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Geodesic coordinates in the de Sitter universe

Ronald Gautreau
Physics Department, New Jersey Institute of Technology, Newark, New Jersey 07102
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We analyze the de Sitter universe using geodesic clock reference systems. Our starting point is the de Sitter universe described with a static metric form where curvature coordinates are used. This curvature metric form contains a horizon across which timelike and null trajectories cannot be followed. We show how to use new geodesic reference systems that will allow trajectories to be followed across the horizon. These geodesic reference systems are analogs to Eddington-Finkelstein and Kruskal-Novikov coordinate systems that have previously been used to analyze the Schwarzschild field. The resulting comoving synchronous coordinate systems result in metrics for the de Sitter universe that are markedly different in form from the standard Robertson-Walker isotropic metric form. This leads to questions of whether it is valid to describe the de Sitter universe with isotropic Robertson-Walker coordinates.

I. INTRODUCTION

In discussing cosmological solutions to the Einstein field equations of general relativity, a usual starting point is to represent a particular universe with a metric written in the Robertson-Walker isotropic form

$$ds^2(r,t)=[1+K(r/2b)^2]^{-2}\times e^{2h(t)}(dr^2+r^2d\Omega^2)-dt^2\ ,$$
$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2\ , \tag{1.1}$$

where $K=-1,0,+1$ defines the intrinsic curvature of the three-dimensional subspace $t=$const, and the constant b is a measure of the radius of curvature of this subspace. The metric (1.1) is of an "isotropic" form in that the spatial part has the form

$$d\sigma^2=dr^2+r^2d\Omega^2\ . \tag{1.2}$$

In this paper we analyze the de Sitter universe which has zero curvature ($K=0$) and where the function $h(t)$ in (1.1) is given by

$$h(t)=t/R_0\ ,\quad R_0=(3/\Lambda)^{1/2}\ , \tag{1.3}$$

where Λ is the cosmological constant in Einstein's vacuum field equations $G^\mu_\nu=\Lambda\delta^\mu_\nu$. We will assume Λ to be positive. Along with the isotropic metric form (1.1), we will also be looking at the de Sitter universe from the point of view of curvature coordinates, where a metric has the form

$$ds^2(R,T)=A(R,T)dR^2+R^2d\Omega^2-B(R,T)dT^2\ . \tag{1.4}$$

Starting from the de Sitter universe expressed in the curvature form (1.4), where A and B are independent of the time coordinate T, we will show how geodesic reference systems can be used to track previously unfollowable timelike and null trajectories across an existing horizon. For the most part, we will take procedures we have previously used for analyzing a Schwarzschild field[1-5] and apply them to the present de Sitter universe. Among other things, when we are done serious questions will be raised about the validity of using (1.1) for the description of a de Sitter universe.

II. FORMS OF THE DE SITTER UNIVERSE

The isotropic metric form of the de Sitter universe is

$$ds^2(r,t)=e^{2t/R_0}(dr^2+r^2d\Omega^2)-dt^2\ . \tag{2.1}$$

We will cast this into a curvature metric form in two steps. We first replace the isotropic radial coordinate r with a curvature radial coordinate R defined by

$$R=re^{t/R_0} \tag{2.2}$$

to transform (2.1) into an intermediate nondiagonal form

$$ds^2(R,t)=[dR-(R/R_0)dt]^2+R^2d\Omega^2-dt^2\ . \tag{2.3}$$

We next diagonalize (2.3) by replacing the isotropic time coordinate t with a curvature time coordinate T defined by

$$T=t+\int_0^R \frac{y/R_0}{[1-(y/R_0)^2]}dy$$
$$=t-(R_0/2)\ln[1-(R/R_0)^2] \quad (2.4)$$

to give us the final curvature metric form of the de Sitter universe as

$$ds^2(R,T)=\frac{dR^2}{1-(R/R_0)^2}+R^2d\Omega^2 -[1-(R/R_0)^2]dT^2 . \quad (2.5)$$

We note that (2.5) has a coordinate singularity at $R=R_0$ which did not exist in the previous metric forms (2.1) or (2.3). Related to this is the singular behavior at $R=R_0$ in the transformation (2.4) that replaces t with T.

We know from Birkhoff's theorem that any spherically symmetric field *in vacuo* is static, which means that it must be possible to find static coordinates that explicitly exhibit this intrinsic static property. Curvature coordinates are these coordinates, for it is seen that the metric coefficients in (2.5) are independent of T. In contrast, the metric coefficients in the isotropic form (2.1) depend upon t, so it is not evident from (2.1) that the de Sitter universe is static.

The de Sitter curvature form (2.5) is a special case of the most general solution to the Einstein vacuum field equations in curvature coordinates, which is

$$ds^2(R,T)=\frac{dR^2}{1-2M/R-(R/R_0)^2}+R^2d\Omega^2 -[1-2M/R-(R/R_0)^2]dT^2 , \quad (2.6)$$

which can be regarded as a combination of a Schwarzschild field in a de Sitter universe. The general curvature form (2.6) cannot be changed into an isotropic form with $g_{44}=-1$, because with the presence of M the field is not homogeneous.

III. TIMELIKE GEODESICS

The timelike solutions to the radial geodesic equations in the (R,T) curvature coordinates of (2.5) are

$$dR/d\tau=m[(R/R_0)^2+k^2-1]^{1/2} , \quad (3.1a)$$
$$dT/d\tau=k[1-(R/R_0)^2]^{-1} , \quad (3.1b)$$

where $m=+1$ or -1 depending, respectively, on whether the particle is moving in the sense of increasing or decreasing R, and k is a parameter representing the energy per unit mass of the geodesic particle. A particle with $k<1$ will have a turning radius. Its motion will be such that it will come in from infinity, stop momentarily at $R_i>0$ given by

$$R_i/R_0=(1-k^2)^{1/2} , \quad (3.2)$$

and then head back to infinity with ever-increasing speed relative to $R=$const points. Thus, a particle released from rest at $R_i>0$ will move toward increasing R. The value $k=1$ corresponds to a transition value. A particle with $k=1$ released at $R=0$ will stay there for all succeeding time, because the trajectory $R=0$ is a timelike geodesic. Particles with $k>1$ correspond to particles that are "shot out" or are "caught" at $R=0$ with a velocity v_0 given by

$$k=(1-v_0{}^2)^{-1/2} , \quad (3.3)$$

as measured by an observer fixed at the timelike geodesic $R=0$.

Upon combining (3.1a) and (3.1b) and integrating, we obtain

$$T-T_c=mk\int_{R_c}^R \frac{dy}{[(y/R_0)^2+k^2-1]^{1/2}[1-(y/R_0)^2]} , \quad (3.4)$$

which shows that a particle will take an infinite amount of the coordinate T to asymptotically approach R_0. However, from (3.1a) by itself we obtain

$$\tau-\tau_c=m\int_{R_c}^R \frac{dy}{[(y/R_0)^2+k^2-1]^{1/2}} , \quad (3.5)$$

which shows that the proper time τ measured by a clock coincident with the geodesic particle records a finite amount of ticks as it reaches and crosses R_0.

Thus, if one plots world lines on an (R,T) space-time diagram, one finds that neither timelike nor null trajectories can be followed across $R=R_0$

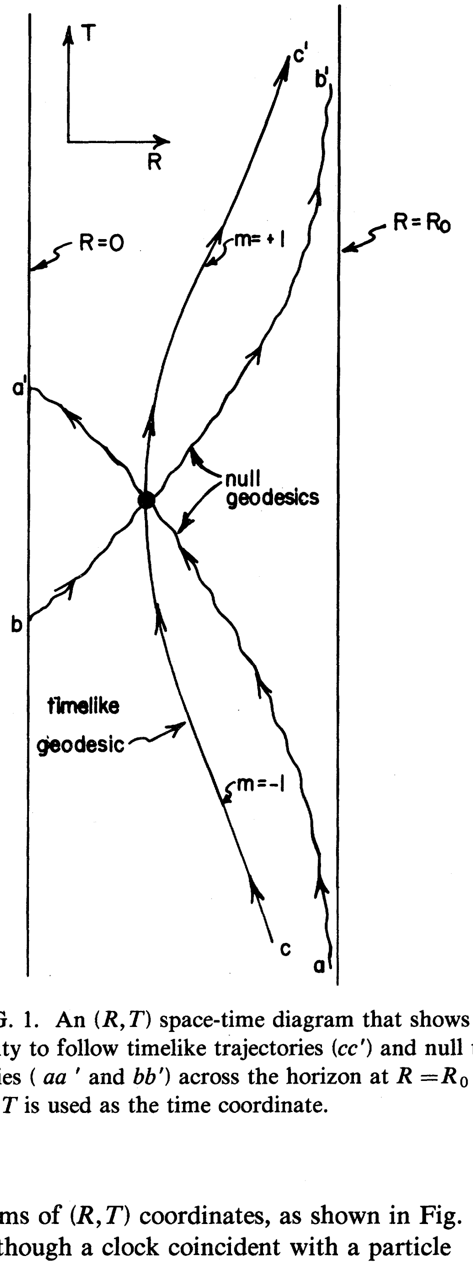

FIG. 1. An (R,T) space-time diagram that shows the inability to follow timelike trajectories (cc') and null trajectories (aa' and bb') across the horizon at $R=R_0$ when T is used as the time coordinate.

in terms of (R,T) coordinates, as shown in Fig. 1, even though a clock coincident with a particle records a finite time as the particle's trajectory asymptotically approaches R_0. This behavior is exactly the same as the inability to follow particles and light signals across the Schwarzschild radius $R=2M$ in a Schwarzschild field when curvature coordinates are used. We shall exploit this similarity and apply methods we have previously used with the Schwarzschild field[1–5] to the de Sitter universe.

We note, though, an important difference between a Schwarzschild field and the de Sitter universe. In a Schwarzschild field, the location $R=0$ has an intrinsic significance in that it is, in an absolute sense, the single center of spherical symmetry. In a de Sitter universe, however, there is nothing special or intrinsic about the point $R=0$. We could equally well refer the universe to any other geodesic observer besides the one located at $R=0$. This other observer, located at some $R'=0$, that is, moving relative to $R=0$, would also find the universe describable by a metric of the form (2.5), and would have a horizon at $R'=R_0$.

IV. THE BREAKDOWN OF THE T COORDINATE

We can account for the coordinate singularity at $R=R_0$ with the following argument, which is an extension of one we have previously used to explain the coordinate singularity at $R=2M$ in the Schwarzschild field.[2–5]

In the curvature form (2.5) of the de Sitter universe, the trajectory $R=0$ is a timelike geodesic, and a particle placed at rest at $R=0$ will remain there forever. Consequently, if we wish to remove a particle from $R=0$ we must necessarily give it some energy, i.e., we must throw it out with a velocity v_0 given by (3.3). A trajectory $R=\text{const}>0$ is not a timelike geodesic, and if we place a particle at such a point and release it, it will move to larger and larger values of R with increasing speed relative to $R=\text{const}$ points. Conversely, if we wish to keep a particle at an $R=\text{const}$ point, we must necessarily exert forces on it.

Consider now a geodesic observer with a local inertial reference system moving radially outward in the de Sitter universe. As this observer moves past $R=\text{const}$ points, he finds the velocities of these points to increase continuously as R gets larger and larger. Eventually, he will measure a particular $R=\text{const}$ point to move with the speed of light $c=1$. This is the radius $R=R_0$, which, because it is a null line, will be measured by any inertial observer to have the speed c, independently of the energy of the inertial observer. Correspondingly, points $R=\text{const}>R_0$ will be measured by any inertial observer to move faster than c, i.e., these trajectories will be spacelike.

Now, the reference system associated with the curvature time coordinate T is composed of clocks located at fixed values of R, appropriately synchronized and with the coordinate time lapse ΔT scaled to the actual clock reading time lapse $\Delta\tau_R$

according to

$$\Delta\tau_R = [1-(R/R_0)^2]^{1/2}\Delta T \ . \tag{4.1}$$

But, from the reasoning given in the previous paragraph, we cannot have a physical object such as a clock located at $R=R_0$ or $R=\text{const}>R_0$, because such trajectories are, respectively, null and spacelike. This means that the rationale of a reference system composed of clocks at fixed values of R, from which the coordinate T is determined, breaks down for $R>R_0$, because there is no real physical way to measure T there. This breakdown shows up in the coordinate singularity in the metric form (2.5) at $R=R_0$ and the imaginary relationship between $\Delta\tau_R$ and ΔT in (4.1) for $R>R_0$.

V. A CLOCK-FACTORY GEODESIC CLOCK REFERENCE SYSTEM

Since the curvature time coordinate T breaks down for $R\geq R_0$ because we cannot have clocks fixed at $R=\text{const}\geq R_0$ points, we must seek some other means to measure time in this region if we want to follow trajectories to $R>R_0$. One way of proceeding is to replace T with a new time measurement based upon times recorded by radially moving geodesic clocks. We are guaranteed that this approach will succeed, because we know from (3.5) that the proper time measured by a geodesic clock stays finite as the clock crosses R_0. An important extra advantage is that we have a good physical feeling for the meaning of the time coordinate we are working with.

We must first specify the way we are going to arrange our geodesic clocks and the synchronization procedure we will employ, i.e., we must first describe our reference system. In this section we will use what might be called a "clock-factory" reference system. In Sec. VII we will describe a different "Novikov-type" reference system.

The reference system used here is based upon a clock factory located at some fixed value of R that sends out geodesic clocks synchronized so that they read the same time as a master clock fixed in the clock factory. We then assign to each event the time reading τ on the particular geodesic clock that happens to be coincident with the event. We have previously used such a clock-factory reference system to analyze the Schwarzschild field.[1-5] We could locate the clock factory at any $R<R_0$, but the simplest and most convenient choice is to put the clock factory at $R=0$. The generalization to other values of R is straightforward.

In order to get the clocks out of the clock factory, we must necessarily throw them out with some velocity v_0, or equivalently with some energy parameter $k>1$ given by (3.3). Thus at the outset we cannot have $k=1$, for this would correspond to clocks staying forever at $R=0$. As will be seen, this will prove important later on. We could equally well consider the symmetric situation where the clocks are caught at $R=0$ with some energy parameter $k>1$. We account for both situations with a constant m in our equations, where $m=+1$ or -1 stands for a reference system where the geodesic clocks are moving in the sense of increasing or decreasing R, respectively. For brevity, however, we will talk only about clocks moving in the sense of increasing R.

Consider a geodesic clock shot out from $R=0$ at some time $T=T_0$. To account for the time dilation between the master clock at $R=0$ that measures T_0 and the moving geodesic clock, we synchronize the geodesic clock by setting its time of release τ_0 to be

$$\tau_0=(1-v_0{}^2)^{-1/2}T_0=kT_0 \ . \tag{5.1}$$

Upon setting the lower limit $R_c=0$ in (3.4) and (3.5), and using (5.1) to eliminate reference to the starting time T_0 of each geodesic clock, we obtain

$$\tau=kT-m\int_0^R \frac{[(y/R_0)^2+k^2-1]^{1/2}}{1-(y/R_0)^2}dy \ . \tag{5.2}$$

This is the desired transformation that replaces T with τ at each event. It is singular at $R=R_0$.

In terms of τ the metric (2.5) assumes the nondiagonal form

$$ds^2(R,\tau)=k^{-2}\{dR-m[(R/R_0)^2+k^2-1]^{1/2}d\tau\}^2+R^2d\Omega^2-d\tau^2 \ , \tag{5.3}$$

which has no coordinate singularities and is regular at all finite values of R. Upon comparing (5.3) with (2.3) we see that, for $m=+1$, the two metric forms would be equal to each other if we set the constant $k=1$. As we have explained above, however, it would be physically impossible to have a reference system as we are here considering if we had $k=1$. We shall return to this point in Sec.

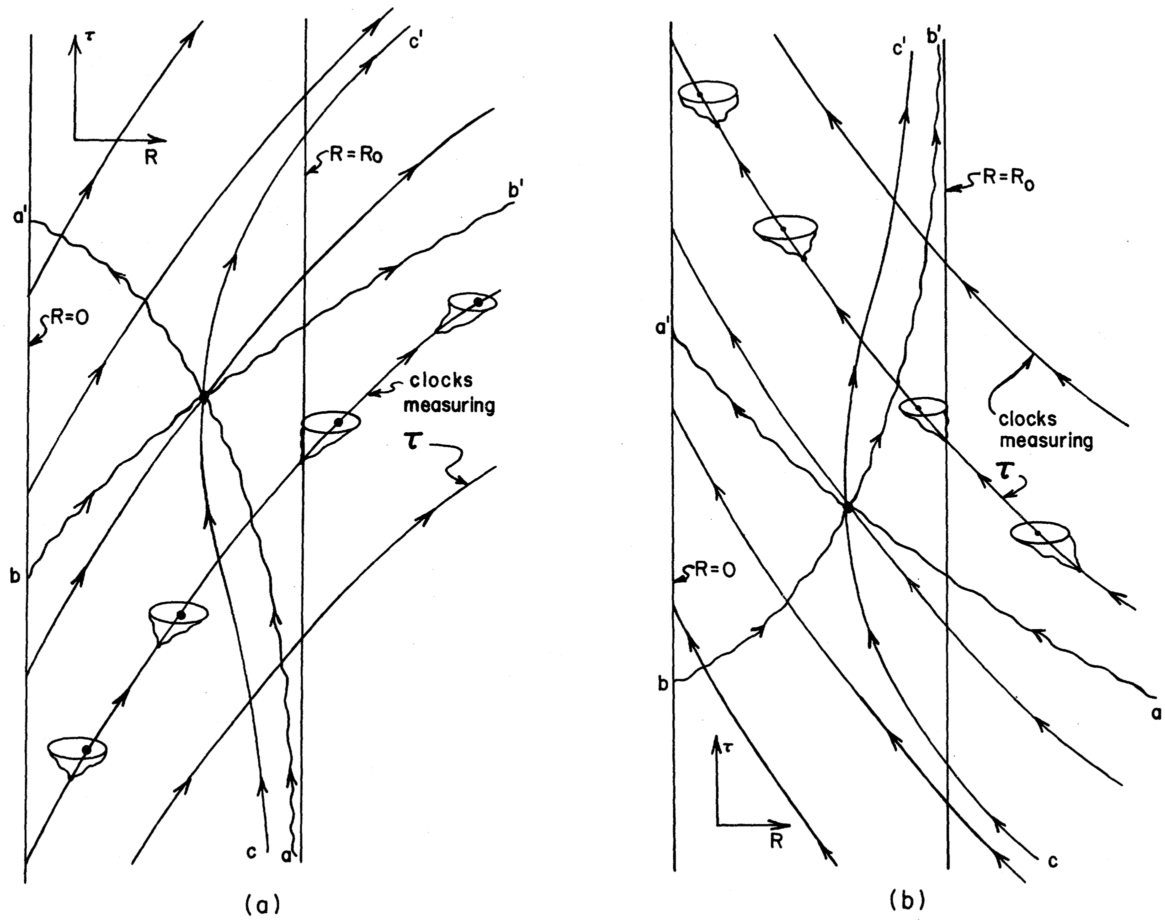

FIG. 2. (a) An (R,τ) space-time diagram where τ is measured by geodesic clocks moving in the sense of increasing R ($m=+1$). The trajectories of Fig. 1 are repeated here. It is seen that trajectories such as the null line bb' and the future part of the timelike trajectory cc' can now be followed across R_0. These trajectories are moving in the same sense as the geodesic clocks measuring τ. Trajectories such as the null line aa' and the past part of the timelike trajectory cc' cannot be followed across R_0, and are qualitatively similar to Fig. 1. These trajectories are moving in the opposite sense to the geodesic clocks measuring τ. Infinite time dilations as R_0 is approached, between the past history of a clock moving along the geodesic trajectory cc' and the oppositely moving geodesic clocks measuring τ, account for the fact that a clock along cc' records a finite number of ticks as it asymptotically approaches R_0. Also shown is the tilting of the null cones emitted along a timelike world line as R_0 is approached and crossed. The portion of the null cone that is moving with opposite sense to the geodesic clocks measuring τ undergoes the extreme tilting as R_0 is crossed. Nothing special happens to the portion of the null cone that is moving in the same sense as the geodesic clocks as R_0 is crossed. The situation upon crossing R_0 is directly analogous to what is referred to in a Schwarzschild field as a "black hole." (b) An (R,τ) space-time diagram where τ is measured by geodesic clocks moving in the sense of decreasing R ($m=-1$). The trajectories of Fig. 1 are repeated here. The discussion here is essentially the same as for (a). Again it is seen that those trajectories that have opposing motion to the geodesic clocks measuring τ are the ones that cannot be followed across R_0. The situation here is analogous to what is referred to in a Schwarzschild field as a "white hole."

VI.

Figures 2(a) and 2(b) show the world lines of Fig. 1 as they appear on an (R,τ) space-time diagram for $m=+1$ and $m=-1$, respectively. It is seen that portions of null and timelike trajectories that were not previously followable across R_0 can now be followed to $R>R_0$. However, there is a "one-way" crossing across R_0. A particle will be followable or not followable across R_0 depending, respectively, on whether the particle is moving in the same sense or opposing sense to the coordinate clocks measuring the time coordinate τ. The proper time along each nonfollowable trajectory remains finite as the trajectory asymptotically approaches R_0. This situation is similar to the one-way crossing of the Schwarzschild radius when Eddington-Finkelstein coordinates are used. Employing the same reasoning we have used previously for the Schwarzschild field,[2–5] we find that the nonfollowability occurs because as R_0 is approached there is an infinite time dilation between the geodesic clocks measuring τ and a clock on a particle moving with motion opposing the geodesic clocks.

If we define proper distance between two τ-simultaneous events as

$$\Delta L = \int_{\tau=\text{const}} ds \,, \tag{5.4}$$

we find from (5.3) that for two radially located events

$$\Delta L = \Delta R / k \,. \tag{5.5}$$

Thus, the proper distance is proportional to the coordinate separation ΔR, and is independent of which side of R_0 the two τ-simultaneous events happen to lie.

An analysis of (5.3) shows that there will be a "tilting" of radial null cones, with one part of a null cone becoming vertical as R_0 is crossed, as shown in Fig. 2. This is similar to what occurs in a Schwarzschild field when Eddington-Finkelstein coordinates are used. It is seen that the part of the null cone that undergoes the vertical tilting at R_0 is the part that had opposing motion to the geodesic clocks measuring τ before R_0 was reached. Nothing exceptional happens at R_0 to that part of the null cone moving in the same sense as the geodesic clocks measuring τ. Thus we have in the de Sitter universe a situation analogous to the "black-hole—white-hole" view of the Schwarzschild field, which we have discussed and criticized elsewhere.[2–5]

Thus far we have a situation with geodesic clocks measuring τ moving relative to $R=$const points, resulting in the nondiagonal metric form (5.3). If we wish, we can "straighten out" this picture by introducing a comoving coordinate that stays constant along the trajectory of each geodesic clock. We will try to define this new spatial coordinate as closely as possible to a spatial coordinate used in a flat-space inertial reference system. To this end, we will "stamp" each clock sent out from the clock factory with a coordinate ρ equal to the proper time elapsed at the clock factory from the time when the "origin clock" at $\rho=0$ was sent out with the reading τ_0. If we fix the resulting $\pm$ signs so that increasing ρ corresponds to increasing R, the transformation replacing R with ρ is

$$k\rho = -m\tau + \int_0^R \frac{dy}{[(y/R_0)^2+k^2-1]^{1/2}} + m\tau_0 \,, \tag{5.6a}$$

which, upon performing the integration, is

$$k\rho = -m\tau + R_0 \ln\left[\frac{R/R_0+[(R/R_0)^2+k^2-1]^{1/2}}{(k^2-1)^{1/2}}\right] + m\tau_0 \,. \tag{5.6b}$$

This results in the nondiagonal metric form (5.3) being changed into the diagonal form

$$ds^2(\rho,\tau) = [(R/R_0)^2+k^2-1]d\rho^2 + R^2 d\Omega^2 - d\tau^2 \,, \tag{5.7}$$

where we now regard R as a function of ρ and τ given by (5.6).

In contrast with the singular transformation (5.2) that replaced T with τ, the transformation (5.6) is nonsingular. In replacing T with τ we made a definite physical change in our reference system, going from clocks located at fixed R's to geodesic clocks moving relative to $R=$const points. In replacing R with ρ via (5.6), though, the reference system is not changed at all. Whether we use R or ρ as our spatial coordinate, the reference system is still composed of geodesic clocks shot out from the clock factory at $R=0$.

Figures 3(a) and 3(b) show space-time diagrams that repeat the (R,τ) space-time diagrams of Figs. 2(a) and 2(b). There is nothing intrinsically different between the corresponding space-time diagrams of Figs. 2 and 3—each is simply a continuous deformation of the other. The trajectories of the geodesic clocks measuring τ that appeared curved in Figs. 2 are vertical lines in Figs. 3. Lines $R=$const that appeared vertical in Figs. 2 are inclined parallel straight lines in Figs. 3. Also, the one-way crossing of R_0 and the tilting of the

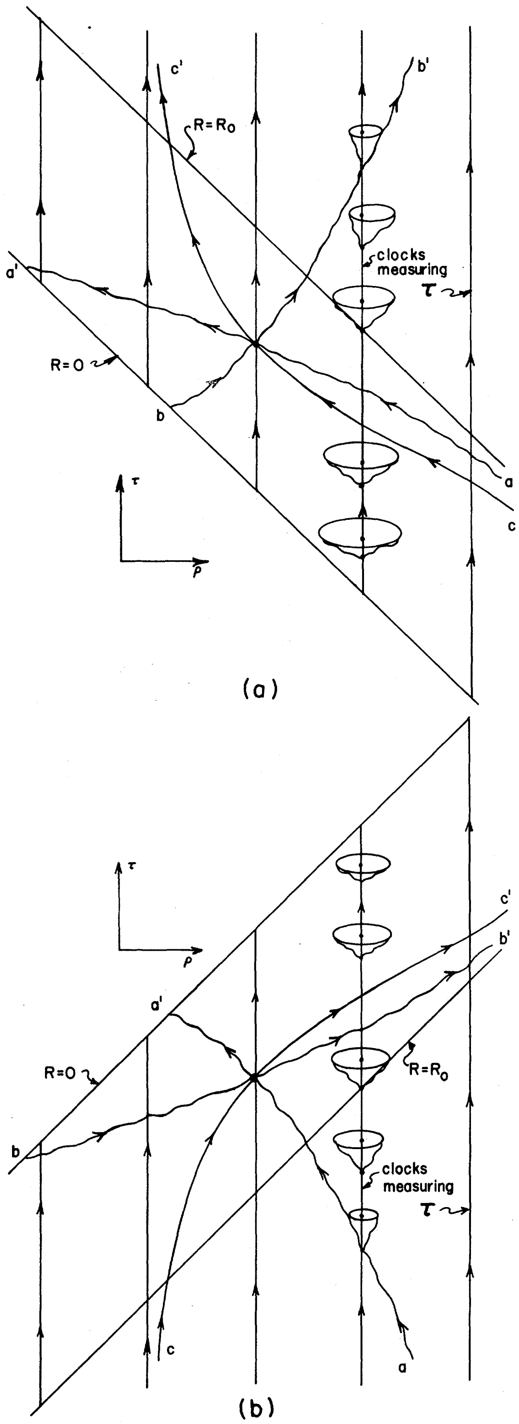

FIG. 3. (a) Repeating Fig. 2(a) in terms of comoving (ρ,τ) coordinates. This figure is simply a continuous deformation of Fig. 2(a), and there is a one-to-one correspondence between world lines on each figure. (b) Repeating Fig. 2(b) in terms of comoving (ρ,τ) coordinates. This figure is simply a continuous deformation of Fig. 2(b), and there is a one-to-one correspondence between world lines on each figure.

null cones remain in Figs. 3. Again, it is because we have not changed our reference system that there is nothing different other than a coordinate deformation between Figs. 2 and 3.

VI. CONVERTING (ρ,τ) COORDINATES TO ISOTROPIC (r,t) COORDINATES

In the diagonal metric form (5.7), where $g_{44}=-1$, the spatial coordinates (ρ,θ,ϕ) are "comoving" or "synchronous" coordinates because a constant value of these coordinates marks the trajectory of a geodesic clock that is measuring the time coordinate τ. The isotropic metric form (2.1), which is supposed to describe the same de Sitter universe and has $g_{44}=-1$, also purportedly makes use of comoving coordinates. We have carefully defined in Sec. V the physical meaning of the coordinates (ρ,θ,ϕ) used in the metric form (5.7). We here describe the procedures necessary to convert the metric form (5.7) to the isotropic form (2.1).

Factoring out the coefficient of $d\Omega^2$, we can write the metric forms (2.1) and (5.7), respectively, as

$$ds^2(r,t)=(re^{t/R_0})^2(dr^2/r^2+d\Omega^2)-dt^2 , \quad (6.1)$$

$$ds^2(\rho,\tau)=R^2\{[(R/R_0)^2+k^2-1]R^{-2}d\rho^2 +d\Omega^2\}-d\tau^2 . \quad (6.2)$$

We see that if (6.1) and (6.2) are to be equal to each other, we must have

$$\frac{dr}{r}=\frac{[(R/R_0)^2+k^2-1]^{1/2}}{R}d\rho . \quad (6.3)$$

Since R is a function of both τ and ρ, as given by (5.6), the only way (6.3) can be achieved is by setting $k=1$ so that the R terms drop out of the right-hand side. But we know from the discussions of Sec. V that if we set $k=1$ we will not have a physically realizable reference system.

Proceeding in a purely mathematical manner, however, we can formally cast (6.2) into (6.1). We rewrite the transformation equation (5.6b) as

$$R/R_0+[(R/R_0)^2+k^2-1]^{1/2}=2Ae^{k\rho/R_0}e^{m\tau/R_0} , \quad (6.4)$$

where the constant A is defined by

$$2A=(k^2-1)^{1/2}e^{-m\tau_0/R_0} . \quad (6.5)$$

For the time being, forget about (6.5) and treat A as some arbitrary constant. Now let $k=1$ in (6.4) to get

$$R/R_0 = Ae^{\rho/R_0}e^{m\tau/R_0}, \tag{6.6}$$

with, correspondingly, the metric (6.2) assuming the form

$$ds^2(\rho,\tau) = R^2(d\rho^2/R_0{}^2 + d\Omega^2) - d\tau^2, \tag{6.7a}$$

or equivalently, using (6.6),

$$ds^2(\rho,\tau) = [R_0 Ae^{\rho/R_0}e^{m\tau/R_0}]^2 \times (d\rho^2/R_0{}^2 + d\Omega^2) - d\tau^2. \tag{6.7b}$$

With $k=1$, (6.3) can be satisfied if r and ρ are related by

$$r/r_0 = e^{\rho/R_0}, \tag{6.8}$$

where r_0 is an arbitrary scaling factor. It is seen from (6.8) that for $r=0$, we have $\rho = -\infty$. From (6.6) we then have

$$R = (AR_0/r_0)re^{m\tau/R_0} \tag{6.9}$$

and (6.7) becomes

$$ds^2(r,\tau) = [(AR_0/r_0)re^{m\tau/R_0}]^2 \times (dr^2/r^2 + d\Omega^2) - d\tau^2. \tag{6.10}$$

Finally, the metric form (6.10) will be equal to the isotropic metric form (6.1) providing we take $AR_0 = r_0$, $\tau = t$, and $m = +1$.

Let us now examine what had to be done in order to make the comoving metric form (6.2) take on the standard isotropic form (6.1). By its very nature the comoving spatial coordinate ρ is not a "radial" coordinate. The trajectories $(\rho,\theta,\phi) = \text{const}$ (including the origin $\rho=0$) mark the locations of geodesic clocks that are moving away from the center of spherical symmetry at $R=0$, relative to which the angles θ and ϕ are measured, as given by the transformation equations (5.6) or (6.4). This was of no concern to us, for in Sec. V we had precisely defined ρ and made it as close as possible to a flat-space spatial coordinate, and we clearly understood its physical meaning.

In order to bring the isotropic coordinate r into the picture, we necessarily had to set the energy parameter $k=1$, which we knew gave us a questionable physical situation. In turn, this required that we deal with ambiguities in the constant A defined by (6.5) and the associated scaling factor $r_0 = AR_0$. The only way to avoid zeros or infinities in the transformations is to let $m\tau_0 \to -\infty$ in (6.5) in such a manner that A remains finite as $k \to 1$. But this seems highly artificial.

Now consider the reverse procedure of going from the isotropic form (6.1) to the curvature form (2.5). This can be accomplished by using (6.9) with $AR_0/r_0 = 1$ to replace r with R together with (5.2) with $k=1$ to replace t with T. This is, in fact, the procedure that we followed in Sec. II in going from (2.1) to (2.5). In this purely mathematical method, there is no need to bring in any mention of the comoving spatial coordinate ρ. However, simply finding a mathematical transformation between coordinates used in different metric forms does not tell the whole picture. Metric forms are, by their very nature, local descriptions. On the other hand, coordinate transformations are global relationships, and as such require additional nonlocal specifications.

VII. NOVIKOV-TYPE GEODESIC CLOCK REFERENCE SYSTEM

In this section we describe a geodesic reference system, different from the clock-factory reference system of Sec. V, that will also allow trajectories of particles and light signals to be followed across R_0. The reference system discussed here is similar to one developed by Novikov[6] to describe the Schwarzschild field, and which has been amplified by us.[2–5]

In a "Novikov-type" reference system, a swarm of geodesic clocks is fired radially from $R=\infty$ through $R=R_0$ to some minimum radius R_i, different for each clock, after which the clocks move again through R_0 back to $R=\infty$. The firing is done such that each clock reaches its minimum radius at the same coordinate time T_0. The swarm of clocks is synchronized such that each clock reads $\tau_0^* = 0$ at the instant when it reaches its minimum turning radius R_i at T_0. The time τ^* assigned to an event is then the time on the particular geodesic clock coincident with that event.

At the outset, though, problems can be expected with this reference system in certain regions of space-time. As was noted in Sec. IV, a geodesic particle cannot be momentarily at rest relative to a point $R = \text{const} > R_0$, because at this instant it would then be moving faster than the speed of light. Therefore, this type of reference system will break down as the turning radii of its geodesic clocks approach R_0. We thus expect mathematical difficulties in our derived expressions as $R_i \to R_0$.

To find the transformation that replaces T with τ^* we can use a straightforward extension of the procedure we used to obtain (5.2), with k now being given by (3.2). The result is

$$\tau^*=[1-(R_i/R_0)^2]^{1/2}(T-T_0)-m\int_{R_i}^{R}\frac{[(y/R_0)^2-(R_i/R_0)^2]^{1/2}}{1-(y/R_0)^2}dy\ , \tag{7.1}$$

where $m=+1$ or -1 corresponds, respectively, to a clock moving in the sense of increasing R ($T>T_0$) or decreasing R ($T<T_0$). In terms of τ^*, the metric (2.5) takes the nondiagonal form

$$ds^2(R,\tau^*)=[1-(R_i/R_0)^2]^{-1}\{dR-m[(R/R_0)^2-(R_i/R_0)^2]^{1/2}d\tau^*\}^2+R^2d\Omega^2-d\tau^{*2}\ , \tag{7.2}$$

where R_i is a function of R and τ^* given by

$$\tau^*=m\int_{R_i}^{R}\frac{dy}{[(y/R_0)^2-(R_i/R_0)^2]^{1/2}} \tag{7.3}$$

from which, upon performing the integration, we obtain

$$\tau^*=mR_0\ln\{R/R_i+[(R/R_i)^2-1]^{1/2}\}\ . \tag{7.4a}$$

The metric form (7.2) is mathematically similar to the metric form (5.3). The physical situations corresponding to the way the times are measured are completely different, though, and

$$k=[1-(R_i^2/R_0^2)]^{1/2}$$

in (7.2) is a variable quantity, while

$$k=(1-v_0^2)^{-1/2}$$

in (5.3) is a constant.

It is seen that the metric form (7.2) contains a coordinate singularity at those values of R and τ^* for which $R_i=R_0$. This marks the trajectory of the limiting or "last" clock in our reference system whose turning radius is $R_i=R_0$. We expected troubles along this limiting trajectory, and the coordinate singularity is a confirmation of our expectations.

If (7.4a) is solved explicitly for $R_i(R,\tau^*)$, one obtains

$$R_i=R\ \mathrm{sech}(m\tau^*/R_0)\ , \tag{7.4b}$$

which allows the metric form (7.2) to be written as

$$ds^2(R,\tau^*)=[1-(R/R_0)^2\mathrm{sech}^2(m\tau^*/R_0)]^{-1}[dR-m(R/R_0)\tanh(m\tau^*/R_0)d\tau^*]^2+R^2d\Omega^2-d\tau^{*2}\ , \tag{7.5}$$

where the metric coefficients are now explicit functions of the coordinates (R,τ^*).

Figure 4 shows an (R,τ^*) space-time diagram that repeats the (R,T) space-time diagram of Fig. 1. It is seen that it is now possible to follow trajectories of particles and light signals across R_0. The lines $R=$const are vertical lines, and each geodesic clock follows a hyperbolic half-loop trajectory given by setting $R_i=$const in (7.4b). The jagged world line represents the coordinate clock with turning radius $R_i=R_0$, and it is the largest coordinate clock loop possible. It is along this line that we have the coordinate singularity in the metric form (7.2) or (7.5). Figure 4 shows that (R,τ^*) coordinates give rise to what might be described as a "hole" in the coordinate system, where certain parts of space-time cannot be reached by the reference system.

Because of the coordinate hole and the corresponding coordinate singularity at $R_i=R_0$, the (R,τ^*) coordinate system is geodesically incomplete, because one can find geodesic trajectories that cannot be followed to $R=\infty$. This incompleteness can be seen with those trajectories in Fig. 4 that seem to "disappear" at the world line of the limiting geodesic, as if they were "swallowed up" by the coordinate hole. A similar situation also exists in a Schwarzschild field when a Novikov reference system is used.[2-5]

Figure 4 also shows the tilting of radial null cones emitted along a timelike trajectory. As with (R,τ) coordinates, the portion of the null cone moving with opposing sense to the geodesic clocks measuring τ^* undergoes extreme tilting as R_0 is crossed in both the inward and outward directions.

If we desire, we can "straighten out" the curved trajectories of the geodesic clocks by introducing a new spatial coordinate that stays constant along the trajectory of each geodesic clock. One such convenient quantity is the turning radius R_i, a

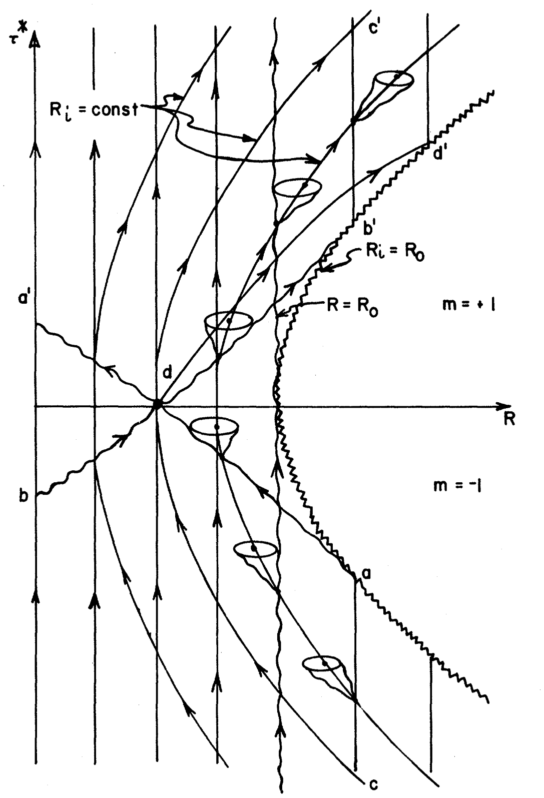

FIG. 4. An (R,τ^*) space-time diagram where τ^* is measured by a swarm of geodesic clocks. The trajectories of Fig. 1 are repeated here. It is seen that the previously unfollowable trajectories of Fig. 1 can now be followed across R_0 when T is replaced by τ^* as a time coordinate. The jagged line corresponds to $R_i=R_0$, where the (R,τ^*) metric form (7.2) or (7.5) has a coordinate singularity. The jagged line marks the physical limitation of the validity of the (R,τ^*) reference system. The reference system is geodesically incomplete, because one can find null trajectories such as bb' and timelike trajectories such as dd' that cannot be followed to $R=\infty$. The trajectory $R=R_0$ is a null line, and trajectories $R=\text{const}>R_0$ are spacelike. Also shown is the tilting of the null cones emitted along a timelike world line. As with the (R,τ) coordinates of Figs. 2, the portion of the null cone that is moving with opposite sense to the geodesic clocks measuring τ^* undergoes the extreme tilting as R_0 is crossed.

number uniquely associated with each clock's trajectory. Increasing values of R_i take us smoothly and monotonically from one coordinate clock to the next, and R_i has the significance that it is directly related to the radial coordinate R. The transformation replacing R with the new coordinate R_i has already been obtained in (7.4). In terms of R_i the nondiagonal metric (7.2) or (7.5) is changed to a diagonal form which can be written either as

$$ds^2(R_i,\tau^*)=\frac{(R/R_i)^2}{1-(R_i/R_0)^2}[dR_i^2+R^2d\Omega^2]-d\tau^{*2}\,, \tag{7.6a}$$

where R is given as a function of R_i and τ^* by (7.4b), or

$$ds^2(R_i,\tau^*)=\cosh^2(\tau^*/R_0)\left[\frac{dR_i^2}{1-(R_i/R_0)^2}+R_i^2d\Omega^2\right]-d\tau^{*2}\,, \tag{7.6b}$$

where the metric coefficients are explicit functions of (R_i,τ^*).

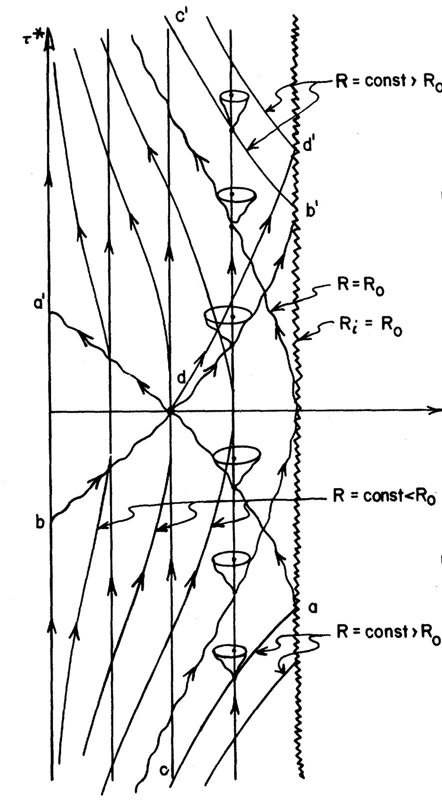

FIG. 5. Repeating Fig. 4 in terms of (R_i,τ^*) coordinates. This figure is simply a continuous deformation of Fig. 4, and there is a one-to-one correspondence between world lines on each figure. In particular, the coordinate singularity at $R_i=R_0$ and the tilting of the null cones remain in this figure.

It should be noted that the transformation (7.4) is nonsingular, indicating that there is nothing new physically in changing from R to R_i as a spatial coordinate. Whether we use (R,τ^*), or (R_i,τ^*) as coordinates, the reference system still consists of the same swarm of geodesic clocks measuring τ^*.

Figure 5 shows an (R_i,τ^*) space-time diagram that repeats in a one-to-one fashion the world lines of the (R,τ^*) space-time diagram of Fig. 4. The coordinate clock trajectories that were curved in Fig. 4 are now straight vertical lines in Fig. 5, while the straight vertical lines $R=\text{const}$ in Fig. 4 appear curved in Fig. 5, with τ^* as a function of R_i given by (7.4). In particular, the line $R_i=R_0$, along which we have a coordinate singularity, remains in both figures. It is seen that there is a one-to-one correspondence between points in Figs. 4 and 5, and that each space-time diagram is simply a continuous deformation of the other. In particular, the coordinate singularity at $R_i=R_0$ and the tilting of the null cones remain in Fig. 5.

VIII. CONVERTING (R_i,τ^*) COORDINATES TO ISOTROPIC (r,t) COORDINATES

The diagonal metric form (7.6) with $g_{44}=-1$ is expressed in terms of comoving or synchronous coordinates (R_i,θ,ϕ), because a constant value of these coordinates marks the trajectory of a geodesic clock that is measuring the time coordinate τ^*. The comoving coordinates (R_i,θ,ϕ), however, have an entirely different meaning from the comoving coordinates (ρ,θ,ϕ) used in the diagonal metric form (5.7), where also $g_{44}=-1$, because the τ^*-reference system is completely different from the τ-reference system. There is, of course, nothing wrong with describing the de Sitter universe with two different reference systems.

The metric form (7.6) is not in the usual standard isotropic form (2.1), where also $g_{44}=-1$. It is possible, though, with appropriate mathematical maneuvers, to cast (7.6) into the form (2.1), as we shall now demonstrate.

We introduce an isotropic radial coordinate r defined in terms of the comoving coordinate R_i by

$$r/r_0=\frac{R_i/R_0}{1+n[1-(R_i/R_0)^2]^{1/2}}\ ,\quad n=\pm 1 \tag{8.1a}$$

$$R_i/R_0=\frac{2r/r_0}{1+(r/r_0)^2}\ , \tag{8.1b}$$

where r_0 is an arbitrary scaling factor, and $n=+1$ or -1 indicates the two signs of the square root. In terms of r, the metric (7.6) takes the isotropic form

$$ds^2(r,\tau^*)=(R/r)^2(dr^2+r^2d\Omega^2)-d\tau^{*2} \tag{8.2a}$$

or

$$ds^2(r,\tau^*)=(2R_0/r_0)^2\frac{\cosh^2(\tau^*/R_0)}{[1+(r/r_0)^2]^2}\times(dr^2+r^2d\Omega^2)-d\tau^{*2}\ . \tag{8.2b}$$

In the form (8.2b) the metric coefficients are explicit functions of the coordinates (r,τ^*). This form has been previously found by Lanczos.[7]

The de Sitter metric (8.2) is in isotropic form, and upon comparison with the standard isotropic Robertson-Walker metric form (1.1) is seen to correspond to a universe with curvature $K=+1$. This is somewhat strange, since we started our analysis from the de Sitter isotropic metric form (2.1) which had zero curvature, $K=0$. The differences in curvature will be resolved below when we describe the procedure necessary to change (8.2) into (2.1).

The coordinate singularity at $R_i=R_0$ that was explicitly displayed in the previous metric forms has disappeared in the isotropic metric form (8.2). This disappearance was produced by the mathematical transformation (8.1). One can question the reason for using the coordinate r in place of R_i. The transformation (8.1) is double valued, with the double-valuedness being indicated by the ± 1 signs of n. To each value of R_i, there corre-

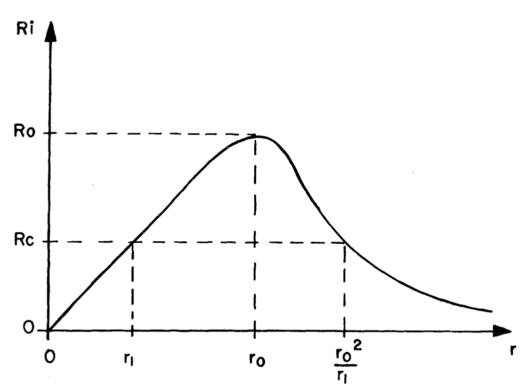

FIG. 6. The relationship between r and R_i as given in Eq. (8.1). It is seen that each value of R_i corresponds to two values r_1 and r_2 related by $r_1r_2=r_0{}^2$, with the value $R_i=0$ corresponding to $r=\infty$ as well as $r=0$.

spond two values of r, and the pairs (r_i, r_2) related by

$$r_1 r_2 = r_0^2 \tag{8.3}$$

will produce the same value of R_i in the transformation (8.1), as shown in Fig. 6. Correspondingly, at each value of τ^* the pairs (r_1, r_2) related by (8.3) will give the same value of R.

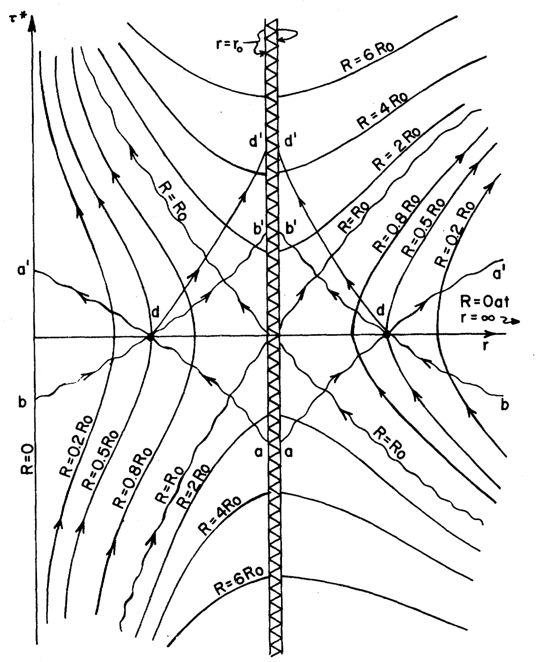

FIG. 7. Repeating Fig. 5 in terms of (r,τ^*) coordinates. It is seen that there is a one-to-one correspondence between the trajectories of Fig. 5 and the left-hand side $r < r_0$ of this figure, each being a continuous deformation of the other. Trajectories $R_i = \text{const}$ and $r = \text{const}$ are, of course, vertical straight lines on both space-time diagrams. However, the double-valued transformation (8.1) at the pairs given by (8.3) permits the possibility of patching onto the left-hand part $r < r_0$ a right-hand part $r > r_0$, such that each trajectory where $r < r_0$ will be imaged where $r > r_0$, with the line $R_i = 0$ corresponding to $r = \infty$. The mathematics is such that each trajectory smoothly and continuously joins some different counterpart trajectory at the jagged dividing line $r = r_0$, corresponding to $R_i = R_0$, marking the division of the double-valuedness, as shown on Fig. 6. Each line $R = \text{const}$ also appears twice, once at the left of r_0 and again at the right of r_0. The trajectory $R = 0$ corresponds to both $r = 0$ and $r = \infty$. The two mappings of the null line $R = R_0$ mark the division between the timelike lines $R < R_0$ and the spacelike lines $R > R_0$. If one were not aware of the double-valuedness, one might easily think that trajectories can be followed across the jagged line r_0.

Figure 7 shows an (r,τ^*) space-time diagram that repeats the trajectories of Fig. 5. Trajectories $R_i = \text{const}$ are, of course, vertical straight lines. But each $R_i = \text{const}$ line appears twice on the (r,τ^*) space-time diagram of Fig. 7, because of the double-valuedness at the pairs (r_1, r_2) given by (8.3). In particular, the line $R_i = 0$ corresponds to $r = \infty$ as well as $r = 0$. The jagged line $R_i = R_0$ at $r = r_0$ is the dividing line between the double-valuedness. Each line $R = \text{const}$ also appears twice, once at the left of $r = r_0$ and again at the right of $r = r_0$, with the null line $R = R_0$ marking the division between the timelike lines $R < R_0$ and the spacelike lines $R > R_0$. The line $R = 0$ corresponds to both $r = 0$ and $r = \infty$. Thus, if we accept $R = \infty$ as describing a point infinitely distant from $R = 0$, it is incorrect to regard $r = \infty$ as being infinitely distant from $r = 0$.

We now come to an interesting question. What if one started analyzing the de Sitter universe with the (r,τ^*) coordinates used in the isotropic metric form (8.2)? Since the coordinate singularity at $R_i = R_0$ does not appear explicitly in (8.2), there is nothing immediately obvious that suggests double-valuedness or points toward $r = r_0$ as having any distinguishing features. Further, the equations for timelike and null trajectories in terms of (r,τ^*) coordinates do not show any strange behavior at $r = r_0$.

Thus, starting from only (8.2) one might arrive at the conclusion that geodesic trajectories can be followed across r_0. For example, if one looks at the two mappings of the single trajectory $R = R_0$ in Fig. 7, it could appear as if one had two separate null trajectories extending from $r = 0$ to $r = \infty$ that cross each other at $r = r_0$ at $\tau^* = 0$. In a similar fashion, one can find a separate null trajectory that can be matched onto the two mappings of the single null line bb' at $r = r_0$ that will make it appear as if one has two different null lines that cross each other at $r = r_0$. A similar matching can be found for timelike trajectories such as dd'.

There are various ways of avoiding the mistake of assuming that trajectories can be followed across $r = r_0$. If one makes a transformation to a new spatial coordinate r' defined by $r' = r_0^2/r$, as suggested by (8.3), one will find that the metric $ds^2(r',\tau^*)$ has exactly the same form as (8.2) with r replaced with r'. This means that invariant quantities such as curvature invariants calculated at r' would have the same value as at r, indicating the double-valuedness.

In our opinion, though, the best way of avoiding incorrect interpretations of a space-time manifold described by a particular metric is to understand how the coordinates are related to physical reference systems, as we have tried to do with the coordinates (R,τ^*), and (R_i,τ^*). Along this line, we have previously pointed out a similar misinterpretation of double-valuedness in the Schwarzschild field when Kruskal-Novikov coordinates are used.[2-5]

The isotropic de Sitter metric form (8.2) with $K=+1$ does not at all look like the isotropic de Sitter metric form (2.1) with $K=0$, although both forms presumably describe the same de Sitter universe. We can, though, change (8.2) into (2.1) with the following mathematical maneuvers. Instead of letting each geodesic clock read $\tau_0^*=0$ when it reaches its turning radius at $R=R_i$, let τ_0^* be equal to some constant, the same for each clock. This changing of the zero of the time measurement causes no conceptual difficulties, and simply means that we will replace τ^* with $(\tau^*-\tau_0^*)$ in all our mathematical expressions. Expressing the hyperbolic function in terms of exponentials, we can then rewrite (8.2b) as

$$ds^2(r,\tau^*)=\left[B\frac{1+e^{-2m(\tau^*-\tau_0^*)/R_0}}{1+(r/r_0)^2}\right]^2(e^{m\tau^*/R_0})^2(dr^2+r^2d\Omega^2)-d\tau^{*2}\ , \tag{8.4}$$

where the constant B is defined by

$$B=(R_0/r_0)e^{-m\tau_0^*/R_0}\ . \tag{8.5}$$

For the time being, forget about (8.5) and treat B as some arbitrary constant. In order for (8.4) to be equal to (2.1), the expression in the square brackets must be unity. One way to accomplish this is to let the scaling factor $r_0\to\infty$ and the synchronization time $\tau_0^*\to-m\,\infty$, so that the metric (8.4) assumes the form

$$ds^2(r,\tau^*)=(Be^{m\tau^*/R_0})^2(dr^2+r^2d\Omega^2)-d\tau^{*2}\ . \tag{8.6}$$

If we now set $B=1$ and $m=+1$, and take $\tau^*=t$, (8.6) will be equal to (2.1).

Let us now examine in more detail what has to be done in order to make (7.6) assume the standard de Sitter isotropic form (2.1). First of all, we had to introduce a double-valued transformation (8.1) in order to bring the isotropic radial coordinate r into the picture. Then we had to let the constants r_0 and τ_0^* take on infinite values. The effect of letting $r_0\to\infty$ is to make the jagged dividing line of Fig. 7 move infinitely far to the right, thereby removing the region of double-valuedness to spatial infinity. This corresponds to letting the radius of curvature of the three-dimensional spherical subspace become infinitely large, which is equivalent to setting $K=0$ in the Robertson-Walker line element (1.1). With $m=+1$, the effect of letting $\tau_0^*\to-\infty$ is equivalent to moving the $\tau^*=0$ axis of Fig. 7 infinitely far to the bottom of the space-time diagram. Essentially only the top left-hand quarter is left on Fig. 7. In turn, the introduction of the infinite constants has to be done in concert so that we always maintain $B=1$ in (8.5).

The whole process seems highly artificial.

IX. SUMMARY AND CONCLUSIONS

When the de Sitter universe is described with the curvature metric form (2.5), it is not possible to follow timelike or null trajectories across the horizon at $R=R_0$ with the curvature coordinates (R,T). We have explained the reason for this in Sec. IV with the physical argument that the curvature time coordinate T is related to times measured by clocks located at fixed values of R, and one cannot have a particle located at a point $R=\text{const}>R_0$ because it would then be moving along a spacelike trajectory faster than the speed of light. To remedy this, we have shown that when one measures time with radially moving geodesic clocks, it is then possible to follow trajectories across R_0. We have discussed two different types of geodesic clock reference systems, the clock-factory τ-reference system of Sec. V and the Novikov-type τ^*-reference system of Sec. VII.

Although each geodesic clock reference system allows trajectories to be followed across R_0, each reference system also has its individual deficiencies. With a clock-factory reference system, trajectories can be followed only one way across R_0, because a particle moving in the opposite sense to the geodesic clocks measuring the time τ will undergo infinite time dilations as R_0 is approached. With a Novikov-type reference system, there is a limiting "last" clock in the swarm of geodesic clocks measuring the time τ^*, namely, the one that has a turning radius $R_i = R_0$. Because of this, there is a "hole" in the reference system, having the effect that certain trajectories are terminated before they reach $R = \infty$. These deficiencies arise because of the structure of each reference system, but we understand the reason for each deficiency.

We have shown that the important maneuver was the replacement of the curvature time coordinate T with a geodesic time coordinate to go from (R,T) coordinates to (R,τ) or (R,τ^*) coordinates. This corresponded to a definite change in the reference system, going from a reference system where clocks are located at fixed values of R to one where the clocks are moving relative to fixed values of R. The metric form in terms of (R,τ) or (R,τ^*) coordinates was nondiagonal. We showed we could diagonalize the metric form by replacing R with a comoving coordinate ρ or R_i to go to (ρ,τ) or (R_i,τ^*) coordinates. The replacement of R with a comoving spatial coordinate did not involve any change of reference system. The reference system still consisted of radially moving geodesic clocks—we merely relabeled the way we described the spatial coordinate. Consequently, the corresponding space-time diagrams were simply continuous deformations of each other, each one telling the same story from a slightly different angle.

At this point we had two perfectly good comoving or synchronous coordinate systems. Values $(\rho,\theta,\phi) = \text{const}$ or $(R_i,\theta,\phi) = \text{const}$ marked the trajectory of a radially moving clock measuring τ or τ^*, and in both diagonal metrics (5.7) and (7.6) $g_{44} = -1$. However, neither metric was equal to the isotropic metric form (2.1), which is usually purported to describe the de Sitter universe in comoving coordinates. In Secs. VI and VIII we showed that in order to cast our comoving metric forms (5.7) and (7.6) into the usual isotropic form (2.1), we had to introduce infinities into our measurements of both space and time.

With our τ- or τ^*-reference systems, we had a perfectly sensible and understandable physical picture. The mathematical machinations involved in going over to the usual isotropic metric form (2.1) cloud this sensible picture by bringing in questionable infinities, which seem to have no purpose other than to transform to the usual metric form (2.1).

The issues we are raising here, however, will not be at all evident if one proceeds purely mathematically, and works backward from (2.5) with (2.4) and (2.2) to get (2.1). This is what is found in the literature. In this manner, there is never any need to make mention of ρ or R_i, or the intermediate nondiagonal metric forms.

If, however, we start from the curvature form (2.5) and work with geodesic reference systems, whose structure we understand, it seems that we cannot attain the usual isotropic form (2.1) with physically realizable reference systems. Because of this, the validity of describing the de Sitter universe with (2.1) can be questioned. In terms of the logic of this paper, we can start from (2.5) and develop the perfectly good comoving metric forms (5.7) and (7.6) without ever bringing (2.1) into the picture.

It thus seems worthwhile to ask why one would want to use the isotropic form (2.1) for describing the de Sitter universe? The form (2.1) is a special case of the general Robertson-Walker metric form (1.1), which was obtained from, among other things, the assumption that the universe is isotropic. "Isotropy" means that the universe will appear the same no matter which direction one looks. This is taken by many authors to mean that the metric describing the universe can be written in a form where the spatial part is proportional to

$$d\sigma^2 = dx^2 + dy^2 + dz^2 , \tag{9.1}$$

or equivalently, with $r^2 = x^2 + y^2 + z^2$,

$$d\sigma^2 = dr^2 + r^2 d\Omega^2 , \tag{9.2}$$

where the $g_{\mu\nu}$ in the complete metric form being independent of the angles θ and ϕ. Implicitly taken for granted with (9.2) is that the coordinate r monotonically increases with distance, and has the range $0 \le r \le \infty$.

It seems that the main reason for starting with (9.2) is the uncritical acceptance of the pioneering work of people such as Riemann and Lobachevski who developed differential geometry. But this work was done in the 1800's, long before we knew about special relativity, which tells us that lengths and times are intertwined, and general relativity, which tells us that lengths and times are affected by gravity. The developers of differential geometry worked with lengths only—the notions of time and gravity never entered into their considerations. There was no four-dimensional space-time, only a three-dimensional (or n-dimensional) space.

It may therefore be entirely too presumptuous to start a discussion of the de Sitter universe with the isotropic coordinates used in (2.1). There are many reasons indicating that a better starting point is the curvature metric form (2.5), some of which are the following.

The curvature coordinates used in (2.5) satisfy the requirements of isotropy; the metric coefficients are independent of θ and ϕ, so the universe appears the same in all directions. Further, there is nothing special about the origin $R=0$. If the de Sitter universe were referred to any other origin corresponding to the location of a geodesic particle, the new metric form would be the same as (2.5). This means that (2.5) satisfies the requirement of homogeneity as well as isotropy.

The de Sitter universe is intrinsically static, and curvature coordinates exhibit this static property explicitly; isotropic coordinates do not. Curvature coordinates such as the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ are proportional to R^n, where n is a positive integer. The area of a sphere $R=\text{const}$, which is an invariant quantity, is $A=4\pi R^2$. With curvature coordinates, the Schwarzschild field can be smoothly joined onto the de Sitter universe, as in (2.6). Curvature coordinates reduce to flat-space Minkowski coordinates as $R\to 0$. Therefore these are the coordinates to use in conjunction with the principle of equivalence, which states that the region locally around $R=0$ (a timelike geodesic) is equivalent to flat space-time.

If, as it seems, it is preferable to describe the de Sitter universe with curvature coordinates instead of isotropic coordinates, one might consider looking at other nonstatic universes from the point of view of curvature coordinates. This we will do in future publications.

ACKNOWLEDGMENT

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