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Realization in phase space of general coordinate transformations

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We realize a subgroup of the full general field-dependent coordinate transformation group for a field theory in a certain phase space. For a pseudo-Riemannian metric in four dimensions, subject to certain conditions, this program leads uniquely to Einstein's theory of gravitation in Hamiltonian form. The framework is also applied to the Nambu-Goto string model, where again reparametrization covariance determines the dynamics.

I. INTRODUCTION

It is well known that the postulate of general covariance, coupled with the demand that the equations of motion for the metric tensor be of second order and derivable from a Lagrangian, leads ineluctably to Einstein's theory of gravitation. Since Dirac and Arnowitt, Deser, and Misner achieved the complete Hamiltonian version of Einstein's theory, the form and the logical role of general covariance have become obscure. We shall show that the demand that general coordinate transformations be realized as canonical transformations in a phase space leads uniquely to Einstein's theory in Hamiltonian form, thus restoring the postulate of general covariance as a foundation principle in gravitation theory.

In a paper published in 1972, Bergmann and Komar[1] recognized that Dirac's lapse and shift generators must generate a subgroup of the group of metric field-dependent coordinate transformations. We regard their work as not quite complete since, as will become apparent, one must show that the constraints fulfill certain differential equations. (Bergmann[2] demonstrated ten years earlier that a second necessary condition is fulfilled, namely, that the commutators must not contain time derivatives. This is the case if they depend on D-invariant quantities only. It is sufficient to show—as Bergmann did in a short elegant argument—that the descriptors may depend arbitrarily on canonical variables which obey the dynamical equations. We shall see that the absence of time derivatives in the commutator actually follows for arbitrary canonical variables.)

Our objective is to find the largest subgroup of the field-dependent coordinate transformation group which may be realized as a canonical transformation group. This line of reasoning was also stimulated in part by work of Hojman, Kuchař, and Teitelboim.[3] Arguing that the variation of canonical variables from hypersurface to hypersurface should be path independent, they derive the algebra of lapse and shift generators and solve these quadratic functional equations for the constraint generators. Their point of departure is, however, a decomposition of coordinate transformations into lapses and shifts. We regard this decomposition as an answer in search of a question; the underlying question is the following: What is the most general field-dependent infinitesimal coordinate transformation whose commutator contains no time derivatives? One of our achievements in this paper is the demonstration that when a metric may be constructed in terms of field variables the lapse plus shift decomposition, with descriptors functionals of the canonical variables, represents the most general transformation with this property.

Since the group property is a restatement of path independence we, of course, recover the usual Poisson-bracket algebra for the Hamiltonian generators. One might argue that nothing is gained beyond the approach of the aforementioned authors since this algebra was sufficient to fix the dynamics. From a practical point of view however, the derivation of the constraints is greatly simplified since in addition to this algebra we obtain linear inhomogeneous functional differential equations for the constraints. Furthermore we need make no extra assumptions on the locality of lapse variations, since they are merely local coordinate transformations. But, most importantly, we would like to argue for the physical necessity that metric-dependent general coordinate transformations be realizable as canonical transformations. If this symmetry were not present, then it would not be possible to construct invariant quantities through the imposition of conditions on canonical variables. We shall discuss this point in our concluding remarks.

The plan of the paper is as follows. In Sec. II we explain the meaning of reparametrization transformations in phase space (we use the terms "reparametrization" and "coordinate transformation" interchangeably). We present our program in a four-step procedure. In the first step we require that the descriptors be D-invariant quantities. In step 2 one introduces "momenta." These are functions of D-invariant variables subject to certain restrictions. Poisson brackets are postulated for "canonical pairs" (notice that we do not have an action from which momenta may be derived). The following steps are partly self-consistency conditions, which may exclude some of the possible choices of "momenta." In step 3 the reparametrizations are compared with infinitesimal transformations in a phase space, locally described by D-invariant fields and the momenta. In comparing the commutator of two reparametrizations with the Poisson bracket of two phase-space transformations we find that the generators of the phase-space transformations must be constraints and we indicate how one derives their Poisson-bracket algebra. Furthermore we find differential equations for the constraints. After obtaining canonical dynamical equations, in step 4 we solve for the constraints. As is clear from this general description of the method, it is typically local.

Instead of dealing directly with general relativity we will formulate and describe each of the steps for an arbitrary theory, covariant under general coordinate transformations, and then apply the procedure to a simple model. A simple illustrative model would be the relativistic free point particle, but this one is too simple and does not deliver certain features which arise in the case of general relativity (in the sense that for the particle some of the steps 1–4 are empty). Instead we choose a two-dimensional field theory to illustrate the method. The computations are somewhat less tedious than the gravitational case and consequently we provide more details. This example, however, distinguishes itself from gravitation through the appearance of relations which correspond to primary constraints in a Hamiltonian formalism. After the imposition of certain conditions, this model represents the closed Nambu-Goto string.

In Sec. III we show that Einstein's sourceless theory of gravitation is the unique reparametrization-covariant Hamiltonian model with polynomial metric field equations of second order in time, and with only one dimensional constant.

We conclude in Sec. IV with a discussion of the physical significance of the field-dependent general coordinate transformation group, and a description of forthcoming work.

Two appendices are added. In Appendix A we discuss the extent to which the group property of transformations derived in step 1 already determines the dynamics of the Nambu-Goto string. Finally, in Appendix B we show that if it is possible to construct more than one metric from dynamical fields, then the demand that reparametrizations be realized as canonical transformations forces these metrics to differ by a D-invariant factor. It is therefore of no physical consequence with respect to which metric one performs the lapse plus shift decomposition.

II. REALIZATION OF THE GENERAL COORDINATE TRANSFORMATION GROUP IN PHASE SPACE

We must first indicate the sense in which we wish to realize the general coordinate transformation group as a transformation group on a certain phase space.

Let $\hat{\Phi}^{\alpha}(y^A)$ be fields, where $y^A(A=0,\ldots,N)$ are coordinates and α is an abstract geometric and/or internal index. Assume that each $\hat{\Phi}^{\alpha}$ has a definite transformation property under general coordinate transformations $y'^A=y^A+\hat{\Psi}^A$.

Let us suppose initially that the fields $\hat{\Phi}^{\alpha}$ are completely arbitrary. Now at each fixed "time" y^0 we select a subset of fields $\hat{\Phi}^i(y^0)$ and functionals $\hat{\Pi}_i[\hat{\Phi}^{\alpha},\partial\Phi^{\alpha}/\partial y^0](y^0)$ which we map, locally, into a symplectic manifold $\Gamma(y^0)$ with coordinates $\Phi^i(y^0),\Pi_i(y^0)$ and canonical symplectic structure $d\Phi^i(y^0)\wedge d\Pi_i(y^0)$. Let us first consider whether it might be possible to realize the one-parameter family of transformations

$$y'^A(y^0)=y^A(y^0)+\hat{\Psi}^A(y^0)$$

as canonical transformations on the respective phase space $\Gamma(y^0)$. Note that the transform of the fields $\hat{\Phi}^i(y^0)$ and $\hat{\Pi}_i(y^0)$ must not depend on $\partial\hat{\Psi}^A(y^0)/\partial y^0$ since a canonical transformation associated with the descriptor $\hat{\Psi}^A(y^0)$ yields a transform dependent only on y^0. We shall discuss this criterion extensively below. Most importantly for our present argument, we observe that time translation is itself an element of the reparametrization group which we wish to realize. Thus if we wish to implement time translations as a canonical transformation within a phase space $\Gamma(y^0)$ we must identify $\Phi^i(y^{0'}),\Pi_i(y^{0'})$ in $\Gamma(y^{0'})$ with the time-translated data

$$\tau(y^{0'},y^0)(\Phi^i(y^0),\Pi_i(y^0))$$

in $\Gamma(y^0)$. We conclude the following. (1) The fields $\hat{\Phi}^{\alpha}$ may not be completely arbitrary, but must obey second-order differential equations in y^0 (second order since the Hamiltonian equations are first order by construction and the $\hat{\Pi}_i$ depend only on first time derivatives by assumption). (2) We have but one phase space with time translation represented by a Hamiltonian flow (the particular flow determined by the particular choice of coordinates y^A). (3) To be self-consistent, general coordinate transformations must map solutions $\hat{\Phi}^\alpha$ of the second-order dynamical equations onto solutions; i.e., these dynamical equations must be generally covariant.

Let us reformulate our original objective, taking into account the preceding remarks. Given a solution $\hat{\Phi}^\alpha$ we construct the corresponding trajectory in phase space $(\Phi^i(y^0),\Pi_i(y^0))$. Under a reparametrization $y'^A=\hat{f}^A(y)$, $\hat{\Phi}^\alpha$ is mapped into a solution $\hat{\Phi}'^\alpha$ with differing functional form and generally differing phase-space trajectory $(\Phi'^i(y^0),\Pi'_i(y^0))$. Now we demand that the transformation

$$(\Phi(y^0),\Pi(y^0))\rightarrow(\Phi'(y^0),\Pi'(y^0))$$

for fixed y^0 be realized by a canonical transformation $f(y^0)$. Thus a particular y-manifold reparametrization is to be implemented by a one-parameter family of canonical transformations. We have attempted to elucidate this idea pictorially in Fig. 1.

Unfortunately the group with descriptors $\hat{\Psi}^A(y)$ may not be realized in this way, since the commutator of two infinitesimal transformations with descriptors $\hat{\Psi}_1^A(y)$ and $\hat{\Psi}_2^A(y)$ at fixed y^0 contains reference to other transformations: it contains time derivatives. Thus we may accomplish our goal only if we generalize the coordinate transformations to include dependence on the dynamical fields:

$$\hat{\Psi}^A=\hat{\Psi}^A[y;\hat{\Phi}^\alpha(y)] .$$

These have been called Q-type transformations by Bergmann and Komar.[1] Our dynamical model is clearly covariant under this larger group.

The descriptor $\hat{\Psi}_3^A$ of the commutator of two transformations with descriptors $\hat{\Psi}_1^A$ and $\hat{\Psi}_2^A$ is

$$\hat{\Psi}_3^A=\hat{\Psi}_{1,B}^A\hat{\Psi}_2^B+\int d^{N+1}y'\frac{\mathscr{D}\hat{\Psi}_1^A}{\mathscr{D}\hat{\Phi}^\alpha(y')}\bar{\delta}_2\hat{\Phi}^\alpha(y') -(1\leftrightarrow 2) . \tag{2.1}$$

Here $\mathscr{D}/\mathscr{D}\hat{\Phi}^\alpha$ is the functional derivative and

$$\bar{\delta}\hat{\Phi}^\alpha\equiv\hat{\Phi}'^\alpha(y)-\hat{\Phi}^\alpha(y) =\delta\hat{\Phi}^\alpha-\hat{\Phi}^\alpha_{,A}\hat{\Psi}^A . \tag{2.2}$$

Now precisely those transformations for which this commutator is local in y^0 and contains no time derivatives may be canonically realized. Note first that $\hat{\Psi}_3^A$ may be local in y^0 only if all descriptors are local in y^0. We shall restrict the functional dependence even further in that we allow the descriptors to depend at most on the first time derivatives of the fields $\hat{\Phi}^\alpha$ [but not excluding "spatial" derivatives, i.e., derivatives with respect to y^a's ($a=1,\ldots,N$) of arbitrary high order]. As will become apparent this restriction will lead to field equations of second order. By dropping this requirement we would be able to produce higher-order field equations.

After all these preliminaries, the first step in our program is the following:

Step 1. Find the most general functional dependence of the descriptors on the fields $\hat{\Phi}^\alpha$ such that (i) they depend at most on first time derivatives of $\hat{\Phi}^\alpha$, and (ii) the commutator of two descriptors does not contain time derivatives of descriptors.

The field theories we consider in this paper are special in the sense that there is a "natural" metric

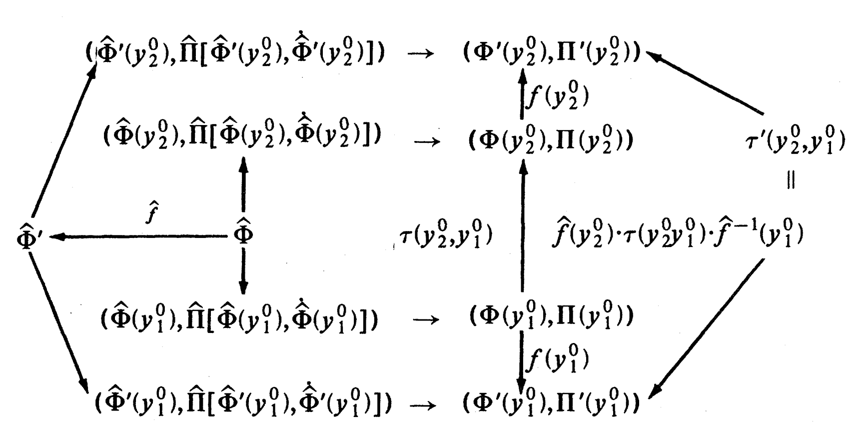

FIG. 1. The realization in phase space of general coordinate transformation symmetries. The "hat" variables represent configuration-velocity-space functionals, while the variables on the right represent points on a trajectory in a symplectic manifold. Were a Lagrangian to exist, the horizontal mappings would correspond to a Legendre transformation.

$g_{AB}[\hat{\Phi}^\alpha]$ which either is a functional of the fields or is a dynamical field itself. (We leave the discussion of theories with more than one "natural" metric to Appendix B.) For these "metric" theories step 1 may be reformulated. Construct

$$n^A = -\frac{g^{A0}}{(-g^{00})^{1/2}} , \tag{2.3}$$

which is the unit future-pointing normal to the $y^0=$const hypersurfaces. Since g_{AB} is a metric we may calculate how n^A transforms under $y^A \to y^A + \hat{\Psi}^A$:

$$\bar{\delta} n^A = -n^A_{,B}\hat{\Psi}^B + n^B \hat{\Psi}^A_{,B} + n_0 e^{AB}\hat{\Psi}^0_{,B} . \tag{2.4}$$

Here

$$e^{AB} = g^{AB} + n^A n^B , \tag{2.5}$$

with the property $e^{A0}=0$, $e^{ab}g_{bc}=\delta^a_c$. Next we perform a "lapse" and "shift" decomposition of

$$\hat{\Psi}^A = \delta^A_a \hat{\xi}^a + n^A \hat{\xi}^0 , \tag{2.6}$$

and calculate the commutator (2.1). Using (2.4) one finds

$$\hat{\Psi}^A_3 = \delta^A_a(\hat{\xi}^a_{1,b}\hat{\xi}^b_2 - e^{ab}\hat{\xi}^0_1\hat{\xi}^0_{2,b} + \bar{\delta}_2\hat{\xi}_1{}^a) - (1\leftrightarrow 2)$$
$$+ n^A(\hat{\xi}^0_{1,a}\hat{\xi}^a_2 + \bar{\delta}_2\hat{\xi}^0_1) - (1\leftrightarrow 2) , \tag{2.7}$$

with

$$\bar{\delta}_2 \hat{\xi}^A_1 = \int d^N y' \left[\frac{\mathscr{D}\hat{\xi}^A_1}{\mathscr{D}\hat{\Phi}^\alpha(y')} \bar{\delta}_2 \hat{\Phi}^\alpha(y') + \frac{\mathscr{D}\hat{\xi}^A_1}{\mathscr{D}\hat{\Phi}^\alpha_{,0}(y')} \bar{\delta}_2 \hat{\Phi}^\alpha_{,0}(y') \right] . \tag{2.8}$$

Observe first that for field-independent descriptors $\hat{\xi}^A$ [which, however, does not mean that the original descriptors $\hat{\Psi}^A$ are field independent as can be seen from (2.6)] the commutator $\hat{\Psi}^A_3$ does not contain time derivatives of descriptors. The extra terms (2.8) in the commutator (2.7) arising for field-dependent $\hat{\xi}^A$'s are the variation of $\hat{\xi}^A_{1(2)}$ under $\hat{\Psi}_{2(1)}$. Step 1 amounts to finding the most general functional dependence of the $\hat{\xi}^A$ such that (i) they depend at most on first-order time derivatives in $\hat{\Phi}^\alpha$ and (ii) $\bar{\delta}\hat{\xi}^A$ does not contain time derivatives of the descriptors $\hat{\Psi}^A$. The last condition is the requirement that the $\hat{\xi}^A$ are D-invariant, a name introduced by Bergmann[2] in the Handbuch der Physik article on general relativity in honor of the concept first formulated by Dirac[4]: a quantity Q is called D-invariant if its transform $Q+\bar{\delta}Q$ does not depend on y^0 derivatives of the descriptors, which geometrically means that Q remain invariant under transformations which leave the coordinates of the hypersurface $y^0=$const fixed.

The descriptors $\hat{\xi}^A$ are D-invariant if they depend on D-invariant quantities only. And by condition (i) we are interested in D-invariants, depending on time derivatives of at most first order (let us refer to them as first-order D-invariants). These may be constructed in the following manner.

(a) Let T be a covariant tensor of arbitrary rank depending on the fields, then the covariant normal derivative of its spatial components,

$$P_{ab\cdots} = T_{ab\cdots;A}n^A , \tag{2.9}$$

is D-invariant, as well as all spatial covariant derivatives thereof [covariant with respect to pure hypersurface transformations (PHT's), i.e., all transformations for which $\Psi^0=0$].

(b) Just from its geometrical meaning, the extrinsic curvature

$$K_{ab} = -n_{a;b} = -n_0\Gamma^0_{ab} \tag{2.10}$$

is D-invariant. We do not have a proof that there are in general no more independent first-order D-invariant variables. We shall show in Sec. III, however, that the extrinsic curvature is the only independent one if g_{AB} is the dynamical field itself. In addition we shall demonstrate that if g_{AB} is an induced metric, then the construction (a) yields all first-order D-invariants.

In summary, step 1 may, for theories with metric, also be formulated as follows: Find all zero- and first-order D-invariants. The descriptors are allowed to depend on these quantities only, and their variation may be written as

$$\bar{\delta}_2\hat{\xi}^A_1 = \frac{\partial\hat{\xi}^A_1}{\partial\hat{\Phi}^i_{,a\cdots b}}\bar{\delta}_2\hat{\Phi}^i_{,a\cdots b} + \frac{\partial\hat{\xi}^A_1}{\partial\hat{P}^I_{,a\cdots b}}\bar{\delta}_2\hat{P}^I_{,a\cdots b} , \tag{2.11}$$

where the $\hat{\Phi}^i$ are those of the original fields which are D-invariant, and the $\hat{P}^I$ are all independent first-order D-invariants.

Instead of immediately turning to the next steps of our program it is perhaps more instructive to consider an example. Consider a field theory in two dimensions with four fields

$$\hat{\Phi}^\alpha(y^0,y^1)\ (\alpha=0,\ldots,3) .$$

Let the $\hat{\Phi}^\alpha$ transform as

$$\bar{\delta}\hat{\Phi}^\alpha = -\hat{\Phi}_{,A}\hat{\Psi}^A , \tag{2.12}$$

so they are obviously D-invariant. We view $\hat{\Phi}^\alpha$ as a parametrized two-surface in a four-dimensional manifold with metric $\eta_{\alpha\beta}$. Then the induced metric in this two-surface is

$$g_{AB}=\eta_{\alpha\beta}\hat{\Phi}^\alpha_{,A}\hat{\Phi}^\beta_{,B} . \tag{2.13}$$

From g_{AB} we construct the normal $n^A[\hat{\Phi}^\alpha_{,A}]$, and with its help the first-order D-invariants

$$\hat{p}^\alpha=\hat{\Phi}^\alpha_{,A}n^A . \tag{2.14}$$

Observe that these are not independent, since

$$\begin{aligned}&\eta_{\alpha\beta}\hat{p}^\alpha\hat{p}^\beta\equiv-1 ,\\ &\eta_{\alpha\beta}\hat{p}^\alpha\hat{\Phi}^\rho_{,1}\equiv 0 .\end{aligned} \tag{2.15}$$

The extrinsic curvature depends on the $\hat{p}^\alpha$. For instance for constant $\eta_{\alpha\beta}$

$$k_{11}=-\eta_{\alpha\beta}\hat{p}^\alpha\hat{\Phi}^\beta_{,11} .$$

Indeed we shall demonstrate, in performing step 1, that every first-order D-invariant quantity can be expressed in terms of the $\hat{p}^\alpha$. Denoting by

$$f^{[l]}=\frac{\partial^l}{(\partial y^1)^l}f$$

the lth derivative with respect to the only spatial parameter y^1, the variation of a descriptor $\hat{\xi}^A_1$, depending on $\hat{\Phi}^\alpha,\hat{\Phi}^\alpha_{,0}$ (and arbitrary spatial derivatives), with respect to $\hat{\Psi}^A_2$ becomes

$$\bar{\delta}_2\hat{\xi}^A_1=\sum_{l'}\frac{\partial\hat{\xi}^A_1}{\partial\hat{\Phi}^{\alpha[l']}}\bar{\delta}_2\hat{\Phi}^{\alpha[l']} +\sum_{l}\frac{\partial\hat{\xi}^A_2}{\partial\hat{\Phi}^{\alpha[l]}_{,0}}\bar{\delta}_2\hat{\Phi}^{\alpha[l]}_{,0} . \tag{2.16}$$

Since the $\bar{\delta}$ variations commute with the derivatives with respect to y^A we find that $\bar{\delta}_2\hat{\Phi}^{\alpha[l']}$ does not contain time derivatives of $\hat{\Psi}^A_2$ and that

$$\bar{\delta}_2\hat{\Phi}^{2[l]}_{,0}=-\sum_{k=0}^{l}\binom{l}{k}\left\{\hat{\Phi}^{2[l-k]}_{,0B}\hat{\Psi}^{B[k]}_2+\hat{\Phi}^{2[l-k]}_{,B}\hat{\Psi}^{B[k]}_{2,0}\right\} .$$

Inserting this expansion into (2.16), and setting equal to zero the sum of terms containing time derivatives of the descriptors, we obtain the conditions

$$0=\sum_{l=J}^{L}\binom{l}{J}\frac{\partial\hat{\xi}^A_1}{\partial\hat{\Phi}^{\alpha[l]}_{,0}}\hat{\Phi}^{\alpha[l-J]}_{,B}\hat{\Psi}^{B[J]}_{2,0} , \tag{2.17}$$

for $J=0,\dots,L$ (where L is the highest order of spatial derivatives of $\hat{\Phi}^\alpha_{,0}$ present in $\hat{\xi}^A_1$). We shall indicate how these equations may be solved inductively. Start with the equation containing the highest derivative in y^1; that is, for $J=L$,

$$0=\frac{\partial\hat{\xi}^A}{\partial\hat{\Phi}^{\alpha[L]}_{,0}}\hat{\Phi}^\alpha_{,B} \tag{2.17'}$$

(we drop the index 1 on $\hat{\xi}^A$). This is the condition that $\partial\hat{\xi}^A/\partial\hat{\Phi}^{\alpha[L]}_{,0}$ is perpendicular to the surface with metric g_{AB} (for which the $\hat{\Phi}^\alpha_{,B}$ are tangents). Now it is easy to find two independent vectors perpendicular to $\hat{\Phi}^\alpha_{,B}$. From the identity

$$\frac{\partial n^A}{\partial\hat{\Phi}^\alpha_{,0}}\hat{\Phi}^\alpha_{,B}\equiv-\delta^A_B n^0 , \tag{2.18}$$

one finds

$$\frac{\partial\hat{p}^\beta}{\partial\hat{\Phi}^\alpha_{,0}}\hat{\Phi}^\alpha_{,B}\equiv 0 . \tag{2.19}$$

Since the $\hat{p}^\beta$ obey the conditions (2.15) one also has

$$\frac{\partial\hat{p}^\beta}{\partial\hat{\Phi}^\alpha_{,0}}\hat{p}_\beta\equiv 0 , \quad \frac{\partial\hat{p}^\beta}{\partial\hat{\Phi}^\alpha_{,0}}\hat{\Phi}_{\beta,1}\equiv 0 ,$$

leaving two independent directions. Therefore

$$\frac{\partial\hat{\xi}^A}{\partial\hat{\Phi}^{\alpha[L]}_{,0}}=\gamma^A_{L\beta}\frac{\partial\hat{p}^\beta}{\partial\hat{\Phi}^\alpha_{,0}}=\gamma^A_{L\beta}\frac{\partial\hat{p}^{\beta[L]}}{\partial\hat{\Phi}^{\alpha[L]}_{,0}} ,$$

with coefficients $\gamma^A_{L\beta}$, which shows that the dependence of $\hat{\xi}^A$ on $\hat{\Phi}^{\alpha[L]}_{,0}$ enters implicitly through dependence on $\hat{p}^{\alpha[L]}$. Next consider the condition on $\hat{\xi}^A$ for $J=L-1$. Applying the same reasoning as before one can show that the $\hat{\Phi}^{\alpha[L-1]}_{,0}$ dependence enters only implicitly through $\hat{p}^{\alpha[L]}$ and $\hat{p}^{\alpha[L-1]}$ dependence. It is straightforward to construct an inductive proof along these lines with the result that every dependence on $\Phi^{\alpha[l]}_{,0}$ is due to the dependence of $\hat{p}^\alpha$ on $\hat{\Phi}^\alpha_{,0}$. So we have the result that our requirements of step 1 uniquely fix the descriptors to be of the form (2.6) with

$$\hat{\xi}^A=\hat{\xi}^A(y;\hat{\Phi}^{\alpha[l']},\hat{p}^{\alpha[l]}) .$$

We express this result in the form $\hat{\xi}^A=\hat{\xi}^A[y;\hat{\Phi}^\alpha,\hat{p}^\alpha]$, where the brackets signify functional dependence on the $y^0=$const hypersurfaces.

Let us now return to the description of our program. With the decomposition (2.6) and descriptors $\hat{\xi}^A$ depending on D-invariant fields $\hat{\Phi}^i$ and first-order D-invariant $\hat{P}^I$ variables we are guaranteed that the commutator of two reparametrizations does not contain time derivatives of the descriptors. The commutator is again of the form (2.6) [cf. Eq. (2.7)], and the $\hat{\xi}^A_3$ in $\hat{\Psi}^A_3=\delta^A_a\hat{\xi}_3{}^a+n^A\hat{\xi}^0_3$ are D-invariant. Nevertheless, transformations corresponding to the descriptors arrived at in step 1 do not constitute a group. As can be seen from (2.11) for local dependence on the $\hat{\Phi}^{\alpha[l]}$ and $\hat{\Phi}^{\alpha[l]}_{,0}$, commutators produce successively higher derivative dependence; $\hat{\xi}^A_3$ will in particular contain terms involving second time derivatives. We interpret these unwanted terms as equations of motion. Then the subset of all transformations with descriptors $\hat{\xi}^A$ depending on at most first-order D-invariants is a subgroup of all field-dependent transformations. This is an example of an "off-shell" algebra, well known from supersymmetric theories. We do not wish to discuss this point further in general, since the isolation of the dynamics simply by the requirement that the permitted reparametrizations form a group shall occur in step 4 of our program. However since these remarks may sound dubious at this stage we try to clarify them through an example in Appendix A.

We turn now to the construction of "phase-space variables." By this we mean those fields and functionals of the fields which are mapped into the phase space with local coordinates Φ^i and Π_i. The variables we are looking for must be D-invariant objects since reparametrizations are to be implemented by canonical transformations.

Step 2. From the original fields $\hat{\Phi}^\alpha$ select only the D-invariant ones (referred to earlier as $\hat{\Phi}^i$). Define the same number of functionals $\hat{\Pi}_i[\hat{\Phi}^i,\hat{p}^I]$, called momenta, depending on $\hat{\Phi}^\alpha$ and the independent first-order D-invariants $\hat{p}^I$. We must assume that there exist no more $\hat{p}^I$ than $\hat{\Phi}^i$. The functionals $\hat{\Pi}_i$ have to be chosen such that the following conditions hold:

(1) The rank of $|\partial\hat{\Pi}_i/\partial\hat{p}^I|$ is equal to the number of p^I's.

(2) The $\hat{\Pi}_i$ transform as densities of weight 1 under PHT's.

(3) The $\hat{\Pi}_i$ obey a set of further restrictions (R) dictated by the physical model under discussion. We shall clarify this point below in our example. Since we wish to map the $\hat{\Phi}^i$ and $\hat{\Pi}_i$ into a symplectic manifold we must make a choice of canonical pairs. The particular choice will be determined by the restrictions (R).

We examine the significance of the conditions (1)–(3) in turn. If condition (1) is fulfilled one may solve the $\hat{\Pi}_i$ for the $\hat{p}^I$ and the $\hat{\Phi}^i$, and it is possible to express the descriptors $\hat{\xi}^A[\hat{\Phi}^i,\hat{p}^I]$ as $\hat{\xi}^A[\hat{\Phi}^i,\hat{\Pi}_i]$. Condition (2) guarantees that the Π's indeed have the transformation property of momenta. Condition (3) and the selection of canonical pairs seem to introduce a great deal of arbitrariness into the program. This is true. However by choosing the restrictions (R) in a suitable way, one is able to limit the number of possible momenta. Further consistency conditions (in later steps) restrict the number of possible local phase-space coordinates even further. Those which remain belong to different dynamical systems. To convince the reader that the program, seemingly intractable at first sight, really works, we return to our example.

All four fields $\hat{\Phi}^\alpha$ are D-invariant; these are the $\hat{\Phi}^i$ variables. As $\hat{P}^I$ variables we only have two of them $\hat{p}^\alpha=\hat{\Phi}^\alpha_{,A}n^A$ at our disposal. So, irrespective of the functional dependence of the four $\hat{\Pi}_i$ the condition (1) implies that the momenta $\hat{\Pi}_i$ and the fields $\hat{\Phi}^i$ are not independent. There are identities among them, which for $\hat{p}^\alpha=\hat{p}^\alpha[\hat{\Pi}_i,\hat{\Phi}^i]$ directly follow from (2.15). The most general scalar densities of weight 1 may be constructed by multiplying scalars by the determinant of the induced metric ($e^{-1/2}$). We obtain a unique dynamical system through the imposition of the following restrictions (R): the metric $\eta_{\alpha\beta}$ is that of Minkowski space-time; the fields $\hat{\Phi}^\alpha$ are components of a Lorentz vector with dimension length ($c=1$); the momenta $\hat{\Pi}_i$ are components of a Lorentz vector with dimension mass; the $\hat{\Phi}^\alpha$ are periodic with period 2π in y^1; the momenta shall not depend on the fields themselves (translation invariance); and there exists only one dimensional parameter λ of dimension mass/length.

From now on we will identify $\hat{\Phi}^i\leftrightarrow\hat{x}^\mu$, and $\hat{\Pi}_i\leftrightarrow\hat{\pi}_\mu$ to be constructed from $\hat{p}^\mu=n^A\hat{x}_{,A}{}^\mu$. Taking for $\hat{\pi}_\mu$ simply $\hat{p}_\mu$ itself is not quite correct, since it does not have the required weight. However

$$\hat{\pi}_\mu=\lambda e^{-1/2}\hat{p}_\mu \tag{2.20}$$

is a choice for the momenta consistent with R. And indeed any other choices for $\hat{\pi}_\mu$ other than (2.20) would either not be solvable for the $\hat{p}_\mu$ and/or involve other dimensional constants. Note that the $\hat{\pi}_\mu$ satisfy identities because of (2.15). As a consequence we shall be concerned with the corresponding surface in phase space determined by $\pi^2+\lambda^2x^2_{,1}=0$ and $\pi\cdot x_{,1}=0$. These relations would correspond to primary constraints in a Hamiltonian formalism derivable from a Lagrangian model. We shall find later on that the choice of the set of restrictions R leads to minimal two-surfaces imbedded in Minkowski space, that is, to the free (closed) Nambu-Goto string. If in R we drop the requirement that the momenta should not depend on the fields themselves, we are led to a string interacting with external fields of different types.

For another theory hidden in our program, take as a restriction the case where $(\hat{\Phi}^0,\hat{\Phi}^1)$ and $(\hat{\Phi}^2,\hat{\Phi}^3)$ are assumed to be components of Lorentz two-vectors. Then we would arrive at a theory of two strings in two dimensions.

We also would like to mention that there is a choice of restrictions such that the program would lead to the parametrized version of two scalar fields in two dimensions (with or without a potential). In this case two of the fields $\hat{\Phi}^\alpha$ would correspond to the coordinates of the deparametrized theory.

Let us now return to our general program. Because of the conditions on the functional dependence of the momenta $\hat{\Pi}_i$ on the first-order D-invariants $\hat{p}^I$ it is possible to express the descriptors $\hat{\xi}^A$ in terms of $\hat{\Pi}_i$ and $\hat{\Phi}^i$. The transformation (2.6) is to be generated in phase space by

$$H(\xi)=\int d^Ny\xi^A\mathcal{H}_A$$
$$=\int d^Ny(\xi^0\mathcal{H}_0+\xi^a\mathcal{H}_a)\ ,$$

with certain generators $\mathcal{H}_0,\mathcal{H}_a$. The ξ^A are the descriptors $\hat{\xi}^A$ expressed in phase-space variables. Our program amounts to the demand that

$$\{H(\xi_2),H(\xi_1)\}=H(\xi_3)$$
$$=\int d^Ny(\xi_3^0\mathcal{H}_0+\xi_3^a\mathcal{H}_a)\ , \tag{2.21}$$

where $\hat{\xi}_3^A$ are given by (2.7). That is, we demand that the Poisson bracket of two phase-space transformations $H(\xi_i)$ is a transformation corresponding to the commutator of two reparametrizations with descriptors $\hat{\xi}_i$. We have

$$\{H(\xi_2),H(\xi_1)\}=\int d^Nyd^Ny'\xi_2^A\xi_1'^B\{\mathcal{H}_A,\mathcal{H}'_B\}+\int d^Nyd^Ny'(\mathcal{H}_A\xi_1'^B\{\xi_2^A,\mathcal{H}'_B\}+\mathcal{H}'_B\xi_2^A\{\mathcal{H}_A,\xi_1'^B\})$$
$$+\int d^Nyd^Ny'\mathcal{H}_A\mathcal{H}'_B\{\xi_2^A,\xi_1'^B\}\ . \tag{2.22}$$

The prime on the ξ's and $\mathcal{H}$'s denotes, for simplicity, their dependence on y'.

In this expression we distinguish three different kinds of contributions: (i) terms with Poisson brackets among generators $\mathcal{H}_A$; (ii) terms with Poisson brackets among descriptors and generators; and (iii) terms with Poisson brackets among descriptors. Because of the arbitrariness of the descriptors these contributions can be handled separately when comparing both sides in (2.22). So let us for a moment assume that the descriptors ξ^A do not depend on $\hat{\Phi}^i,\hat{P}^I$. Thus we require that

$$\int d^Nyd^Ny'\xi_2^A\xi_1'^B\{\mathcal{H}_A,\mathcal{H}'_B\}=\int d^Ny[(\xi_{1,a}^0\xi_2^a)\mathcal{H}_0+(\xi_{1,b}^a\xi_2^b-e^{ab}\xi_1^0\xi_{2,b}^0)\mathcal{H}_a]-(1\leftrightarrow2)\ . \tag{2.23}$$

Here we took the ξ_3^A from (2.7); observe that for this special subset of transformations the $\bar{\delta}_i\hat{\xi}_j^A$ are absent.

In comparing both sides of (2.23) we derive the Poisson-bracket algebra of the generators $\mathcal{H}_A$, namely,

$$\{\mathcal{H}_0,\mathcal{H}'_0\}=(e^{ab}\mathcal{H}_b+e'^{ab}\mathcal{H}'_b)\partial_a\delta(y-y')\ ,$$
$$\{\mathcal{H}_a,\mathcal{H}'_0\}=\mathcal{H}_0\partial_a\delta(y-y')\ , \tag{2.24}$$
$$\{\mathcal{H}_a,\mathcal{H}'_b\}=(\mathcal{H}'_a\partial_b+\mathcal{H}_b\partial_a)\delta(y-y')\ .$$

This is of course a well-known algebra, written down first by Dirac[5] thirty years ago as the algebra of constraints for parametrized theories, and typical for any reparametrization-invariant theory with descriptors of the form (2.6).[6]

Next, allow the $\hat{\xi}^A$ to depend on D-invariant quantities. Again referring to (2.7) and (2.8) we note that ξ_3 contains derivatives with respect to $\hat{\Phi}^i$ and $\hat{P}^I$ (and spatial derivatives thereof). However these derivatives enter only linearly. On the other hand there are terms of type (iii) in (2.22) which as Poisson brackets are quadratic in these derivatives. So we conclude that the generators $\mathcal{H}_A$ must vanish: the physically attainable phase space is that hypersurface in phase space satisfying the constraints $\mathcal{H}_A=0$ (in addition to possible "primary" constraints).

Although the $\mathcal{H}_A$ vanish we still must compare those terms on both sides of (2.22) which are linear in the constraints:

$$\int d^Nyd^Ny'\mathcal{H}_A\xi_1'^B\{\xi_2^A,\mathcal{H}'_B\}-(1\leftrightarrow2)=\int d^Ny\mathcal{H}_A\bar{\delta}_2\xi_1^A-(1\leftrightarrow2)$$

or

$$\int d^Ny'd^Ny''\xi_2'^B\left[\frac{\mathcal{D}\xi_1^A}{\mathcal{D}\Phi''^i}\frac{\mathcal{D}\mathcal{H}'_B}{\mathcal{D}\Pi''_i}-\frac{\mathcal{D}\xi_1^A}{\mathcal{D}\Pi''_i}\frac{\mathcal{D}H'_B}{\mathcal{D}\Phi''^i}\right]=-\frac{\partial\xi_1^A}{\partial\Phi^i_{,ab}\cdots}\bar{\delta}_2\Phi^i_{,ab}\cdots-\frac{\partial\xi_1^A}{\partial\Pi_{i,ab}\cdots}\bar{\delta}_2\Pi_{i,ab}\cdots\ . \tag{2.25}$$

Now we are able to formulate the next step.

Step 3. Compare the coefficients of ξ_2^B on both sides of (2.25). This will yield a set of first-order differential equations for the constraints.

Example. For the string we have to compare both

$$\int dy^{1'}dy^{1''}\xi_2'^B\frac{\mathscr{D}\xi_1^A}{\mathscr{D}x''^\mu}\frac{\mathscr{D}\mathscr{H}'_B}{\mathscr{D}\pi''_\mu}=-\sum_{l=1}\frac{\partial\xi_1^A}{\partial x^{\mu[l]}}\bar{\delta}_2x^{\mu[l]} \tag{2.26}$$

and

$$\int dy^{1'}dy^{1''}\xi_2'^B\frac{\mathscr{D}\xi_1^A}{\mathscr{D}\pi''_\mu}\frac{\mathscr{D}\mathscr{H}'_B}{\mathscr{D}x''^\mu}=\sum_{j=0}\frac{\partial\xi_1^A}{\partial\pi_\mu^{[j]}}\bar{\delta}_2\pi_\mu^{[j]}\ . \tag{2.27}$$

Let us first evaluate (2.26). Substitute

$$\frac{\mathscr{D}\xi_1^A}{\mathscr{D}x''^\mu}=\sum_{l=1}(-1)^l\frac{\partial\xi_1^A}{\partial x^{\mu[l]}}\left[\frac{\partial}{\partial y^{1''}}\right]^l\delta(y^1-y^{1''})\ ,\quad \frac{\mathscr{D}\mathscr{H}'_B}{\mathscr{D}\pi''_\mu}=\sum_{j=0}(-1)^j\frac{\partial\mathscr{H}'_B}{\partial\pi'^{[j]}_\mu}\left[\frac{\partial}{\partial y^{1''}}\right]^j\delta(y^{1'}-y^{1''})$$

into the integrand. After integration by parts (2.26) becomes

$$\sum_{j,l}(-1)^{j+1}\frac{\partial\xi_1^A}{\partial x^{\mu[l]}}\left[\frac{\partial}{\partial y^1}\right]^{j+l}\left[\xi_2^B\frac{\partial\mathscr{H}_B}{\partial\pi_\mu^{[j]}}\right]=\sum_l\frac{\partial\xi_1^A}{\partial x^{\mu[l]}}\bar{\delta}_2x^{\mu[l]}\ . \tag{2.26'}$$

Since

$$\bar{\delta}_2x^{\mu[l]}=(-x^\mu_{,1}\xi_2^1-p^\mu\xi_2^0)^{[l]}\ , \tag{2.28}$$

the comparison of both sides in (2.26') yields

$$\sum_j(-1)^j\left[\xi_2^1\frac{\partial\mathscr{H}_1}{\partial\pi_\mu^{[j]}}\right]^{[j+l]}=(x^\mu_{,1}\xi_2^1)^{[l]}\ ,\quad \sum_j(-1)^j\left[\xi_2^0\frac{\partial\mathscr{H}_0}{\partial\pi_\mu^{[j]}}\right]^{[j+l]}=(p^\mu\xi_2^0)^{[l]}\ ,$$

and therefore

$$\frac{\partial\mathscr{H}_A}{\partial\pi_\mu^{[j]}}=0\ ,\quad j\geq1\ , \tag{2.26a}$$

$$\frac{\partial\mathscr{H}_1}{\partial\pi_\mu}=x^\mu_{,1}\ , \tag{2.26b}$$

$$\frac{\partial\mathscr{H}_0}{\partial\pi_\mu}=p^\mu\ . \tag{2.26c}$$

Before evaluating (2.27) we point out that although at the moment we deal with the string, Eqs. (2.26a) and (2.26b) would be the same for any theory with scalar fields, and only the right-hand side of (2.26c) changes depending on $p^\mu(\pi,x)$ because the transformation $\bar{\delta}$ has the same form (2.28) for any scalar field.

The analog to (2.26') is

$$\sum_{j,l}(-1)^{j+1}\frac{\partial\xi_1^A}{\partial\pi^{\mu[l]}}\left[\frac{\partial}{\partial y^1}\right]^{j+l}\left[\xi_2^B\frac{\partial\mathscr{H}_B}{\partial\xi_\mu^{[j]}}\right]=\sum_l\frac{\partial\xi_1^A}{\partial\pi^{\mu[l]}}\bar{\delta}_2\pi^{\mu[l]}\ . \tag{2.27'}$$

For the variation of $\hat{\pi}_\mu$ one finds

$$\bar{\delta}\hat{\pi}_\mu=-\hat{\pi}_{\mu,1}\hat{\xi}^1-\hat{\pi}_\mu\hat{\xi}^1_{,1}-a_\mu\hat{\xi}^0-b_\mu\xi^0_{,1}\ , \tag{2.29}$$

with

$$a_\mu=n^0\left[\frac{\hat{\pi}_\mu n^A}{n^0}\right]_{,A}+\frac{1}{n^0}n^0_{,1}b_\mu\ ,\quad b_\mu=\lambda e^{1/2}x_{\mu,1}\ .$$

The first two terms are obvious since $\hat{\pi}_\mu$ transforms as a density of weight $+1$ under PHT's; the others (a_μ and b_μ) have to be calculated from the functional form of $\hat{\pi}_\mu$. Inserting (2.29) into (2.27') yields the conditions

$$\sum_j (-1)^{j+1}\left[\xi_2^1\frac{\partial\mathscr{H}_1}{\partial x^{\mu[j]}}\right]^{[j+l]}=(\pi_{\mu,1}\xi_2^1+\pi_\mu\xi_{2,1}^1)^{[l]}=(\pi_\mu\xi_2^1)^{[l+1]}\ ,$$

$$\sum_j (-1)^{j+1}\left[\xi_2^0\frac{\partial\mathscr{H}_0}{\partial x^{\mu[j]}}\right]^{[j+l]}=(a_\mu\xi_2^0+b_\mu\xi_{2,1}^0)^{[l]}=[(a_\mu-b_{\mu,1})\xi_2^0]^{[l]}+(b_\mu\xi_2^0)^{[l+1]}\ ,$$

which result in

$$\frac{\partial\mathscr{H}_1}{\partial x^\mu}=0\ , \tag{2.27a}$$

$$\frac{\partial\mathscr{H}_1}{\partial x^\mu_{,1}}=\pi_\mu\ , \tag{2.27b}$$

$$\frac{\partial\mathscr{H}_1}{\partial x^{\mu[l]}}=0\ ,\ \ l\geq 2\ , \tag{2.27c}$$

$$\frac{\partial\mathscr{H}_0}{\partial x^\mu}=\lambda\left[\frac{1}{n^0}e^{1/2}x_{\mu,1}\right]_{,1}-n^0\left[\frac{\pi_\mu n^A}{n^0}\right]_{,A}\ , \tag{2.27d}$$

$$\frac{\partial\mathscr{H}_0}{\partial x^\mu_{,1}}=\lambda x_{\mu,1}e^{1/2}\ , \tag{2.27e}$$

$$\frac{\partial\mathscr{H}_0}{\partial x^{\mu[l]}}=0\ ,\ \ l\geq 2\ . \tag{2.27f}$$

Again notice that the differential equations for $\mathscr{H}_1$ are the same for any theory with scalar fields, since only the transformation property of $\hat{\pi}_\mu$ under PHT's is decisive, and not its specific functional dependence. The equations (2.26a)–(2.26c) and (2.27a)–(2.27f) for the constraints together with the alegbra (2.24) and some further self-consistency conditions are enough to uniquely determine the constraints and the dynamics.

Notice that Eq. (2.27d) contains a time derivative of the momentum. The derivation of the relations (2.26a)–(2.26c) and (2.27a)–(2.27f) points out what one would get in general in performing step 3. Namely, the equations for $\mathscr{H}_a$ will be differential equations, whereas there will be at least one equation which involves $\partial\mathscr{H}_0/\partial\Phi^{i[k]}$ (for some k) together with an expression containing time derivatives of momenta. Since we demand that all reparametrizations with descriptors of the form (2.6) be realizable as canonical transformations, we shall obtain a unique expression for such time derivatives in terms of derivatives of the constraints.

We wish to find the field-dependent transformation which translates in time fields in a given coordinate chart. We furthermore wish to implement this transformation in phase space in order to find the canonical equations of motion. It has to be stressed at this point that an infinitesimal time translation with descriptors of the form $\Psi^A=\epsilon\delta_0^A$ *may not* be canonically implemented. Such transformations do not belong to the subgroup derived in step 1. We shall however find a field-dependent descriptor which generates time translations of fields in a particular parametrization.

First write

$$\hat{\Psi}^A=\epsilon\delta_0^A=\hat{\xi}^a\delta_a^A+\hat{\xi}^0 n^A\ . \tag{2.30}$$

Recall that the normal n^A depends on the dynamical fields $\hat{\Phi}^\alpha$. Given a particular solution $\hat{\Phi}^2(y)$ it is of course possible to solve (2.30) for the descriptors:

$$\hat{\xi}^0(y)=\frac{\epsilon}{n^0(y)}\ ,\ \ \hat{\xi}_a(y)=-\frac{\epsilon n^a(y)}{n^0(y)}\ . \tag{2.30$'$}$$

We claim that a suitable parametrization may be chosen such that all solutions $\hat{\Phi}^\alpha$ of the reparametrization-covariant dynamical equations yield the same explicit functions $n^A(y)$, i.e., $n^A(\Phi(y)=n^A(\Phi'(y))$. It is even true that given a solution $\Phi(y)$ in a particular parametrization it is possible to find a parametrization such that the transformed $n^A(\Phi(y))$ may be made equal to any given *arbitrary* functions. This is a well-known result and we shall present only a short geometric argument (cf. Ref. 7). We consider a manifold with metric $g_{AB}(\Phi)$. Choose an arbitrary spacelike hypersurface and fix the y^0 coordinate to be constant, with points on the hypersurface fixed by the y^a. Pick an infinitesimal neighboring spacelike hypersurface with orthogonal separation $-N_\perp{}^2(\Delta t)^2$ where $N_\perp>0$ but otherwise arbitrary. Choose coordinates $y^a(t+\Delta t)=y^a+N^a\Delta t$ with separation $g_{ab}N^aN^b(\Delta t)^2$, N^a arbitrary. Then the metric g_{AB} is given by

$$g_{AB}=\begin{bmatrix} N_aN^a-N_\perp{}^2 & N_a\\ N_a & g_{ab}\end{bmatrix}\ ,\qquad g^{AB}=\frac{1}{N_\perp{}^2}\begin{bmatrix}-1 & N^a\\ N^a & N_\perp{}^2e^{ab}-N^aN^b\end{bmatrix}\ , \tag{2.31}$$

where

$$e^{ab}g_{ac}=\delta_c^a\ ,\ \ N^a=e^{ab}N_b\ .$$

The normal to the $y^0=$const hypersurface is then given by

$$n^A=N_\perp{}^{-1}\delta_0^A-N^aN_\perp{}^{-1}\delta_a^A\ . \tag{2.32}$$

The n^A may thus be fixed arbitrarily (as the $N_\perp$ and N^a are arbitrary) if we assume that under a reparametrization the Φ's and therefore the n^A's remain solutions of the dynamical equations.

We proceed as follows. Pick four explicit functions $\epsilon\hat{\xi}^A(y)$. Then the descriptor

$$\hat{\Psi}^A=\hat{\xi}^a(y)\delta^0_a+n^A\hat{\xi}^0(y)$$

describes time translations of those fields with parametrizations such that (2.30′) are fulfilled. In particular we have

$$\dot{\Phi}^i=\left\{\Phi^i,\int d^Ny'(N^{a'}\mathscr{H}'_a+N'_\perp\mathscr{H}'_0)\right\},$$
$$\dot{\Pi}_i=\left\{\Pi_i,\int d^Ny'(N^{a'}\mathscr{H}'_a+N'_\perp\mathscr{H}'_0)\right\}, \tag{2.33}$$

where the $N^a,N_\perp$ are given explicit functions $N^A(y)$.

Example. Due to (2.26a) and (2.27c)–(2.27f) we find that

$$\dot{x}^\mu=N^1\frac{\partial\mathscr{H}_1}{\partial\pi_\mu}+N_\perp\frac{\partial\mathscr{H}_0}{\partial\pi_\mu} \tag{2.34}$$

and

$$\dot{\pi}^\mu=\left[N_\perp\frac{\partial\mathscr{H}_0}{\partial x_{\mu,1}}\right]_{,1}+\left[N^1\frac{\partial\mathscr{H}_1}{\partial x_{\mu,1}}\right]_{,1}-N_\perp\frac{\partial\mathscr{H}_0}{\partial x_\mu}\,. \tag{2.35}$$

Equation (2.34) is merely the phase-space analog of our definition (2.20) of the momentum functional $\hat{\pi}^\mu$. Referring to equations (2.26b) and (2.26c) we find

$$\dot{x}^\mu=N^1x^\mu_{,1}+N_\perp(x_{,1}{}^2)^{-1/2}\pi^\mu$$

or

$$\pi^\mu=\lambda(x_{,1}{}^2)^{1/2}(n^0\dot{x}^\mu+n^1x^\mu_{,1})\,.$$

The relation (2.35) is to be substituted into Eq. (2.27d). We are now prepared to carry out the final step.

Step 4. Substitute the expression for $\dot{\Pi}^i$ obtained above into the relations derived in step 3, thereby achieving a set of differential equations for the constraints $\mathscr{H}_A$. Replace the field variables $n^A(\Phi)$ by the explicit functions

$$\delta^A_0N_\perp{}^{-1}(y)-\delta^A_aN_\perp{}^{-1}(y)N^a(y)\,.$$

Find the most general solution for the constraints $\mathscr{H}_A$.

Example. Substituting (2.27b) and (2.27c) into (2.35) we find that

$$\dot{\Pi}^\mu=(N_\perp\lambda x^\mu_{,1}e^{1/2})_{,1}+(N^1\Pi^\mu)_{,1}-N_\perp\frac{\partial\mathscr{H}_0}{\partial x_\mu}\,.$$

Inserting this expression into (2.27d) we obtain an identity. The unique solution of the remaining equations (2.26) and (2.27) consistent with the algebra (2.24) is

$$\mathscr{H}_1=\pi^\mu x_{\mu,1}$$

and

$$\mathscr{H}_0=\frac{\lambda}{2}(x_{,1}{}^2)^{-1/2}(\lambda^2\pi^2+x_{,1}{}^2)\,.$$

These are nothing but the phase-space analogs of the relations among the momentum functionals following from the definition of the D-invariant objects $\hat{p}^2$ [cf. Eq. (2.14)].

Observe that $\mathscr{H}_1$ has the form $\mathscr{H}_1=\Pi_{,2}\Phi^2_{,1}$ for any scalar theory, since the Eqs. (2.26a) and (2.26b) and (2.27a) and (2.27b) contain no reference to the functional dependence of the momenta on the D-invariant $\hat{p}^2$.

Finally we note that the resulting dynamical equations (2.34) and (2.35) are those of the closed Nambu-Goto string in Hamiltonian form.

III. GRAVITATION

The fields are now the pseudo-Riemannian metric components $\hat{g}_{\alpha\beta}(y^A)$, the $y^A\to x^\mu$ parametrizing four-dimensional space-time. If $\delta x^\mu=\hat{\Psi}^\mu$ the metric transforms as

$$\bar{\delta}\hat{g}_{\alpha\beta}=-(\hat{\Psi}_{\alpha;\beta}+\hat{\Psi}_{\beta;\alpha})\,, \tag{3.1}$$

where the semicolon denotes covariant derivatives with respect to the four-metric. From

$$\bar{\delta}\hat{g}_{\alpha\beta}=-\hat{g}_{\alpha\gamma}\hat{\Psi}^\gamma_{,\beta}+\hat{g}_{\beta\gamma}\hat{\Psi}^\gamma_{,\alpha}+\hat{g}_{\alpha\beta,\gamma}\hat{\Psi}^\gamma)\,, \tag{3.1′}$$

one immediately can see that the three-metric $\hat{g}_{ab}$ is D-invariant ($\bar{\delta}\hat{g}_{ab}$ does not contain derivatives with respect to x^0), whereas the components $\hat{g}_{00}$ and $\hat{g}_{0a}$ are not. It will prove useful to express the metric $\hat{g}_{\alpha\beta}$ in terms of the lapse and shift functions $\hat{N}_\perp$ and $\hat{N}^a$, and the induced metric on the hypersurfaces [cf. (2.31)].

The three-metric is D-invariant, while $\hat{N}_\perp$ and $\hat{N}_a$ are, respectively, a scalar and covectors under PHT's.

The extrinsic curvature is

$$K_{ab}=(2\hat{N}_\perp)^{-1}(\hat{N}_{a\,|\,b}+\hat{N}_{b\,|\,a}-\hat{g}_{ab,0})\,, \tag{3.2}$$

where the vertical bar denotes the covariant derivatives with respect to the hypersurface metric. According to our general remarks in the previous section it is a D-invariant quantity, as one of course might also verify explicitly.

Let us now perform step 1 of our program in order to see that K_{ab} (and spatial derivatives thereof) is the only first-order D-invariant. We again make the decomposition $\hat{\Psi}^\mu=\delta^\mu_a\hat{\xi}^a+n^\mu\hat{\xi}^0$. The descriptors $\hat{\xi}^\mu$ may depend on $\hat{g}_{\alpha\beta,a\cdots b}$ (that is, the metric and spatial derivatives of the metric) and $\hat{g}_{\alpha\beta,0a\cdots b}$ (spatial derivatives of $\hat{g}_{\alpha\beta,0}$). Express everything in terms of the three-metric and lapse and shift functions,

$$\hat{\xi}^\mu=\bar{\xi}^\mu(\hat{g}_{ab,c\cdots d},\hat{N}_{a,c\cdots d},\hat{N}_{\perp,c\cdots d},\hat{g}_{ab,0c\cdots d},\hat{N}_{a,0c\cdots d},\hat{N}_{\perp,0c\cdots d})$$

where of course the order of spatial derivatives for each variable may be different). Now by (3.2) it is possible to express the "velocity" $\hat{g}_{ab,0}$ in terms of the extrinsic curvature, the metric $\hat{g}_{ab}$, lapse and shifts, and spatial derivatives of lapse and shifts. We have

$$\hat{\xi}^\mu=\bar{\bar{\xi}}^\mu(\hat{g}_{ab,c\cdots d},\hat{N}_{a,c\cdots d},\hat{N}_{\perp,c\cdots d},K_{ab,c\cdots d},\hat{N}_{a,0c\cdots d},\hat{N}_{\perp,0c\cdots d})\ .$$

For simplicity we drop the two bars on $\bar{\bar{\xi}}^\mu$ and regard $\hat{\xi}^\mu$ as a function of the above variables. Inserting this form together with the decomposition (2.6) into (2.1), we obtain a commutator $\hat{\Psi}^\mu_3$ containing terms with second time derivatives, terms with first time derivatives, and terms with no time derivatives of the descriptors $\hat{\xi}^\mu$.

First we isolate those which contain second time derivatives of $\hat{\xi}^\mu$ and set the sum equal to zero, thereby obtaining

$$0=\sum_{n=0}\frac{\partial\hat{\xi}_1{}^\mu}{\partial\hat{N}_{a,0b_1\cdots b_n}}(\hat{g}_{a\nu}\hat{\Psi}^\nu_{2,00})_{,b_1\cdots b_n}+\sum_{n=0}\frac{\partial\hat{\xi}^\mu_1}{\partial\hat{N}_{\perp,0b_1\cdots b_n}}[(-\hat{g}^{00})^{-1/2}\delta^0_\nu\hat{\Psi}^\nu_{2,00}]_{,b_1\cdots b_n}$$

for n indices $b_1,\ldots,b_n$ (minus another term for which the indices 1 and 2 are exchanged). From this it follows that

$$\frac{\partial\hat{\xi}^\mu_1}{\partial N_{a,0b_1\cdots b_n}}=\frac{\partial\hat{\xi}^\mu_1}{\partial N_{\perp,0b_1\cdots b_n}}=0\ ,\quad \text{all } n\ , \tag{3.3}$$

which implies that the descriptors are not allowed to depend on time derivatives of the lapse and shift functions.

Next we set the sum of terms containing first time derivatives of the $\hat{\xi}^\mu$ equal to zero to obtain

$$0=\sum_{n=0}^{Z}\frac{\partial\hat{\xi}^\mu_1}{\partial\hat{N}_{a,b_1\cdots b_n}}(\hat{g}_{a\nu}\hat{\Psi}^\nu_{2,0})_{,b_1\cdots b_n}+\sum_{n=0}^{Z'}\frac{\partial\hat{\xi}^\mu_1}{\partial\hat{N}_{\perp,a_1\cdots a_n}}[(-g^{00})^{-1/2}\delta^0_\nu\hat{\Psi}^\nu_{2,0}]_{,a_1\cdots a_n}\ . \tag{3.4}$$

So we immediately conclude that

$$\frac{\partial\hat{\xi}_1{}^\mu}{\partial\hat{N}_{a,b_1\cdots b_n}}=\frac{\partial\hat{\xi}_1{}^\mu}{\partial\hat{N}_{\perp,b_1\cdots b_n}}=0\ ,\quad\text{all } n\ . \tag{3.5}$$

Consequently we deduce that all $\hat{N}_\perp,\hat{N}_a$, and $\hat{g}_{ab,0}$ dependence in the descriptors enters in K_{ab} dependence and arbitrary spatial derivatives thereof. The extrinsic curvature K_{ab} is the only D-invariant object with first-order time derivatives. So the general form of our descriptors in the decomposition (2.6) is

$$\hat{\xi}^\mu=\hat{\xi}^\mu(x^\nu;g_{ab},K_{ab},\ \text{arbitrary spatial derivatives})$$

or

$$\hat{\xi}^\mu=\hat{\xi}^\mu[x^\nu;g_{ab},K_{ab}]\ .$$

Here the K_{ab} play the role of the quantities $\hat{P}^I$ defined in Sec. II.

Step 2. From the original fields $\hat{g}_{\alpha\beta}$ only six are D-invariant; these are the $\hat{g}_{ab}$. They represent the $\hat{\Phi}^i$ of our program. We would like to define six momenta $\hat{\pi}^{ab}$. These should be functionals of g_{ab} and K_{ab}. From these we construct a symmetric contratensor density of rank 2 and weight 1,

$$\begin{aligned}\hat{\pi}^{ab}&=\gamma^{-1}g^{1/2}(\alpha e^{ab}e^{mn}-\beta e^{am}e^{bn})K_{mn}\\&=\gamma^{-1}g^{1/2}(\alpha e^{ab}K-\beta K^{ab})\ .\end{aligned} \tag{3.6}$$

Assuming we have at our disposal one dimensional constant, Newton's gravitational constant $\gamma,\hat{\pi}^{ab}$ above is the only momentum of the correct dimension one may construct which is polynomial in the fields (one might try adding a nonpolynomial term $\gamma^{-1}g^{1/2}e^{ab}R^{1/2}$, for example). The constants α and β are dimensionless parameters. The condition that $\hat{\pi}^{ab}(\hat{g}_{mn},\hat{K}_{mn})$ can be solved for K_{mn} leads to certain restrictions on these parameters. Take the trace on both sides of (3.6),

$$\hat{\pi}^a_a=\pi=\gamma^{-1}g^{1/2}(d\alpha K-\beta K)\ ,$$

where d is the dimension of the hypersurface (let us leave this dimension open for a while, and only later fix it to $d=3$). If $d\alpha-\beta\neq 0$ the previous expression allows us to write K in terms of the metric fields and $\hat{\pi}$. Inserting this into (3.6) gives

$$\hat{\pi}^{ab}=\alpha(d\alpha-\beta)^{-1}e^{ab}\pi-\gamma^{-1}g^{1/2}\beta K^{ab}$$

and so only for $\beta\neq 0$ do we have

$$K^{ab}=\gamma g^{-1/2}\beta^{-1}[\alpha(d\alpha-\beta)^{-1}e^{ab}\pi-\pi^{ab}] \; . \quad (3.7)$$

Without loss of generality we may take $\beta=1$, then the only restriction on α is $\alpha\neq d^{-1}$.

We are now prepared to derive differential equations for the constraints $\mathscr{H}_\mu$.

Step 3. Defining

$$V_{\mu ab}^{i\cdots j}:=\frac{\partial\mathscr{H}_\mu}{\partial\pi^{ab}_{,i\cdots j}} \; , \quad (3.8)$$

we calculate

$$\int d^3x' d^3x'' \xi_2^\mu \frac{\mathscr{D}\xi_1^\nu}{\mathscr{D}g''_{ab}}\frac{\mathscr{D}\mathscr{H}'_\mu}{\mathscr{D}\pi''^{ab}}=\sum_{Z',Z''}(-1)^{Z'}\frac{\partial\xi_1^\nu}{\partial g_{ab,i_1\cdots i_{Z'}}}\partial_{i_1}\cdots\partial_{i_{Z'}}\partial_{m_1}\cdots\partial_{m_{Z''}}(\xi_2^\mu V_{\mu ab}^{m_1\cdots m_{Z''}}) \quad (3.9)$$

for Z' indices $i_1,\ldots,i_{Z'}$ and Z'' indices $m_1,\ldots,m_{Z''}$. This has to be compared with

$$-\sum_Z \frac{\partial\xi_1^\nu}{\partial g_{ab,i_1\cdots i_Z}}\bar{\delta}_2 g_{ab,i_1\cdots i_Z} \; . \quad (3.10)$$

In the decomposition (2.6) the variation of the metric fields becomes

$$\bar{\delta}g_{ab}=2K_{ab}\xi^0-g_{ab,c}\xi^c-(g_{ac}\xi^c_{,b}+g_{bc}\xi^c_{,a}) \; ,$$

so the comparison of (3.9) with (3.10) amounts to

$$\sum_{Z,Z'}(-1)^{Z'+1}\frac{\partial\xi_1^\nu}{\partial g_{ab,i_1\cdots i_Z}}\partial_{i_1}\cdots\partial_{i_Z}\partial_{m_1}\cdots\partial_{m_{Z'}}(\xi_2^0 V_{0ab}^{m_1\cdots m_{Z'}})=\sum_Z\frac{\partial\xi_1^\nu}{\partial g_{ab,i_1\cdots i_Z}}(2K_{ab}\xi_2^0)_{,i_1\cdots i_Z} \quad (3.11a)$$

and

$$\sum_{Z,Z'}(-1)^{Z'}\frac{\partial\xi_1^\nu}{\partial g_{ab,i_1\cdots i_Z}}\partial_{i_1}\cdots\partial_{i_Z}\partial_{m_1}\cdots\partial_{m_{Z'}}(\xi_2^c V_{cab}^{m\cdots n})=\sum_Z\frac{\partial\xi_1^\nu}{\partial g_{ab,i_1\cdots i_Z}}(g_{ab,c}\xi_2^c+2g_{c(a}\xi^c_{2,b)}) \; . \quad (3.11b)$$

From these we infer the differential equations

$$V_{0ab}=\frac{\partial\mathscr{H}_0}{\partial\pi^{ab}}=-2K_{ab} \; , \quad (3.12)$$

$$V_{0ab}^{m_1\cdots m_Z}=0 \; , \quad Z>0 \; , \quad (3.13)$$

$$V_{cab}=\frac{\partial\mathscr{H}_c}{\partial\pi^{ab}}=-2\,{}^3\Gamma_{cab} \; , \quad (3.14)$$

$$V_{cab}^m=\frac{\partial\mathscr{H}_c}{\partial\pi^{ab}_{,m}}=-g_{ca}\delta_b^m-g_{cb}\delta_a^m \; , \quad (3.15)$$

$$V_{cab}^{m_1\cdots m_Z}=0 \; , \quad Z>1 \; . \quad (3.16)$$

Similarly comparing in (2.25) coefficients of derivatives of the descriptors with respect to momenta one obtains conditions for the derivatives of the constraints with respect to the metric. The calculation is laborious, but straightforward, so we shall simply give the results:

$$\frac{\partial\mathscr{H}_l}{\partial g_{ij,m_1\cdots m_Z}}=0 \; , \quad Z>1 \; , \quad (3.17)$$

$$\frac{\partial\mathscr{H}_l}{\partial g_{ij,n}}=-\pi^{in}\delta_l^j-\pi^{jn}\delta_l^i+\pi^{ij}\delta_l^n \quad (3.18)$$

$$\frac{\partial\mathscr{H}_l}{2g_{ij}}-\left[\frac{\partial\mathscr{H}_l}{\partial g_{ij,n}}\right]_{,n}=-\pi^{ij}_{\;\;,l} \quad (3.19)$$

and

$$\frac{\partial\mathscr{H}_0}{\partial g_{ij,m_1\cdots m_Z}}=0 \; , \quad Z>2 \; , \quad (3.20)$$

$$\frac{\partial\mathscr{H}_0}{\partial g_{ij,kl}}=-\tfrac{1}{2}g^{1/2}(\tilde{G}^{ikjl}+\tilde{G}^{iljk})+(\alpha-1)g^{1/2}e^{ij}e^{kl} \; , \quad (3.21)$$

$$\frac{\partial\mathscr{H}_0}{\partial g_{ij,k}}+2\left[\frac{\partial\mathscr{H}_0}{\partial g_{ij,kl}}\right]_{,l}=g^{1/2}\,{}^3\Gamma^k_{mn}[\tilde{G}^{imjn}-(\alpha-1)e^{ij}e^{mn}] \; , \quad (3.22)$$

where

$$\tilde{G}^{ikjl}=e^{ik}e^{jl}-e^{ij}e^{lk} ,$$

and finally

$$N^l\frac{\partial\mathscr{H}_l}{\partial g_{ab}}-\left[N^c\frac{\partial\mathscr{H}_l}{\partial g_{ab,c}}\right]_{,c}+N_\perp\frac{\partial\mathscr{H}_0}{\partial g_{ab}}-\left[N_\perp\frac{\partial\mathscr{H}_0}{\partial g_{ab,c}}\right]_{,c}+\left[N_\perp\frac{\partial\mathscr{H}_0}{\partial g_{ab,cd}}\right]_{,cd}=\dot{\pi}^{ab} . \quad (3.23)$$

From the demand (2.33) that time translations be realized as canonical transformations, we observe that (3.23) is identically satisfied. Just as in our earlier example the comparison of coefficients does not determine the derivative of $\mathscr{H}_0$ with respect to the fields: $\partial\mathscr{H}_0/\partial g_{ab}$.

Step 4. Equations (3.17)—(3.19), along with the Poisson-bracket relation (2.24c) may be easily integrated. We find

$$\mathscr{H}_l=-2\pi_l{}^j{}_{|j} . \quad (3.24)$$

We recognize Eq. (2.24b) as the condition that $\mathscr{H}_0$ transforms as a scalar density under PHT's. Referring to (3.20)—(3.22) we observe that $\mathscr{H}_0$ must contain a scalar term constituted solely out of g_{ab}'s up to and including second derivatives. The only such scalar is the curvature scalar R. Since

$$\frac{\partial R}{\partial g_{ij,kl}}=-\tfrac{1}{2}(\tilde{G}^{ikjl}+\tilde{G}^{iljk}) ,$$

we deduce from (3.21) that $\alpha=1$. Indeed since

$$\frac{\partial R}{\partial g_{ij,k}}+2\left[\frac{\partial R}{\partial g_{ij,kl}}\right]_{,l}={}^3\Gamma^k_{mn}\tilde{G}^{imjn}$$

and $\mathscr{H}_0$ by (3.13) contains no $\pi^{ij}{}_{|k}$ dependence, the remaining terms in $\mathscr{H}_0$ must depend on undifferentiated g_{ab}'s and they must be quadratic in π^{ab} by (3.12). Thus we find

$$\mathscr{H}_0=\gamma^{-1}(\tfrac{1}{2}g^{-1/2}G_{ijkl}\pi^{ij}\pi^{kl}-g^{1/2}R) ,$$

with

$$G_{ijkl}=g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl} .$$

This $\mathscr{H}_0$ does indeed fulfill the Poisson-bracket relation (2.24a). We should point out here that this derivation of $\mathscr{H}_0$ has been greatly simplified over that due to Hojman *et al.*,[3] since we have assumed a functional dependence for the momentum variables $\hat{\pi}^{ab}$, and as a consequence of our demand that the reparametrization group be canonically implementable, we have obtained linear differential equations containing $\partial\mathscr{H}_0/\partial g_{ab,c\cdots d}$.

If we suppose we have available another dimensional constant Λ of dimension mass $(\text{length})^{-3}$, then it is not possible to construct any further momenta of the correct dimension. However, a term of the correct dimension, $\Lambda g^{1/2}$, may be added to $\mathscr{H}_0$ without affecting equations (3.12)—(3.23) since $\partial\mathscr{H}_0/\partial g_{ab}$ does not appear in these relations. Since $\mathscr{H}_0$ contains no derivatives of π^{ab}, this new $\mathscr{H}_0$ continues to fulfill the algebra (2.24a).

The equations of motion following from the Hamiltonian

$$H=\int d^3x\,(N^a\mathscr{H}_a+N_\perp\mathscr{H}_0) ,$$

with N^a and $N_\perp$ arbitrary functions,

$$\mathscr{H}_0=\gamma^{-1}(\tfrac{1}{2}g^{-1/2}G_{ijkl}\pi^{ij}\pi^{kl}-g^{1/2}R)+\Lambda g^{1/2} ,$$

and $\mathscr{H}_a$ given by (3.24), are Einstein's equations in Hamiltonian form, with cosmological constant Λ.

IV. CONCLUSION

We have formulated a general program for realizing general coordinate tranformations as canonical transformations for systems with Bose fields and dynamical equations of at most second order. We have applied the technique to the relativistic string and to general relativity. In a subsequent article[10] we shall extend the program to include gauge transformations. From the examples it is obvious that the local symmetries alone do not provide a unique dynamics. There may be some arbitrariness in choosing the momenta and/or in imposing the restrictions R. Indeed the dynamics become *defined* by the transformation properties of the fields, the functional dependence of the momenta in terms of zero- and first-order D invariants and the restrictions R. The examples also demonstrate that the choice of momenta is limited in that the generators $\mathscr{H}_\mu$ of canonical transformations must obey certain differential equations, including a Poisson-bracket algebra. Every choice of momenta which is consistent in this sense corresponds to a dynamical system. Therefore any arbitrariness left over is welcome. Even a consistent set of momenta does not specify the dynamics uniquely; one must pose further restrictions. In the case of general relativity the restriction to only one dimensional constant excludes a cosmological term. Although we did not begin with an action, clearly the dynamical equations we ultimately obtain may be derived from the Lagrangian density

$$\mathscr{L}[\hat{\Phi},\hat{\dot{\Phi}}]=\hat{\Pi}_a(\hat{\Phi},\hat{\dot{\Phi}})\hat{\dot{\Phi}}^a$$

(recall that $\mathscr{H}\approx 0$). We view our approach as a physically and group-theoretically well-motivated alternative to the Dirac-Bergmann algorithm applied to the foregoing Lagrange density (cf. Ref. 8 for a review of the Dirac-Bergmann Formalism).

Now let us argue that the realizability of a subgroup of the field-dependent reparametrization group is a necessary element in the reduction of a second-order reparametrization-covariant theory to deterministic Hamiltonian form. Our objective is not only to reduce the theory to first-order form, but to classify all solutions via physically inequivalent initial-data sets (hence the term deterministic). We note first that due to the general covariance, a given set of fields and velocities on the initial hypersurface will not uniquely evolve off the surface. We would like our theory to be such that time evolution may be uniquely fixed through conditions on the initial data (otherwise we will *a fortiori* not have achieved an initial-value formulation).

An equivalent statement of this demand is that we wish to be able to construct observables of the theory in terms of initial data. A quantity is obviously not an observable (invariant under coordinate transformations) if its evolution off an initial hypersurface is not unique.

Since we may hope to achieve unique evolution through conditions on initial data and since, furthermore, we do not want to eliminate any solutions through these conditions, we must demand that the general transformation coordinate group be implementable as a transformation group on initial data, i.e., realizable as canonical transformations. Coordinate conditions on initial data should then ideally select one and only one representative data set from each equivalence class under these coordinate transformations.

Thus we are assured that coordinates may be fixed in the string model and in gravitation theory through the imposition of conditions on canonical coordinates (of course, one must also prescribe the lapse $N_\perp$ and the shifts N^a). The group element with infinitesimal descriptors of the form

$$\Psi^\mu=\delta^\mu_a\xi^a(g,\pi)+n^\mu\xi^0(b,\pi)$$

transforms canonical coordinates into invariant functionals (in general nonlocal). The form of the coordinate conditions is determined by a canonical group element.

To illustrate this point, and also to motivate a logical generalization of the preceding analysis, we consider an example from electromagnetism. Suppose we require that the electromagnetic potentials fulfill the Coulomb-gauge condition $\partial_a A_a=0$. Then there exists a corresponding field-dependent gauge transformation Λ which transforms any given field A into a functional $\widetilde{A}(A)$ satisfying the gauge condition. In particular

$$\widetilde{A}_a=A_a-\partial_a\Delta^{-1}\partial_b A_b \ ,$$

so

$$\Lambda(A)=\Delta^{-1}\partial_b A_b \ .$$

This particular gauge transformation is implementable in phase space, and it suggests that here, as in reparametrization-covariant theories, the enlarged field-dependent gauge group plays a crucial role. It is precisely this group which one would wish to unitarily implement in the quantum theory to obtain gauge equivalence. One may broaden the concept of D-invariance to include invariance under instantaneously vanishing gauge transformations, and thereby obtain conditions which coupled metric and gauge fields must satisfy if this enlarged group is to be realized in phase space. This will be the topic of the following paper.[9]

Open relativistic strings are of considerable physical interest, but the dynamical equations may not be consistently derived from a Lagrangian.[10] However reparametrization-covariant Hamiltonian equations may be derived applying the method described above, but only for parametrizations which are singular at the ends of the string. This problem is discussed in another paper.[11]

Of course, the program outlined in Sec. II, and applied to the string and to gravitation may also be applied to any reparametrization-covariant theory. For instance, a relativistic particle or string, with spin described by fermionic variables, typically has second-class constraints, and although the steps 1–4 may be performed, the arguments are somewhat different from those in first-class examples. Also, relativistic particles, interacting with themselves, with external or with dynamical fields may be fruitfully treated within this framework. One may also apply the program to obtain alternative strings.[12] In fact, it is clear that depending upon the choice of initial fields and momenta, one may apply the method to describe the Hamiltonian dynamics of the parametrized version of at least all theories derivable from a Lagrangian. It is possible to treat theories with higher-than-first-order field equations, too. This is only a rather incomplete list of topics; work on some of them is in progress and will appear elsewhere.

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APPENDIX A

In this appendix we would like to indicate, for the induced metric example, how the demand that the field-dependent reparametrizations with descriptors $\hat{\xi}^A$ depend on D-invariants of at most first order implies information on the dynamics of the fields $\hat{\Phi}^\alpha$.

In the commutator $\hat{\Psi}_3^A$ of two reparametrizations as given by (2.7) there appears a term $\bar{\delta}_2\hat{\xi}_1^A-\bar{\delta}_1\hat{\xi}_2^A$. The relevant transformations constitute a group only if this is a D-invariant of first order. For the example with scalar fields we have

$$\bar{\delta}_2\hat{\xi}_1^A=\sum_{l=0}^{L}\frac{\partial\hat{\xi}_1^A}{\partial\hat{\Phi}^{i[l]}}\bar{\delta}_2\hat{\Phi}^{i[l]}+\sum_{l=0}^{L'}\frac{\partial\hat{\xi}_1^A}{\partial\hat{p}^{i[l]}}\bar{\delta}_2\hat{p}^{i[l]} \tag{A1}$$

with $\hat{p}^i$ given by (2.14). Now

$$\bar{\delta}_2\hat{\Phi}^i=-\Phi^i_{,1}\xi_2^1-p^i\xi_2^0 \tag{A2}$$

is a first-order D-invariant, and so are $\bar{\delta}_2\hat{\Phi}^{i[l]}$. It remains to check whether the second term in (A1) has the required form, too. Because of the arbitrariness of the descriptors we have to investigate $\bar{\delta}\hat{p}^{i[l]}$. From the definition of $\hat{p}^i$ and (A2) one finds

$$\bar{\delta}\hat{p}^i=-\hat{p}^i_{,1}\xi^1-\hat{\Phi}^i_{,1}e^{11}\hat{\xi}^0_{,1}-n^An^B\hat{\Phi}^i_{,A;B}\hat{\xi}^0 , \tag{A3}$$

where the semicolon denotes the covariant derivative with respect to g_{AB}. The first two terms are D-invariants of first order, in the last one the expression $n^An^B\hat{\Phi}_{,A;B}$ prevents $\bar{\delta}\hat{p}^i$ from being a first-order D-invariant. Since

$$n^An^B\hat{\Phi}^i_{,A;B}\equiv g^{AB}\hat{\Phi}^i_{,A;B}+e^{11}\hat{\Phi}^i_{,A}\Gamma^A_{11}$$

and

$$\hat{\Phi}^i_{,A}\Gamma^A_{11}\equiv(\hat{p}^j\hat{\Phi}_{j,11})\hat{p}^j+e^{11}(\hat{\Phi}^j_{,1}\hat{\Phi}_{j,11})\hat{\Phi}^i_{,1} ,$$

the term preventing the closure of the reparametrization algebra is proportional to $g^{AB}\hat{\Phi}^i_{,A;B}$. Of course we are not asked to set this to zero; it is sufficient to equate this to a D-invariant function,

$$g^{AB}\hat{\Phi}^i_{,A;B}=f^i(y^A;\hat{\Phi}^j,\Phi^{j[l]}_{,1},\hat{p}^{j[m]}) , \tag{A4}$$

and we declare this as our equation of motion. As long as f^i is not specified these will, of course, be empty. However, observe that there are restrictions on f^i. The left-hand side of (A4) is perpendicular to $\hat{\Phi}_{,A}$. Therefore, consistency demands that $f^i\hat{\Phi}_{i,A}=0$. Hence

$$f^i=\hat{\gamma}_j\frac{\partial\hat{p}^j}{\partial\hat{\Phi}_{i,0}}$$

with certain coefficients $\hat{\gamma}_j$. (The proof that $\partial\hat{p}^j/\partial\Phi_{i,0}$ are perpendicular to $\Phi^i_{,A}$ is given in Sec. II.) In calculating that expression one finds that

$$\begin{aligned}f^i&=\gamma_j(\delta^{ji}+\hat{p}^j\hat{p}^i-e^{11}\hat{\Phi}^j_{,1}\hat{\Phi}^i_{,1})\\&\equiv\gamma_j(\delta^{ji}-g^{AB}\hat{\Phi}^j_{,A}\hat{\Phi}^i_{,B}) ,\end{aligned} \tag{A5}$$

where γ_j are D-invariant scalars of at most first order.

For the special case of the string the only scalar at hand is zero, and therefore we obtain the equations of motion

$$g^{AB}x^\mu_{,A;B}=0 ,$$

which are identical with the field equations derived from the Nambu-Goto action.

APPENDIX B

The program of realizing the general coordinate transformations in a phase space has been performed in general in Sec. II for theories for which a metric can be defined in terms of the fields. In this appendix we would like to argue that the outcome is independent of the metric one prefers to choose. Or, more precisely, if it is possible to define different metrics the program can only be worked out consistently if the metrics are related, i.e., if there are constraints among the fields right from the beginning.

Assume that it is possible to define different metrics $g^{(i)}_{AB}$ in terms of the fields. One way of constructing a normal is the following. Define a linear combination

$$G_{ab}(\lambda)=\lambda_i g^{(i)}_{AB} , \tag{B1}$$

with scalars λ_i, and λ denoting the i-tuples $(\lambda_1,\lambda_2,\dots)$. The normal vector with respect to $G_{AB}(\lambda)$ is

$$n^A(\lambda)=-\frac{G^{A0}(\lambda)}{[-G^{00}(\lambda)]^{1/2}} . \tag{B2}$$

Now compare two "lapse" and "shift" decompositions

$$\begin{aligned}\Psi^A&=\delta^A_a\xi^a(\lambda)+n^A(\lambda)\xi^0(\lambda)\\&=\delta^A_a\xi^a(\lambda')+n^A(\lambda')\xi^0(\lambda') ,\end{aligned} \tag{B3}$$

where the λ signifies that the descriptors depend on $n^A(\lambda)$:

$$\begin{aligned}\xi^0(\lambda)&=\frac{1}{n^0(\lambda)}\Psi^0 ,\\ \xi^a(\lambda)&=\Psi^a-\frac{n^a(\lambda)}{n^0(\lambda)}\Psi^0 .\end{aligned} \tag{B4}$$

In the main text we proved (in discussing step 1) that the demand that the commutator of two reparametrizations does not contain time derivatives of the Ψ^A is equivalent to the requirement that the descriptors ξ^A are D-invariant. The proof rested solely on the transformation property of the normal which in turn is derived from the tensor character of the metric. Furthermore, D-invariance has a meaning independent of the decomposition. Hence the

$\xi^A(\lambda)$ and $\xi^A(\lambda')$ can for each A differ only by a D-invariant factor. Consider first the time components. From

$$n^0(\lambda)\xi^0(\lambda)=n^0(\lambda')\xi^0(\lambda')$$

and

$$\xi^0(\lambda)=u(\lambda,\lambda')\xi^0(\lambda')$$

we find

$$n^0(\lambda')=un^0(\lambda)\ , \tag{B5}$$

and therefore

$$G^{00}(\lambda')=u^2G^{00}(\lambda)\ .$$

Next, for the spatial indices we have the identity

$$\xi^a(\lambda)-\xi^a(\lambda')=n^a(\lambda')\xi^0(\lambda')-n^a(\lambda)\xi^0(\lambda)$$
$$=[n^a(\lambda')-n^a(\lambda)u]\xi^0(\lambda')\ ,$$

where the previous result has been used. The left-hand side being D-invariant, the right-hand side will only be D-invariant if $n^a(\lambda')-un^a(\lambda)$ is D-invariant, which implies

$$n^a(\lambda')=un^a(\lambda)\ . \tag{B6}$$

Hence we deduce that

$$G^{0a}(\lambda')=u^2G^{0a}(\lambda)\ .$$

Now compare the variation of the normal for different λ's:

$$\bar{\delta}n^A(\lambda)=-n^A(\lambda)_{,B}\Psi^B+n^B(\lambda)\Psi^A_{,B}$$
$$-\frac{1}{n^0(\lambda)}e^{AB}(\lambda)\Psi^0_{,B}\ , \tag{B7}$$

with

$$e^{AB}(\lambda)=G^{AB}(\lambda)+n^A(\lambda)n^B(\lambda)\ .$$

The variation $\bar{\delta}n^A(\lambda')$ must be of the same form as (B7). Comparing coefficients of $\Psi^A_{,b}$ we find that u must transform as a scalar, and

$$e^{AB}(\lambda')=u^2e^{AB}(\lambda)\ .$$

In conclusion we find that

$$G_{AB}(\lambda')=u^2(\lambda,\lambda')G_{AB}(\lambda)\ , \tag{B8}$$

where u is a D-invariant scalar.

There exists another way to construct vectors in terms of different metrics. Define

$$n^A_{(i)}=-\frac{g^{A0}_{(i)}}{(-g^{00}_{(i)})^{1/2}}$$

and

$$\underline{n}^A(\lambda)=\lambda_i n^A_{(i)}\ .$$

Notice that $\underline{n}^A(\lambda)$ and $n^A(\lambda)$ of (B2) are not linearly related since the definition of the normals in terms of the metric is nonlinear. Nevertheless one can show by the same reasoning as above that $\underline{n}^A(\lambda)$ and $\underline{n}^A(\lambda')$ must differ by a D-invariant scalar u,

$$\underline{n}^A(\lambda)=u\underline{n}^A(\lambda')\ ,$$

which means that

$$(\lambda'_i-u\lambda_i)n^A_i=0\ ,$$

and demonstrates that the $n^A_{(i)}$ are linearly dependent. This again forces us to conclude that the different metrics $g^{(i)}_{AB}$ differ by a D-invariant scalar.

It might sound strange that the program of realizing reparametrizations as canonical transformations enforces constraints on the different metrics. But this is what happens for instance in the first-order formalism for the string. Consider the fields x^μ ($\mu=0,\ldots,3$) and h_{AB} where the latter is a symmetric tensor ($A,B=1,2$). The induced metric is

$$g_{AB}=x^\mu_{,A}x_{\mu,B}\ .$$

In constructing normals one may either take g_{AB} or h_{AB}. By the arguments above, however, we know right from the beginning that

$$h_{AB}=u^2g_{AB}\ .$$

One obtains the same condition in deriving Euler-Lagrange equations from the action

$$A=\int dy^0dy^1h^{AB}x^\mu_{,A}x_{\mu,B}(-h)^{1/2}\ ,$$

where the h_{AB} and x^μ are varied independently.

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