_{\nu\alpha}g_{\mu\beta})Y+\epsilon_{\mu\nu\alpha\beta}Z+g_{\mu\alpha}(S^{(2)}_{\nu\beta}+ia^{(2)}_{\nu\beta})-g_{\nu\alpha}(S^{(2)}_{\mu\beta}+ia^{(2)}_{\nu\beta})$$
$$+g_{\nu\beta}(S^{(2)}_{\mu\alpha}+ia^{(2)}_{\mu\alpha})-g_{\mu\beta}(S^{(2)}_{\alpha\nu}+ia^{(2)}_{\nu\alpha})+i(\epsilon_{\mu\nu\alpha\lambda}S^{(3)\lambda}_{\beta}-\epsilon_{\mu\nu\beta\lambda}S^{(3)\lambda}_{\alpha}-\epsilon_{\alpha\beta\mu\lambda}S^{(3)\lambda}_{\nu}+\epsilon_{\alpha\beta\nu\lambda}S^{(3)\lambda}_{\mu})\ . \qquad (11)$$

All the irreducible fields are arranged to be real; of these X, Y, and Z are scalars, A_μ, B_μ, U_μ, and V_μ are evidently vectors, $a^{(1,2)}$ are antisymmetric, and $S^{(1,2,3)}$ are symmetric and traceless, corresponding to the representation (1,1). Finally, $\tilde{C}_{\alpha\mu\nu}$ and $\tilde{d}_{\alpha\mu\nu}$ correspond to $(\frac{3}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})$, and $\tilde{f}_{\mu\nu\alpha\beta}$ corresponds to (2,0) + (0,2).

Once again ϕ_i, ϕ_-, ψ_{-i}, and ψ_{+-} satisfy equations of motion. Therefore, one must have three equations of constraint involving these variables alone. The primary constraints have the form

$$(1+2f_{1212})\psi_{12}-2f_{12-k}\psi_{+k}-2iC_{-12}\phi_+=(\partial_1\phi_2-\partial_2\phi_1)-2iC_{k12}\phi_k+2iC_{+12}\phi_-+2f_{12+k}\psi_{-k}\ , \qquad (12)$$

$$2(f_{-i12}\psi_{12}-f_{-i-j}\psi_{+j}-iC_{--i}\phi_+)+(\delta_{ij}-2f_{-i+j})\psi_{-j}=\partial_-\phi_i-\partial_i\phi_--2iC_{j-i}\phi_j+2iC_{+-i}\phi_-\ , \qquad (13)$$

$$2(iC^*_{--j}\psi_{+j}-iC^*_{-12}\psi_{12}-\tfrac{1}{2}T_{--}\phi_+)=\partial_-\psi_{+-}+(m^2+T_{-+})\phi_--T_{-i}\phi_i+(\partial_i-2iC^*_{-+i})\psi_{-i}\ . \qquad (14)$$

Insofar as (12) will always contain the dependent component ψ_{12}, it should be possible to remove all the dependent components from Eqs. (13) and (14). This is possible only if these components appear in the same linear combination as in Eq. (2). In view of the tensor nature of the external fields f, C, and T, this condition is seen, after some thought, to reduce to the requirement that none of the dependent components should appear in Eqs. (13) and (14), i.e., one must have

$$f_{-i12}=0=f_{-1-2}=C_{--i}=C_{-12}=T_{--}\ . \qquad (15)$$

It is immediately verified from Eqs. (9)–(11) that this condition is identically satisfied, in the case of couplings to any of the scalar fields, to the antisymmetric tensor $a^{(1)}_{\alpha\beta}$ and to the vectors A_μ and B_μ, while the remaining types of couplings lead to the violation of (15) in some Lorentz frame.

Let us compare the consistency of the light-cone theories with the causal behavior in ordinary space-time in the case of each of the couplings. The existing results for various couplings are as follows.

(i) The self-coupling of the Takahashi-Palmer field, a coupling of the type $\tilde{f}$, is noncausal.[12]

(ii) The "minimal coupling" of the Takahashi-Palmer field, as studied in Ref. 12, is easily seen to involve a term of the $a^{(2)}$ type with $a^{(2)}_{\mu\nu}\propto F_{\mu\nu}$, and is noncausal. When the $a^{(2)}$ term is subtracted, one gets a theory equivalent to the minimally coupled Proca field, which is indeed causal.

(iii) Coupling to a symmetric tensor $S^{(1)}$ is found to be noncausal in Ref. 10.

(iv) The anomalous "quadrupole moment" coupling studied in Ref. 10 involves terms of the types $\tilde{C}$ and $S^{(1)}$ and is noncausal.

(v) The minimal coupling as well as the Pauli moment terms of the type $a^{(1)}$ are known to be causal (cf. Ref. 10).

We see that one again, theories with noncausal modes suffer from a loss of constraints on the light cone, while those which remain causal do not. Thus one would suspect that couplings involving terms of the types $\tilde{d}_{\alpha\mu\nu}$, $S^{(2,3)}_{\mu\nu}$, U_α, or V_α lead to noncausality whereas those involving B_μ or any of the scalars X, Y, and Z remain causal. Rather straightforward calculations along the lines of Refs. 10 and 12 indeed verify this conjecture.

IV. SPIN-1 FIELD COUPLED TO THE DIRAC FIELD

In the previous section, all the fields other than the Proca field were treated as external. The consistency situation may change if these are also required to satisfy their own field equations. It is clear that if a certain coupling leads to inconsistencies in the external-field case, the situation will not be remedied by including the dynamics of these hitherto external fields. On the other hand, couplings which are consistent in the external-field case may become inconsistent when additional equations are imposed.

In fact, several interactions of the Proca field ϕ_μ with a Dirac field ψ were examined by Shamaly and Capri[12] from the viewpoint of causal propagation. They found the interactions given by $g\phi_\mu\bar{\psi}\gamma^\mu\psi$, $ig\phi_\mu\bar{\psi}\gamma_5\gamma^\mu\psi$, $g\phi_\mu\phi^\mu\bar{\psi}\psi$, $ig\phi_\mu\phi^\mu\bar{\psi}\gamma_5\psi$, $g\bar{\psi}\gamma^\mu\psi\phi_\nu\partial_\mu\phi^\nu$, and $g\bar{\psi}\sigma^{\mu\nu}\psi(\partial_\mu\phi_\nu)$ to be causal, and the ones corresponding to $ig\bar{\psi}\gamma^\mu\gamma_5\psi(\partial_\mu\phi_\nu\phi^\nu)$, $g\bar{\psi}\psi(\partial_\mu\phi^\mu)$, and $ig\bar{\psi}\gamma_5\psi(\partial_\mu\phi^\mu)$ to be noncausal.

We now analyze these interactions on the light cone. The situation is seen to be as follows. The Dirac equation

$$(-i\gamma^\mu\partial_\mu+M)\psi=\frac{\partial\mathscr{L}_I}{\partial\bar{\psi}}$$

is decomposed into the upper and lower components[14] by multiplying by $\frac{1}{2}\gamma^-$ and $\frac{1}{2}\gamma^+$, respectively, resulting in

$$-i\partial_+\psi^{(+)}+\tfrac{1}{2}\gamma^-(-i\gamma_i\partial_i+M)\psi^{(-)}=\tfrac{1}{2}\gamma^-\frac{\partial\mathscr{L}_I}{\partial\bar{\psi}}\,, \tag{16}$$

$$-i\partial_-\psi^{(-)}+\tfrac{1}{2}\gamma^+(-i\gamma_i\partial_i+M)\psi^{(+)}=\tfrac{1}{2}\gamma^+\frac{\partial\mathscr{L}_I}{\partial\bar{\psi}}\,. \tag{17}$$

Thus the two-component object $\psi^{(+)}$ represents the dynamical part of the Dirac field, while $\psi^{(-)}$ involves the two dependent components and is defined by the constraint (17). As for the Proca field, the six dynamical components are the same as in the free-field case, so one again needs three constraints involving these alone. Equations (6) and (7) are unchanged; however, Eq. (8) becomes

$$\partial_i\psi_{-i}+\partial_-\psi_{+-}+m^2\phi_-=\frac{\partial\mathscr{L}_I}{\partial\phi^-}\,. \tag{18}$$

For consistency, this must involve only the dynamical components. This can happen only if either $\partial\mathscr{L}_I/\partial\phi^-$ is free of all dependent components, including $\psi^{(-)}$ or if the elimination of $\psi^{(-)}$ using (17) results in an equation involving only the dynamical components (including $\psi^{(+)}$). The situation is easily examined for each of the couplings listed above; the details will be omitted. One finds that in each case (except for the interactions $g\phi_\mu\phi^\mu\bar{\psi}\psi$ and $ig\phi_\mu\phi^\mu\bar{\psi}\gamma_5\psi$) the correspondence between loss of causality and loss of constraints on the light cone persists. In the case of the remaining two interactions, one finds that Eq. (18), upon the elimination of $\psi^{(-)}$, contains the dependent component ϕ_+, and therefore the Proca field has four degrees of freedom. These interactions, however, are "singular" in ordinary spacetime. To see this, one notes that the effect of, e.g., the first of these interactions is to make the substitution

$$m^2\rightarrow m^2-g\bar{\psi}\psi\equiv M^2(x)$$

in the equations for the Proca field. At the spacetime points where $M^2(x)=0$ the Proca field becomes effectively massless, and there are no equations to determine the longitudinal components of ϕ_k or ϕ_0.

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