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Geometric constraints on nonsingular, momentarily static, axisymmetric systems in general relativity

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This paper attempts to examine the relationship between system size and gravitational collapse for the case of axial symmetry. The approach here is to construct noncollapsing systems, with momentarily static matter interiors and static vacuum exteriors, and to find limitations on the validity of the construction. Specifically, the exteriors are static, axisymmetric, asymptotically flat, vacuum geometries, described by Weyl solutions of the Einstein field equations. These solutions have singular sources (naked singularities, except for the Schwarzschild solution); here, regions of the Weyl solutions containing the singularities are replaced by momentarily static material bodies. These are described by axisymmetric solutions of Brill's time-symmetric initial-value equation with non-negative energy density, joining smoothly to the Weyl geometries at the bodies' boundaries. The consistency requirements of such a construction limit the choice of surfaces in the exterior geometry suitable as matter/vacuum boundaries; general constraints on the boundary location and geometry are derived here. For the explicit examples of the Γ metric and the Bach-Weyl ring metric as exteriors, these constraints forbid the boundary surface to be arbitrarily near the Weyl singularity. The "hoop conjecture" demands, roughly, that the largest circumference of the boundary surface of such a noncollapsing system always exceed a limit of the order of the system's mass. The specific examples studied here are all consistent with the hoop conjecture, but they show that the boundary constraints derived in this paper are not in general related to boundary surface size and thus that these constraints do not embody the hoop conjecture.

I. INTRODUCTION

In astrophysical calculations and speculations about black holes one usually takes for granted several "articles of faith" that relativity theorists have not yet proved with any rigor.[1] These include the hypothesis of cosmic censorship, the rapid-loss-of-hair conjecture, and the hoop conjecture. Of these, the one for which we have the least concrete evidence is the hoop conjecture. This states that a black hole forms when and only when a mass M gets compacted into a region with circumference in any direction $C \lesssim 4\pi M$, so a hoop of that circumference can be slipped over the region and rotated through 360°.[2,3] This statement of the conjecture is deliberately imprecise, but it indicates the form which a rigorous result linking system size and black-hole formation is expected to take. The proof of such a result is also likely to require certain physical constraints such as a positive-energy-density condition. The motivation of this paper is to seek insight into this size-constraint problem by considering a special case.

More specifically, I restrict attention to axisymmetric systems and approach the problem not by examining black-hole formation but the opposite—I ask what conditions must obtain for a material system to be noncollapsing. Specifically, I consider a bounded matter system (oocupying the "interior" region I) which is axisymmetric and momentarily static; the latter embodies noncollapse and implies that the system can be described with Brill's time-symmetric initial-value formalism.[4] The exterior region E, i.e., the region outside the light cones of the interior at the moment of stasis, is required to be axisymmetric, fully (not just momentarily) static, asymptotically flat, and vacuum. These conditions embody the absence of gravitational waves and imply that the exterior is a slice of a Weyl solution of the vacuum Einstein equations.[5] This shows that the boundary surface of the matter interior lies outside the absolute event horizon (if there is one), since the Weyl solutions are devoid of horizons except possibly at the edge of the Weyl coordinate patch.[6,7] I further impose the physical conditions that the local energy density be everywhere non-negative, and that no physical singularities occur. I then formulate the question thus: Given a specific Weyl solution, and assuming a general interior geometry satisfying the above conditions and matched smoothly to the Weyl exterior, what constraints are imposed on the matter/vacuum boundary surface? Do these constraints have any bearing on the size of the boundary surface, which the hoop conjecture suggests should be "larger in all directions" than $\sim 4\pi M$?

I am aware of one previous calculation of this sort: a cursory study by Thorne[8] of constraints on interior solutions for the Weyl-type gravitational field of a thin-ring torus. Thorne's calculation showed that the location of the interior's surface in the Weyl exterior is bounded away from the immediate neighborhood of the Weyl toroidal singularity. However, this gave no substantial insight into the hoop conjecture.

My analysis of constraints on momentarily static, axisymmetric systems proceeds as follows: In Sec. II, I introduce the time-symmetric initial-value formalism which forms the basis of my calculations, and I derive the junction conditions for matching interior and exterior geometries. In Sec. III, I describe the exterior and interior geometries, and using the initial-value equation and the junction conditions I derive a constraint on the matter/vacuum boundary surface; in Sec. IV, I utilize an alternative description of the interior to derive a second such boundary constraint, particularly suited to toroidal systems. In Secs. V and VI, I apply the boundary constraints of Secs. III and IV, respectively, to simple examples of Weyl exterior geometries, and examine the implications of these constraints and their possible interpretations. In Sec. VII, I discuss the possible extension of these results to exterior geometries more general than the Weyl solutions.

The principal conclusions of this analysis are as follows: There *do* exist constraints on the location of the matter/vacuum boundary in the Weyl exterior, for noncollapsing systems as described here. For systems with toroidal topology, Eq. (4.11) represents a rigorous constraint. For spherical-topology systems, constraints are given by Eqs. (3.24), (3.30), and (3.43), although the derivation of the most generally applicable of these, Eq. (3.43), relies on an unproved assumption (see Appendix C). It may well be possible to close this gap in the derivation, though I have not been able to do so.

Applied to the spherical-topology Γ metric (Sec. V) and to the toroidal Bach-Weyl ring metric (Sec. VI), my constraints imply the existence of a forbidden region near the Weyl singularity, within which the boundary of the matter system cannot lie. These examples further show that the constraints are not, in general, related in any obvious way to a minimum size for the matter system, and do not in any obvious sense embody the hoop conjecture. On the other hand, I have found no violation of the hoop conjecture in these examples; more precisely, the examples do not test my constraints against the hoop conjecture, because none of the candidate boundary surfaces in the Γ-metric or ring-metric exteriors have arbitrarily small circumference in all directions.

Although I have not accomplished the original goal of this research—to prove a special case of the hoop conjecture or to find a counterexample to it—the formalism I have used and the results I have obtained here may prove useful in the hands of other researchers. Specifically, further manipulations of this formalism may yield additional boundary constraints for noncollapsing systems which are stronger than the ones I have derived, more generally applicable, or more amenable to interpretation as size constraints or manifestations of the hoop conjecture. It may also be possible to clarify the geometric meaning of the constraints derived here, perhaps by applying them to additional explicit examples of Weyl geometries.

II. GOVERNING EQUATIONS AND JUNCTION CONDITIONS

A. Initial-value equations

The requirements of a momentarily static interior and a fully static exterior allow this problem to be treated using the time-symmetric initial-value formalism,[4] the hypersurface of constant time at the moment of interior stasis being time-symmetric. The three-dimensional geometry of the system on this hypersurface (hereafter denoted Σ) is governed by the single initial-value equation

$$ {}^{(3)}R = 16\pi\epsilon \ , \tag{2.1}$$

where ${}^{(3)}R$ is the three-dimensional curvature scalar and ϵ is the locally measured energy density. I further assume the weak energy condition

$$ \epsilon \geq 0 \text{ throughout } \Sigma \tag{2.2}$$

and the absence of any physical singularity on Σ. The approach I take is to restrict all calculations to the hypersurface Σ and to study the two-dimensional boundary surface between its interior and exterior regions. The above relations and assumptions determine the matching conditions across and constraints upon that boundary.

B. Junction conditions across a two-surface

The derivation of junction conditions across a two-surface in Σ is similar to that of junction conditions across a three-dimensional hypersurface in space-time.[9] Let $\mathscr{T}$ be a two-dimensional surface in

Σ. The first step in this derivation is to express the three-dimensional curvature scalar ${}^{(3)}R$ in the vicinity of $\mathscr{T}$ in terms of the intrinsic and extrinsic curvatures of $\mathscr{T}$. This may be done by contracting the Gauss-Codazzi equations written in Gaussian normal coordinates, in a manner analogous to that for the higher-dimensional calculation cited above. After some manipulation, one obtains

$$ {}^{(3)}R = {}^{(2)}R + 2\partial(\mathrm{Tr}S)/\partial n - (\mathrm{Tr}S)^2 - \mathrm{Tr}(S^2) , \quad (2.3) $$

where ${}^{(2)}R$ is the curvature scalar for the two-dimensional geometry of $\mathscr{T}$, $\mathrm{Tr}S = S^{\alpha}{}_{\alpha}$ is the trace of the extrinsic curvature of $\mathscr{T}$, $\mathrm{Tr}(S^2) = S^{\alpha}{}_{\beta}S^{\beta}{}_{\alpha}$ is the trace of its square (the sums over α and β extend over the two dimensions of $\mathscr{T}$), and $\partial/\partial n$ is the derivative with respect to proper distance normal to $\mathscr{T}$. The second step of the derivation is to integrate Eq. (2.1) over an infinitesimal interval of proper length across $\mathscr{T}$ in the normal direction, using the above result for ${}^{(3)}R$. That the intrinsic geometry of $\mathscr{T}$ be well defined requires that the metric restricted to $\mathscr{T}$ and the curvature scalar ${}^{(2)}R$ be continuous across $\mathscr{T}$, and consequently that $\mathrm{Tr}S$ and $\mathrm{Tr}(S^2)$ have no δ-function discontinuities at $\mathscr{T}$. Given the assumption that the energy density ϵ contains no singular surface layer, this integration thus implies the junction condition

$$ \Delta(\mathrm{Tr}S) \equiv \lim_{\delta\to 0} (\mathrm{Tr}S)\,|_{n=-\delta}^{+\delta} = 0 , \quad (2.4) $$

where n is the proper distance normal to the surface $\mathscr{T}$.

In summary, the junction conditions across a two-surface $\mathscr{T}$ in the time-symmetric hypersurface Σ are the intrinsic geometry of $\mathscr{T}$ must be continuous across the surface, and (in the absence of a singular surface layer) the trace of the extrinsic curvature of $\mathscr{T}$ must be likewise continuous.

III. DERIVATION OF A BOUNDARY CONSTRAINT

My approach to the derivation of constraints on the two-dimensional matter/vacuum boundary surface is to write the three-dimensional curvature scalar ${}^{(3)}R$ in the interior region of Σ as a total divergence plus a nonpositive quantity; Eq. (2.1) and the inequality (2.2) then imply that the divergence so obtained must be non-negative. By integrating this divergence over the interior volume and invoking the assumption of nonsingularity to apply the divergence theorem, I obtain surface integrals over the boundary which are constrained to be non-negative. Applying the above junction conditions to these integrals yields integrals, involving exterior quantities, which likewise are required to be non-negative. To carry out this approach it is necessary to describe the interior and exterior geometries of Σ with appropriate coordinate systems.

A. Exterior coordinate system, metric, and field equations.

Since the exterior region is a slice of a static, axially symmetric, vacuum four-geometry, it can be described in complete generality by the Weyl formalism.[5] The four-metric of the exterior spacetime can be put in the form

$$ ds_E{}^2 = -\exp[2\psi_E(\rho_E,z_E)]dt^2 + \exp\{2[\gamma_E(\rho_E,z_E) - \psi_E(\rho_E,z_E)]\}(d\rho_E{}^2 + dz_E{}^2) + \rho_E{}^2\exp[-2\psi_E(\rho_E,z_E)]d\varphi^2 . \quad (3.1) $$

Restricted to Σ, this gives the three-metric

$$ d\sigma_E{}^2 = \exp[2(\gamma_E - \psi_E)](d\rho_E{}^2 + dz_E{}^2) + \rho_E{}^2\exp(-2\psi_E)d\varphi^2 \quad (3.2) $$

(the subscript E denotes "exterior"). For metric (3.1) in vacuum, the Einstein field equations reduce to

$$ \partial^2\psi_E/\partial\rho_E{}^2 + (1/\rho_E)\partial\psi_E/\partial\rho_E + \partial^2\psi_E/\partial z_E{}^2 = 0 , \quad (3.3) $$

$$ \partial\gamma_E/\partial\rho_E = \rho_E[(\partial\psi_E/\partial\rho_E)^2 - (\partial\psi_E/\partial z_E)^2] , \quad (3.4) $$

$$ \partial\gamma_E/\partial z_E = 2\rho_E(\partial\psi_E/\partial\rho_E)(\partial\psi_E/\partial z_E) . \quad (3.5) $$

It is also required that $\gamma_E = 0$ for $\rho_E = 0$ to avoid a conical singularity on the z (symmetry) axis, and that far from a bounded source ψ_E approach the Newtonian potential,

$$ \lim_{r\to\infty} \psi_E = -M/r + O(M^3/r^3) , \quad (3.6) $$

where $r \equiv (\rho_E{}^2 + z_E{}^2)^{1/2}$ and M is the total gravitational mass of the system as measured at infinity. Condition (3.6) ensures that the metric (3.1) has the appropriate asymptotic behavior at infinity.[10] Equation (3.3) means that ψ_E is a harmonic potential in a Euclidean "background space" with cylindrical coordinates (ρ_E,φ,z_E). Since γ_E can be determined from ψ_E by integrating Eqs. (3.4) and (3.5), the entire exterior geometry is specified if ψ_E, or its fictitious "source" in the flat background space, is given.

The matter/vacuum boundary surface can be de-

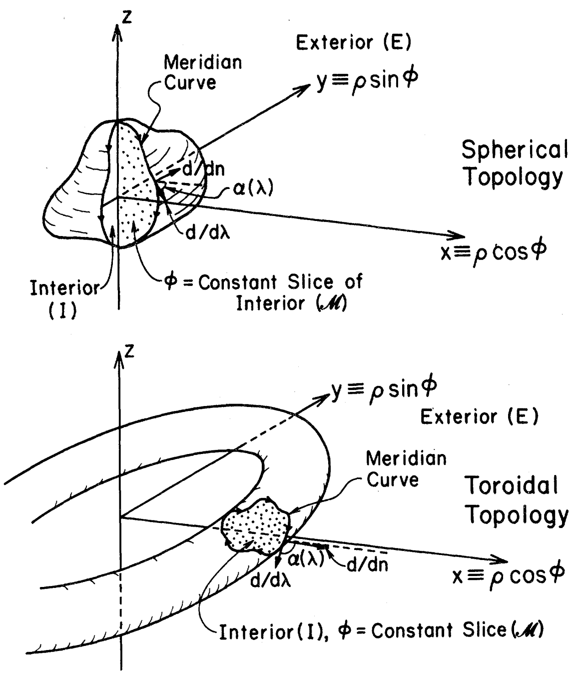

FIG. 1. Possible boundary and interior geometries, illustrated in exterior (Weyl) coordinate space. The configuration of the vectors $d/d\lambda$, d/dn, and the angle $\alpha(\lambda)$ appears the same whether these quantities are defined from the interior (I) or the exterior (E).

fined in terms of the exterior Weyl coordinates by specifying a meridian (φ=constant) curve for the surface. In general, two cases of interest arise. In the "spherical topology" case, the interior region includes a segment of the symmetry axis. The boundary meridian in this case is an open curve with end points on the symmetry axis; I assume the curve intersects the axis orthogonally at its ends so that the boundary surface has no cusps. The other case is that of "toroidal topology"; here the symmetry axis lies wholly outside the matter (interior) region. In this case the boundary meridian is a closed curve, which I assume to be simple, i.e., nonself-intersecting. It is convenient to specify the meridian for either case parametrically, using proper length λ along the meridian as parameter and defining the meridian with two functions $\rho_E = R(\lambda)$, $z_E = Z(\lambda)$; I take these to be twice differentiable. With λ and φ as coordinates on the boundary surface, the intrinsic geometry of the surface approached from the exterior is given by the two-metric

$$ds_2{}^2 = d\lambda^2 + R^2(\lambda) \times \exp[-2\psi_E(R(\lambda),Z(\lambda))]d\varphi^2 \,. \tag{3.7}$$

The coordinate basis vectors tangent to the boundary are $\partial/\partial\varphi$ and

$$d/d\lambda = R'(\lambda)\partial/\partial\rho_E + Z'(\lambda)\partial/\partial z_E \,, \tag{3.8}$$

where here and below primes denote derivatives with respect to λ. The normal to the surface is

$$d/dn = -Z'(\lambda)\partial/\partial\rho_E + R'(\lambda)\partial/\partial z_E \,. \tag{3.9}$$

I choose the direction of increasing λ on the meridian so that (3.9) gives the outward-directed normal; this orientation of λ is analogous to the orientation of the coordinate θ in ordinary three-dimensional spherical coordinates. The metric (3.2), restricted to a boundary meridian, shows that both $d/d\lambda$ and d/dn are unit vectors. Figure 1 illustrates possible boundary geometries, coordinates, and associated vectors.

The trace of the boundary surface extrinsic curvature, for use in the junction conditions of Sec. II B, can be calculated directly given the above boundary coordinates, vectors, and metric. I obtain

$$\begin{aligned}\mathrm{Tr}S_E &= \frac{R'Z''-Z'R''}{R'^2+Z'^2} + Z'/R + d(2\psi_E-\gamma_E)/dn \\ &= -\alpha_E' + Z'/R + d(2\psi_E-\gamma_E)/dn \,,\end{aligned} \tag{3.10}$$

where

$$\alpha_E(\lambda) \equiv -\tan^{-1}(Z'/R') \tag{3.11}$$

is the angle between the vectors $\partial/\partial\rho_E$ and $d/d\lambda$, as indicated in Fig. 1. [Since the metric (3.2), restricted to a φ=constant surface, is conformally flat, α_E is this angle as measured both in the physical space and in the background space.] Here, as above, the subscript E denotes quantities calculated on the exterior side of the boundary.

B. Interior coordinate system, metric, and field equations

Because the interior geometry is only momentarily static, it is not constrained as strongly as is the exterior; it can be described in several ways. One simple description uses a three-metric similar to that of the vacuum Weyl metric:

$$d\sigma_I{}^2 = \exp\{2[\gamma_I(\rho_I,z_I)-\psi_I(\rho_I,z_I)]\}(d\rho_I{}^2+dz_I{}^2) + \rho_I{}^2\exp[-2\psi_I(\rho_I,z_I)]d\varphi^2 \,, \tag{3.12}$$

where here and below the subscript I denotes "interior." Since g_{00} is not specified, no generality is lost in this description.

Under the assumptions made here, it is always possible to cover the interior region with coordinates (ρ_I,φ,z_I) such that the metric takes the form (3.12).

The φ coordinate derives from the axial symmetry. The existence of these interior coordinates hinges on the existence of coordinates (ρ_I, z_I) in which the metric has the above-indicated isothermal (conformally flat) form on a two-dimensional φ=constant slice of the interior. The uniformization theorem for Riemann surfaces (Appendix A) guarantees the existence of such an isothermal coordinate patch covering this slice, provided that the slice is simply connected. Since vacuum regions may be included as part of the interior if necessary, the φ=constant slices of the interior can be assumed simply connected without loss of generality. The uniformization theorem further ensures that these coordinates (ρ_I, z_I) can be chosen so that their values fill any desired bounded, simply connected region in the plane R^2. (Here this choice is limited by regularity requirements of the full three-dimensional interior coordinate system; in particular, $\rho_I=0$ on the symmetry axis is required, with $\rho_I>0$, say, off that axis, in order that γ_I and ψ_I be nonsingular.) For example, for a toroidal-topology interior, the coordinates (ρ_I, z_I) might be chosen to fill a unit disk in the right half of the plane; for a spherical-topology interior, the right half of the unit disk centered on the origin is convenient. The latter choice makes the coordinates regular even at the "corners" of the interior slice, where the meridian meets the symmetry axis (see Appendix A).

Since the metric (3.12) has the same form as (3.2), the description of the boundary surface from the interior is similar to that from the exterior. The boundary meridians are defined by two functions, $\rho_I=P(\lambda)$ and $z_I=\mathscr{Z}(\lambda)$. The intrinsic geometry of the boundary is given by the two-metric

$$ds_2{}^2=d\lambda^2+P^2(\lambda)\times\exp[-2\psi_I(P(\lambda),\mathscr{Z}(\lambda))]d\varphi^2 . \quad (3.13)$$

The coordinate basis vectors tangent to the boundary are $\partial/\partial\varphi$ and

$$d/d\lambda=P'(\lambda)\partial/\partial\rho_I+\mathscr{Z}'(\lambda)\partial/\partial z_I . \quad (3.14)$$

The normal vector is

$$d/dn=-\mathscr{Z}'(\lambda)\partial/\partial\rho_I+P'(\lambda)\partial/\partial z_I . \quad (3.15)$$

With the orientation of λ already specified from the exterior, I make (3.15) the outward-directed normal by choosing the appropriate sign for the coordinate z_I. As above, both d/dn and $d/d\lambda$ are unit vectors. The configuration of interior coordinates and vectors is as shown in Fig. 1. The trace of the boundary extrinsic curvature is calculated as before, with the result

$$\begin{aligned}\mathrm{Tr}S_I&=\frac{P'\mathscr{Z}''-\mathscr{Z}'P''}{P'^2+\mathscr{Z}'^2}+\mathscr{Z}'/P+d(2\psi_I-\gamma_I)/dn\\&=-\alpha_I'+\mathscr{Z}'/P+d(2\psi_I-\gamma_I)/dn \end{aligned}\quad (3.16)$$

with

$$\alpha_I(\lambda)\equiv-\tan^{-1}(\mathscr{Z}'/P') . \quad (3.17)$$

The potentials $\gamma_I(\rho_I,z_I)$ and $\psi_I(\rho_I,z_I)$ in (3.12) need not satisfy equations like (3.3), (3.4), and (3.5), which are consequences of the vacuum Einstein field equations. The functions γ_I and ψ_I are constrained only by the initial-value equation (2.1). The calculation of the scalar curvature ${}^{(3)}R$ for the geometry described by the metric (3.12) is straightforward, and yields

$$\begin{aligned}{}^{(3)}R_I&=2\{2[\partial^2\psi_I/\partial\rho_I{}^2+(1/\rho_I)(\partial\psi_I/\partial\rho_I)+\partial^2\psi_I/\partial z_I{}^2]-(\partial^2\gamma_I/\partial\rho_I{}^2+\partial^2\gamma_I/\partial z_I{}^2)\\&\quad-[(\partial\psi_I/\partial\rho_I)^2+(\partial\psi_I/\partial z_I)^2]\}\exp[2(\psi_I-\gamma_I)]\\&=16\pi\epsilon\geq0 .\end{aligned}\quad (3.18)$$

C. A boundary-constraint inequality

It is convenient to express (3.18) in terms of the covariant gradient ∇ for the three-metric (3.12). If this is done, a little rearrangement yields

$$\tfrac{1}{2}{}^{(3)}R_I+(\nabla\psi_I)^2=e^{-\psi_I}\nabla\cdot\{e^{\psi_I}[\nabla(2\psi_I-\gamma_I)+(\gamma_I/\rho_I)\nabla\rho_I]\} , \quad (3.19)$$

where all the dot products in this equation are given by the metric (3.12). This form is suitable for deriving a boundary constraint via the approach outlined at the beginning of this section. Combined with (2.1) and (2.2), (3.19) implies

$$\nabla\cdot\{e^{\psi_I}[\nabla(2\psi_I-\gamma_I)+(\gamma_I/\rho_I)\nabla\rho_I]\}=e^{\psi_I}[8\pi\epsilon+(\nabla\psi_I)^2]\geq0 . \quad (3.20)$$

I integrate this expression over the interior region I and apply the divergence theorem to the left-hand expression, obtaining a surface integral over the boundary ∂I:

$$\oint_{\partial I}d^2Ae^{\psi_I}[d(2\psi_I-\gamma_I)/dn+(\gamma_I/\rho_I)d\rho_I/dn]=\int_I d^3Ve^{\psi_I}[8\pi\epsilon+(\nabla\psi_I)^2]\geq0 . \quad (3.21)$$

The metric (3.13) gives $d^2A = d\lambda P(\lambda) \times\exp[-\psi_I(P(\lambda),\mathscr{Z}(\lambda))]d\varphi$ and the operator (3.15) applied to ρ_I gives $d\rho_I/dn = -\mathscr{Z}'$. Dividing out the φ integral, I obtain

$$\int_0^{\lambda_{\max}} d\lambda[P(\lambda)d(2\psi_I-\gamma_I)/dn - \mathscr{Z}'(\lambda)\gamma_I] = [1/(2\pi)]\int_I d^3V e^{\psi_I}[8\pi\epsilon+(\nabla\psi_I)^2] \geq 0 , \tag{3.22}$$

where the left-hand expression is a line integral taken over the boundary meridian, with all quantities evaluated at $\rho_I = P(\lambda)$, $z_I = \mathscr{Z}(\lambda)$.

The interior coordinates which give rise to the metric form (3.12) and thence to the inequality (3.22) are not unique, since the region of the plane filled by the coordinates (ρ_I, z_I) can be freely chosen (subject only to the requirement of simple connectivity, and the conditions $\rho_I = 0$ on the symmetry axis, $\rho_I > 0$ elsewhere). Any two such coordinate systems must be related by a conformal transformation on the coordinates (ρ_I, z_I), i.e., if (ρ_I, z_I) and $(\tilde{\rho}_I, \tilde{z}_I)$ are two sets of isothermal coordinates on a φ=constant slice of the interior, then $\tilde{\rho}_I \pm i\tilde{z}_I$ must be an injective analytic function of $\rho_I + iz_I$ (allowing for a possible sign change for the z coordinate). Under these transformations, hereafter termed "gauge transformations," the potentials ψ_I and γ_I are not invariant; only the combination $\rho_I e^{-\psi_I} = ||\partial/\partial\varphi||$ and scalars constructible from the geometry (3.12) are. Thus the splitting off of the $(\nabla\psi_I)^2$ term in (3.19) and the weighting of the volume integrand in (3.21) by e^{ψ_I} are gauge dependent, making the boundary-constraint inequality (3.22) gauge dependent.

To isolate this gauge dependence, and because exterior quantities are more completely and simply determined than interior ones, it is convenient to express (3.22) in terms of exterior quantities wherever possible. The junction conditions of Sec. II B require that the two-metric (3.7) be continuous with (3.13), and that the extrinsic curvature trace (3.10) be continuous with (3.16). Solving the latter condition for $d(2\psi_I - \gamma_I)/dn$ and substituting the result into (3.22) gives

$$\int_0^{\lambda_{\max}} d\lambda\{P[d(2\psi_E-\gamma_E)/dn + Z'/R - \alpha_E' + \alpha_I'] - \mathscr{Z}'(1+\gamma_I)\} \geq 0 . \tag{3.23}$$

This constraint can be rendered a bit more tractable by specifying a choice of gauge. Let $\mathscr{M}$ be a φ=constant slice of the interior I; the boundary of $\mathscr{M}$, $\partial\mathscr{M}$, consists of a meridian for toroidal-topology interiors, a meridian plus a segment of the symmetry axis for the spherical-topology case. Let ${}^{(2)}\nabla$ denote the covariant derivative on $\mathscr{M}$ corresponding to the restriction of the metric (3.12) to $\mathscr{M}$. Let $\tilde{\rho}_I$ be the solution to the covariant Laplace equation ${}^{(2)}\nabla^2\tilde{\rho}_I = 0$, where the Laplacian is also constructed from (3.12) restricted to $\mathscr{M}$, subject to the following boundary conditions: on the boundary meridian, $\tilde{\rho}_I = R(\lambda)$, as defined in Sec. III A; if $\partial\mathscr{M}$ contains a segment of the symmetry axis, $\tilde{\rho}_I = 0$ on that segment. Exactly one such $\tilde{\rho}_I$ always exists, since if $D \subset R^2$ is the region of the plane filled by the interior coordinates (ρ_I, z_I), finding $\tilde{\rho}_I$ is equivalent to solving the Dirichlet problem on D with the corresponding boundary conditions on ∂D. Given $\tilde{\rho}_I$, the corresponding $\tilde{z}_I$ is determined by the Cauchy-Riemann equations, except for its overall sign and a possible overall translation. The sign is fixed by the requirement that (3.15) represent the outward normal to ∂I; the translation is unimportant. Thus for any interior geometry considered here there exists a unique set of interior coordinates ("matched" coordinates) in which the metric has the form (3.12) and the radial coordinate $\tilde{\rho}_I$ matches the exterior radial coordinate ρ_E at the boundary, provided the map $(\tilde{\rho}_I, \tilde{z}_I):\mathscr{M} \to R^2$ [or equivalently, $(\tilde{\rho}_I, \tilde{z}_I):D \to R^2$] is injective. A sufficient condition for this, relying only on exterior quantities, is that if the meridian exterior radial coordinate $R(\lambda)$ has only one local maximum, then injectivity of the matched coordinates is guaranteed for any interior geometry (see Appendix B). In matched interior coordinates, (3.23) takes the form

$$\int_0^{\lambda_{\max}} d\lambda\{R[d(2\psi_E-\gamma_E)/dn - \alpha_E' + \alpha_I'] + Z' - \mathscr{Z}'(1+\gamma_I)\} \geq 0 . \tag{3.24}$$

Here α_I is given by (3.17), subject to the matching condition $P(\lambda) = R(\lambda)$; $\mathscr{Z}(\lambda)$ and $\gamma_I(P(\lambda),\mathscr{Z}(\lambda))$ are those appropriate to the matched interior coordinates $(\tilde{\rho}_I, \tilde{z}_I)$.

The inequality (3.24) may be further transformed by treating some of the terms as integrals in the Weyl "background" space, i.e., the exterior coordinate space with a flat Euclidean metric. Let dl be background-space length along the meridian, and ${}^{(B)}d/dn$ the unit outward normal derivative at the boundary surface in the background space. Since the metric (3.2), restricted to a φ=constant surface, is conformally flat, the scale factor between physical-space and background-space meridian lengths is the same as that between physical-space and background-space normal distances. Thus $d\lambda(d/dn) = dl({}^{(B)}d/dn)$. Consequently, I can write

$$\int_0^{\lambda_{\max}} d\lambda\, R(d\psi_E/dn) = [1/(2\pi)] \oint_{\partial I} d^2A_B\, {}^{(B)}d\psi_E/dn\ , \tag{3.25}$$

where d^2A_B is the background-space area measure. But the integral on the right side of (3.25) can be evaluated via Gauss's law, i.e., by integrating (3.3) in the background space between ∂I and a coordinate sphere at infinity, applying the flat-space divergence theorem and utilizing (3.6). I obtain

$$\int_0^{\lambda_{\max}} d\lambda\, R(d\psi_E/dn) = 2M\ . \tag{3.26}$$

A similar transformation begins with

$$\int_0^{\lambda_{\max}} d\lambda\, R(d\gamma_E/dn) = [1/(2\pi)] \oint_{\partial I} d^2A_B\, {}^{(B)}d\gamma_E/dn\ . \tag{3.27}$$

The flat-space divergence theorem can be applied to the integral on the right side of this equation [Eqs. (3.4), (3.5), and (3.6) imply that γ_E is of order $(M/r)^2$ as $r\to\infty$, so the integral over the sphere at infinity vanishes], with the result

$$\int_0^{\lambda_{\max}} d\lambda\, R(d\gamma_E/dn) = -[1/(2\pi)] \int_E d^3V_B (\partial^2\gamma_E/\partial\rho_E{}^2 + \partial^2\gamma_E/\partial z_E{}^2) - [1/(2\pi)] \int_E d^3V_B (1/\rho_E)(\partial\gamma_E/\partial\rho_E)\ . \tag{3.28}$$

The integrals on the right are over the exterior volume E, with d^3V_B the coordinate-space volume measure. It follows from Eqs. (3.4), (3.5), and (3.3) that the integrand of the first integral on the right equals the flat-space divergence of the vector field $-\psi_E{}^{(B)}\nabla\psi_E$, where ${}^{(B)}\nabla$ is the flat-space gradient. The integrand of the second integral is just the flat-space divergence of the vector field $(\gamma_E/\rho_E)(\partial/\partial\rho_E)$. Applying the divergence theorem to both integrals and converting the resulting surface integrals into line integrals in the physical space, I obtain

$$\int_0^{\lambda_{\max}} d\lambda\, R(d\gamma_E/dn) = -\int_0^{\lambda_{\max}} d\lambda [R\psi_E(d\psi_E/dn) + \gamma_E Z']\ . \tag{3.29}$$

Using this and (3.26) in (3.24) gives

$$4M + \int_0^{\lambda_{\max}} d\lambda \{R[\psi_E(d\psi_E/dn) - \alpha_E' + \alpha_I'] + Z'(1+\gamma_E) - \mathscr{Z}'(1+\gamma_I)\} \geq 0\ . \tag{3.30}$$

The boundary constraint (3.30) still depends on the interior geometry, but only through the coordinate derivative $\mathscr{Z}'$ and its derivative $\mathscr{Z}''$ along the meridian. These appear in α_I', as per (3.17), and in the last term of the integrand. The metric (3.12), specialized to a boundary meridian and in matched coordinates, yields the condition

$$1 = \exp\{2[\gamma_I(R(\lambda),\mathscr{Z}(\lambda)) - \psi_I(R(\lambda),\mathscr{Z}(\lambda))]\}\{[R'(\lambda)]^2 + [\mathscr{Z}'(\lambda)]^2\}\ . \tag{3.31}$$

The junction conditions of Sec. II B require that the boundary two-metrics (3.7) and (3.13) be continuous; coupled with the coordinate-matching condition $P(\lambda) = R(\lambda)$, this means ψ_I is continuous with ψ_E at the boundary. Thus γ_I at the boundary depends only on $\mathscr{Z}'$ and exterior quantities, i.e.,

$$\gamma_I(R(\lambda),\mathscr{Z}(\lambda)) = \psi_E(R(\lambda),Z(\lambda)) - \tfrac{1}{2}\ln\{[R'(\lambda)]^2 + [\mathscr{Z}'(\lambda)]^2\}\ . \tag{3.32}$$

This remaining dependence of (3.30) on interior quantities is dependence on the actual interior geometry rather than gauge dependence, since the choice of matched coordinates fixes the gauge. This may be seen by reexpressing (3.30) in terms of the matched interior coordinate ρ_I (dropping the tilde). This coordinate is uniquely and invariantly defined, as above, as the solution to ${}^{(2)}\nabla^2\rho_I = 0$ on $\mathscr{M}$, with boundary conditions $\rho_I = R(\lambda)$ on the boundary meridian and $\rho_I = 0$ on the symmetry axis if $\partial\mathscr{M}$ contains a segment thereof. Since $d\rho_I/dn = -\mathscr{Z}'$ on the meridian by (3.15), and similarly for $d\rho_E/dn$, inequality (3.30) may be written

$$4M + \int_0^{\lambda_{\max}} d\lambda \{R[\psi_E(d\psi_E/dn) + \alpha_I' - \alpha_E'] + (1+\gamma_I)(d\rho_I/dn) - (1+\gamma_E)(d\rho_E/dn)\} \geq 0\ . \tag{3.33}$$

Here α_I is defined as the angle from $\partial/\partial\rho_I$ to $d/d\lambda$, measured in accord with the orientation indicated in Fig. 1; α_E is similarly defined. In terms of ρ_I and ρ_E, this means

$$\alpha_I = \tan^{-1}[(d\rho_I/dn)/(d\rho_I/d\lambda)]\ ,$$
$$\alpha_E = \tan^{-1}[(d\rho_E/dn)/(d\rho_E/d\lambda)]\ . \tag{3.34}$$

Also γ_I is given on the meridian by

$$\gamma_I=\psi_E-\tfrac{1}{2}\ln[(d\rho_I/d\lambda)^2+(d\rho_I/dn)^2] . \quad (3.35)$$

Hence the dependence of the boundary constraint (3.33) on interior quantities appears only through the function ρ_I. I have not been able to eliminate this dependence from this constraint in general.

The usefulness of the constraint (3.30) is in identifying surfaces in any given exterior geometry which are forbidden as boundaries of systems satisfying the $\epsilon\geq 0$ and nonsingularity assumptions made here. The interior coordinates, specifically the coordinate derivative $\mathscr{Z}'$ for matched coordinates, must be specified in some way to evaluate the inequality (3.30); then, surfaces violating the inequality are forbidden. Surfaces satisfying the inequality may or may not be acceptable boundaries, since they may bound interiors satisfying (3.20) but violating the $\epsilon\geq 0$ condition.

D. Double-matched coordinates

The boundary constraint (3.30) becomes very simple for cases in which the matched interior coordinates are also "double matched," i.e., they satisfy $\mathscr{Z}(\lambda)=Z(\lambda)$ as well as $P(\lambda)=R(\lambda)$. In such cases γ_I and γ_E are continuous at ∂I as well as ψ_I and ψ_E, and of course $\alpha_I'=\alpha_E'$. Inequality (3.30) reduces to

$$4M+\int_0^{\lambda\max} d\lambda\, R\psi_E(d\psi_E/dn)\geq 0 . \quad (3.36)$$

If in addition the boundary surface is chosen to be an equipotential surface of ψ_E, then the integral can be evaluated via (3.26). The boundary constraint becomes

$$\psi_E\,|_{\partial I}\geq -2 \quad (3.37)$$

for a double-matched system with a ψ_E-equipotential boundary.

E. Maximization with respect to $\mathscr{Z}'$

It is also possible in certain cases to eliminate the interior-coordinate dependence of (3.30) by maximizing the left side of the inequality with respect to the interior function $\mathscr{Z}'$. I denote the left side of (3.30) $\mathscr{I}_1[\mathscr{Z}']$, and regard it as a functional of $\mathscr{Z}'$; different functions $\mathscr{Z}'$ correspond to different interior geometries since matched interior coordinates are assumed in (3.30). If there exists a function $\mathscr{Z}_0'(\lambda)$ for which $\mathscr{I}_1$ is maximized, for a given exterior geometry and boundary surface, then the maximum value of $\mathscr{I}_1$ can serve to identify a forbidden surface. Specifically, a surface with $\mathscr{I}_1[\mathscr{Z}_0']<0$ is forbidden as a matter/vacuum boundary for any interior geometry under the assumptions made here.

It is a straightforward variational problem to extremize $\mathscr{I}_1$ with respect to $\mathscr{Z}'$. The first variation is

$$\delta\mathscr{I}_1=\int_0^{\lambda\max}[\tfrac{1}{2}\ln(R'^2+\mathscr{Z}'^2)-\psi_E](\delta\mathscr{Z}')d\lambda$$
$$=-\int_0^{\lambda_{\max}}\gamma_I(\delta\mathscr{Z}')d\lambda \quad (3.38)$$

and the second variation is

$$\delta^2\mathscr{I}_1=\int_0^{\lambda_{\max}}\frac{\mathscr{Z}'}{R'^2+\mathscr{Z}'^2}(\delta\mathscr{Z}')^2 d\lambda . \quad (3.39)$$

Thus the choice of $\mathscr{Z}'$ for which $\mathscr{I}_1$ is extremal is given by

$$\mathscr{Z}_0'(\lambda)=\pm\{\exp[2\psi_E(R(\lambda),Z(\lambda))]-[R'(\lambda)]^2\}^{1/2} , \quad (3.40)$$

provided of course that

$$[R'(\lambda)]^2\leq\exp[2\psi_E(R(\lambda),Z(\lambda))] \quad (3.41)$$

holds on the entire boundary meridian. This choice of $\mathscr{Z}_0'$ is equivalent to

$$\gamma_I(R(\lambda),\mathscr{Z}_0(\lambda))=0 . \quad (3.42)$$

The choice of sign in (3.40) is fixed by the orientation of $d/d\lambda$ and d/dn on the boundary meridian. For those cases in which the negative square root applies along the entire meridian, (3.39) implies that $\mathscr{Z}_0'$ gives a maximum of $\mathscr{I}_1$. This is actually a local maximum in the space of functions $\mathscr{Z}'$; establishing it as a global maximum poses some difficulties (see Appendix C). In using this boundary constraint, I assume the global maximality of $\mathscr{I}_1[\mathscr{Z}_0']$.

If condition (3.41) holds on the boundary meridian, and if the negative sign in (3.40) is admitted by the topology over the entire meridian, then the boundary constraint takes the form

$$\mathscr{I}_1[\mathscr{Z}_0']=4M+\int_0^{\lambda_{\max}} d\lambda\{R[\psi_E(d\psi_E/dn)-\alpha_E'+\alpha_I^{(0)\prime}]+Z'(1+\gamma_E)-\mathscr{Z}_0'\}\geq 0 \quad (3.43)$$

with $\mathscr{Z}_0'$ given by (3.40) and $\alpha_I^{(0)\prime}$ given by

$$\alpha_I^{(0)\prime}=-\frac{R'\mathscr{Z}_0''-\mathscr{Z}_0'R''}{R'^2+\mathscr{Z}_0'^2}$$
$$=(d/d\lambda)[\tan^{-1}(-\mathscr{Z}_0'/R')] . \quad (3.44)$$

Surfaces violating (3.43) are forbidden. Because of the aforementioned conditions necessary to establish the existence and maximality of $\mathscr{Z}_0'$, this form of the boundary constraint is most readily applied to systems of spherical topology, rather than toroidal systems with closed meridians.

IV. A SECOND BOUNDARY CONSTRAINT

A boundary constraint distinct from (3.30) or (3.43) can be derived via the same procedure as in Sec. III, starting from a slightly different interior description. I obtain the new boundary constraint by maintaining gauge invariance throughout the calculation.

A. Alternative interior description

The interior metric can be written

$$d\sigma_I{}^2 = \exp[2Q(\rho_I,z_I)](d\rho_I{}^2+dz_I{}^2) + \exp[2\beta(\rho_I,z_I)]d\varphi^2 \ . \qquad (4.1)$$

The existence of coordinates in which the metric takes this form is guaranteed by the uniformization theorem, as in Sec. III B. Since ρ_I does not appear as a factor in $g_{\varphi\varphi}$ here, the restrictions $\rho_I=0$ on the symmetry axis, $\rho_I>0$ elsewhere are not needed in this description. However, regularity of the geometry on the symmetry axis requires that the function β be singular there (i.e., that e^{β} vanish), if the axis passes within the interior region. The form of the metric (4.1) is preserved under gauge transformations of the type discussed in Sec. III C; here the function β is gauge invariant, while Q is gauge dependent.

The boundary surface is specified as before, by two functions $\rho_I=P(\lambda)$, $z_I=\mathscr{Z}(\lambda)$; I make no assumption of matched coordinates here. The boundary's intrinsic geometry is given by the two-metric

$$ds_2{}^2=d\lambda^2+\exp[2\beta(P(\lambda),\mathscr{Z}(\lambda))]d\varphi^2 \ . \qquad (4.2)$$

The coordinate basis vectors on the boundary are $\partial/\partial\varphi$ and $d/d\lambda$ as given by (3.14); the normal vector d/dn is given by (3.15). The trace of the boundary extrinsic curvature is again calculated directly, with the result

$$\mathrm{Tr}S_I=-[\alpha_I'+d(Q+\beta)/dn] \ , \qquad (4.3)$$

where α_I is as given by (3.17).

The scalar curvature ${}^{(3)}R$ for the geometry described by (4.1) is given by

$${}^{(3)}R_I=-2e^{-2Q}\{[(\partial/\partial\rho_I)^2+(\partial/\partial z_I)^2](Q+\beta) + (\partial\beta/\partial\rho_I)^2+(\partial\beta/\partial z_I)^2\} \ . \qquad (4.4)$$

B. Derivation of the alternate boundary constraint

In terms of the covariant derivative ∇ and covariant divergence corresponding to the metric (4.1), Eq. (4.4) takes the form

$${}^{(3)}R_I=-2[\nabla^2(Q+\beta)-\nabla Q\cdot\nabla\beta] \ , \qquad (4.5)$$

where the dot product in the last term is also that of (4.1). This expression for ${}^{(3)}R_I$ can be rearranged for integration over the interior volume in different ways; I maintain gauge invariance by rewriting it in the form

$$-[\tfrac{1}{2}{}^{(3)}R_I+(\nabla\beta)^2]=e^{\beta}\nabla\cdot[e^{-\beta}\nabla(Q+\beta)] \ . \qquad (4.6)$$

Thus by the initial-value equation (2.1), I obtain

$$\nabla\cdot[e^{-\beta}\nabla(Q+\beta)]=-e^{-\beta}[8\pi\epsilon+(\nabla\beta)^2]\leq 0 \ . \qquad (4.7)$$

The divergence on the left is gauge invariant because the quantity on the right is.

I derive the desired boundary constraint by integrating (4.7) over the interior volume and applying the divergence theorem to obtain a surface integral. Because of the singularity of β on the symmetry axis, that axis must be excluded from the integration volume if this is to be done. For spherical-topology systems, this means that the resulting surface integral consists of two terms, an integral over the boundary surface plus an integral over an infinitesimal "sheath" about the symmetry axis. But the integral over the sheath has the form

$$\int_{\mathrm{sheath}} d^2A[e^{-\beta}d(Q+\beta)/dn] = 2\pi\int_{\mathrm{sheath}} d\lambda\, d(Q+\beta)/dn \ . \qquad (4.8)$$

In general, dQ/dn will be finite at the symmetry axis. But $d\beta/dn=e^{-\beta}d(e^{\beta})/dn$ diverges to $-\infty$ there, since e^{β} vanishes while $d(e^{\beta})/dn$ approaches -1 by elementary flatness (the negative sign appears because the outward normal from the interior at the sheath points toward the symmetry axis). With the sheath term negative and infinite, the integral over the actual matter/vacuum boundary surface is not constrained at all by the inequality in (4.7). Thus the boundary constraint to be derived by integrating (4.7) is useful only for toroidal-topology systems, in which the interior region contains no segment of the symmetry axis.

For such toroidal systems, integrating (4.7) over the interior volume and using the divergence theorem yields

$$\int_0^{\lambda\max} d\lambda\, d(Q+\beta)/dn = -[1/(2\pi)] \int_I d^3V e^{-\beta}[8\pi\epsilon + (\nabla\beta)^2] \le 0 \; . \tag{4.9}$$

By the junction condition (2.4), the trace (4.3) must be equal to (3.10). Solving this equality for $d(Q+\beta)/dn$ in (4.9) gives

$$\int_0^{\lambda_{\max}} d\lambda [d(2\psi_E - \gamma_E)/dn + Z'/R - \alpha_E' + \alpha_I'] = [1/(2\pi)] \int_I d^3V e^{-\beta}[8\pi\epsilon + (\nabla\beta)^2] \ge 0 \; . \tag{4.10}$$

The integral of α_I' around the boundary meridian gives 2π for an arbitrary regular interior; the integral of α_E' cannot be so given for all cases but can be evaluated in any given case (see Appendix D). Thus the boundary constraint has the form

$$\int_0^{\lambda_{\max}} d\lambda [d(2\psi_E - \gamma_E)/dn + Z'/R - \alpha_E'] \ge -2\pi \; . \tag{4.11}$$

This has the same significance as the constraint (3.24) or (3.30); it identifies surfaces in a given exterior geometry which are forbidden as matter/vacuum boundaries under the $\epsilon \ge 0$ and nonsingularity assumptions. Surfaces for which (4.11) is violated are forbidden; the suitability of surfaces for which it is obeyed is undetermined. Although this condition, in contrast to that of Sec. III, has the disadvantage that it can only be applied to toroidal systems, it has the desirable feature that it can be evaluated using only exterior quantities, without further assumptions.

V. APPLICATION OF THE BOUNDARY CONSTRAINTS: A SPHERICAL-TOPOLOGY EXAMPLE

The boundary constraint derived in Sec. III can be examined by applying it to simple examples of Weyl exterior geometries. The simplest of these, in terms of the Weyl formalism, is the Curzon metric, for which the background-space source is a point monopole.[11] (This geometry is quite distinct from the Schwarzschild solution for a point monopole in the *physical* space; the Curzon geometry is not spherically symmetric.) Condition (3.43) may be applied to this geometry, but no particularly interesting results are obtained. A slightly more complicated set of Weyl geometries, those with a line source in the background space, does reveal some important features of the constraint.

A. The Γ metrics

Specifically, the background source for these geometries is a line monopole, of linear density $\Gamma/2$, extending from $z_E = -a$ to $z_E = +a$ on the symmetry axis in the background space. This source is fictional; its linear density is the physical mass M of the system, measured at infinity, divided by its coordinate length, so that $\Gamma a = M$. Equation (3.3), the Laplace equation for ψ_E, is easily solved for such a source in prolate spheroidal coordinates (u,v,φ), related to the Weyl coordinates (ρ_E, z_E, φ) by $\rho_E = a \sinh u \sin v$, $z_E = a \cosh u \cos v$, with $u \in [0, +\infty)$, $v \in [0, \pi]$. In these coordinates the Weyl equations (3.3), (3.4), and (3.5) have the solution

$$\psi_E = \Gamma \ln[\tanh(u/2)] \; , \tag{5.1}$$

$$\gamma_E = -(\Gamma^2/2) \ln \left[1 + \frac{\sin^2 v}{\sinh^2 u} \right] . \tag{5.2}$$

The resulting spacetime metric is[12]

$$\begin{aligned} ds_E{}^2 &= -\tanh^{2\Gamma}(u/2) dt^2 + \tanh^{-2\Gamma}(u/2) \left[1 + \frac{\sin^2 v}{\sinh^2 u} \right]^{-\Gamma^2} [d\rho_E{}^2 + dz_E{}^2] + \rho_E{}^2 \tanh^{-2\Gamma}(u/2) d\varphi^2 \\ &= -\tan^{2\Gamma}(u/2) dt^2 + (M/\Gamma)^2 \sinh^2 u \tanh^{-2\Gamma}(u/2) \left[1 + \frac{\sin^2 v}{\sinh^2 u} \right]^{1-\Gamma^2} [du^2 + dv^2] \\ &\quad + (M/\Gamma)^2 \sinh^2 u \tanh^{-2\Gamma}(u/2) \sin^2 v \, d\varphi^2 \; . \end{aligned} \tag{5.3}$$

The time-symmetric hypersurface Σ of concern here is given by any constant-t hypersurface in this geometry.

The metric (5.3) describes a family of geometries parametrized by Γ. If $\Gamma = 1$, so $a = M$, the geometry is just the Schwarzschild geometry[13]; the usual Schwarzschild coordinates (t,r,θ,φ) are related to the above (t,u,v,φ) coordinates by $r = 2M \cosh^2(u/2)$, $\theta = v$. If $\Gamma \in (0,1)$, so $a > M$, the source is more elongated in the z direction than in the spherical case; I term such geometries "prolate." Similarly, if $\Gamma \in (1, +\infty)$, $a < M$, the source is more compressed in the z direction than in the spherical case; these geometries I label "oblate." In the limit $\Gamma \rightarrow +\infty$, $a \rightarrow 0$, the Γ metric becomes the Curzon metric.[13]

B. The spherical-topology boundary constraint

The surfaces of constant u in the Γ metrics, equipotentials of ψ_E, provide a convenient one-parameter family of surfaces to which to apply the criterion (3.43). All the necessary quantities can be calculated from the metric (5.3) and the relations between (ρ_E, z_E) and (u,v). The $\mathscr{I}_1$-maximizing interior coordinate derivative $\mathscr{Z}'_0$, as per (3.40), is given by

$$\mathscr{Z}'_0 = -\tanh^{\Gamma}(u/2)\left[1-\left[1+\frac{\sin^2 v}{\sinh^2 u}\right]^{\Gamma^2-1} \times \cos^2 v\right]^{1/2}. \tag{5.4}$$

This exists for all u if $\Gamma \leq 1$; if $\Gamma > 1$, for small u the argument of the square root will be negative for v near 0 and π. I therefore apply criterion (3.43) only to the spherical (Schwarzschild) and prolate Γ metrics. The functional $\mathscr{I}_1[\mathscr{Z}'_0]$ of (3.43) takes the form

$$\begin{aligned}\mathscr{I}_1[\mathscr{Z}'_0;u] = {} & 2M\{2-(\Gamma+1/\Gamma)\cosh u-(1/\Gamma-\Gamma)\sinh^2 u \ln[\coth(u/2)]\} \\ & +(M/\Gamma)\sinh u \int_0^{\pi} dv \left\{\sin^2 v\left[1+\frac{\sin^2 v}{\sinh^2 u}\right]^{(\Gamma^2-3)/2}\left[1+\frac{1-\Gamma^2\cos^2 v}{\sinh^2 u}\right]\left[1-\left[1+\frac{\sin^2 v}{\sinh^2 u}\right]^{\Gamma^2-1}\cos^2 v\right]^{-1/2}\right. \\ & \left. +\left[1+\frac{\sin^2 v}{\sinh^2 u}\right]^{(1-\Gamma^2)/2}\left[1-\left[1+\frac{\sin^2 v}{\sinh^2 u}\right]^{\Gamma^2-1}\cos^2 v\right]^{+1/2}\right\}. \end{aligned} \tag{5.5}$$

The notation $\mathscr{I}_1[\mathscr{Z}'_0;u]$ indicates that the functional $\mathscr{I}_1[\mathscr{Z}']$, evaluated at $\mathscr{Z}'=\mathscr{Z}'_0$, is a function of the u value on the surface to be evaluated. Forbidden surfaces are those with $\mathscr{I}_1[\mathscr{Z}'_0;u]<0$. If $\Gamma=1$, the integral in (5.5) reduces to an elementary form; the resulting expression for $\mathscr{I}_1$ is

$$\begin{aligned}\mathscr{I}_1^{(S)}[\mathscr{Z}'_0;u] &= 4M(1+\sinh u-\cosh u) \\ &= 4M(1-e^{-u}),\end{aligned} \tag{5.6}$$

where the superscript (S) denotes the Schwarzschild case. This result means that for the Schwarzschild exterior geometry, none of the $u=$constant surfaces, with $u \geq 0$, are forbidden. That is, none of the surfaces of constant Schwarzschild radial coordinate r, with $r \geq 2M$, are forbidden. This is in accord with the existence of an exact interior solution which can be matched to the Schwarzschild exterior at any sphere of Schwarzschild radius $r \geq 2M$. Specifically, the Schwarzschild exterior four-geometry can be matched to a closed Friedmann interior geometry to describe an expanding or collapsing sphere of matter with uniform density and zero pressure.[14] At the moment of maximum expansion of the Friedmann interior, the geometry is momentarily static and time-symmetric; all of the hypotheses underlying the boundary-constraint derivation apply here. The Schwarzschild coordinate radius of the matter/vacuum boundary at the moment of stasis can be freely chosen to be any value $r \geq 2M$. The interior Friedmann metric on the hypersurface of time symmetry can be cast in the form (3.12), and the coordinates (ρ_I, z_I) can be chosen so that at the boundary surface ρ_I coincides with ρ_E of the Weyl coordinates for the Schwarzschild exterior. In that case one finds that $\gamma_I=0$ holds throughout the interior, which means, as in Sec. III E, that the interior coordinate derivative on the boundary, $\mathscr{Z}'$, coincides with the $\mathscr{I}_1$-maximizing function $\mathscr{Z}'_0$. That is, the $\mathscr{I}_1$-maximizing interior coordinate used to derive (5.6) actually occurs in the Friedmann/Schwarzschild system.

For prolate Γ metrics, with $\Gamma<1$, an analytic evaluation of the integral in (5.5) is not possible. It can, however, be studied with approximate and numerical calculations. I find that $\mathscr{I}_1$ has the limiting behavior

$$\lim_{u\rightarrow 0}\mathscr{I}_1[\mathscr{Z}'_0;u]=(2M/\Gamma)[-(1-\Gamma)^2+B(1-\Gamma^2/2,1-\Gamma^2/2)(u/2)^{\Gamma^2}+O(u^{2-\Gamma^2})], \tag{5.7}$$

where B is the beta function. Thus at $u=0$, $\mathscr{I}_1[\mathscr{Z}'_0;u]$ takes the value $-(2M/\Gamma)(1-\Gamma)^2$, which is negative for all $\Gamma \in (0,1)$. The function $\mathscr{I}_1[\mathscr{Z}'_0;u]$ increases monotonically with u, approaching the limit $4M$ as $u \rightarrow +\infty$. Consequently in every prolate Γ geometry there is a value u_0 such that $\mathscr{I}_1[\mathscr{Z}'_0;u_0]$ is zero, and $\mathscr{I}_1[\mathscr{Z}'_0;u]$ is negative for $u<u_0$, positive for $u>u_0$. If the higher-order terms in u are neglected, Eq. (5.7) implies that the zero-crossing value u_0 is given by

$$u_0=2[(1-\Gamma)^2/B(1-\Gamma^2/2,1-\Gamma^2/2)]^{1/\Gamma^2} . \tag{5.8}$$

The neglect of the higher-order terms is valid only if u is small and $2(1-\Gamma^2)$ is large; numerical calculations indicate that the fractional error in the value of u_0 given by (5.8) approaches 20% for Γ values near 1, is less than 1% for $\Gamma<0.7$, and vanishes as $\Gamma\to 0$. By the boundary constraint (3.43), surfaces with $u<u_0$ are forbidden as boundary surfaces.

The constraint imposed by the condition $\mathscr{I}_1[\mathscr{Z}'_0;u]\geq 0$ on $u=$constant boundary surfaces in the prolate Γ metrics takes a simple form in the limit in which the background-space source is very large, i.e., the limit $a\to\infty$ or $\Gamma\to 0$. Equations (5.1) and (5.8) imply, in this limit,

$$\lim_{\Gamma\to 0}\psi_E(u_0)=-2+O(\Gamma) . \tag{5.9}$$

Since ψ_E is a monotone increasing function of u, the boundary constraint $\mathscr{I}_1[\mathscr{Z}'_0;u]\geq 0$, i.e., $u\geq u_0$, becomes equivalent to (3.37). This occurs because in the limit $\Gamma\to 0$, the "double match" of coordinates discussed in Sec. III D is achieved; by Eq. (5.2), $\gamma_E\to 0$ as $\Gamma\to 0$, which means $\mathscr{Z}'_0\to Z'$ in that limit, as may also be seen directly.

C. Interpretation of the boundary constraint: Sizes and the hoop conjecture

The results of this example calculation indicate that the boundary constraints of Sec. III do not admit of interpretation as simple size constraints. The three simplest measures of the size of the $u=$constant surfaces in the Γ geometries, consistent with axial symmetry, are the polar circumference $C_P=2\lambda_{\max}$, the equatorial circumference C_E, and the proper area A. These are given by

$$C_P(u)=(2M/\Gamma)\sinh^{\Gamma^2}u\,\tanh^{-\Gamma}(u/2)\times\int_0^{\pi}(\sinh^2u+\sin^2v)^{(1-\Gamma^2)/2}dv , \tag{5.10}$$

$$C_E(u)=(2\pi M/\Gamma)\sinh u\,\tanh^{-\Gamma}(u/2) , \tag{5.11}$$

$$A(u)=(2\pi M^2/\Gamma^2)\sinh^{1+\Gamma^2}u\tanh^{-2\Gamma}(u/2)\times\int_0^{\pi}(\sinh^2u+\sin^2v)^{(1-\Gamma^2)/2}\sin v\,dv . \tag{5.12}$$

These equations show that in any prolate Γ metric, for $u\ll 1$, C_P behaves as $u^{\Gamma^2-\Gamma}$, C_E as $u^{1-\Gamma}$, and A as $u^{(1-\Gamma)^2}$. Thus as $u\to 0$, $C_P\to\infty$, while $C_E\to 0$ and $A\to 0$ in these prolate geometries.

The form of the boundary constraints of Sec. III suggests that these constraints might constitute lower bounds on C_P, i.e., that surfaces with C_P values less than some minimum might be forbidden. But among $u=$constant surfaces in a prolate Γ metric, the forbidden surfaces are those with $0\leq u<u_0$. Since $C_P\to\infty$ as $u\to 0$, the set of forbidden surfaces contains members with C_P values larger than any specified bound, larger than the C_P value of any given nonforbidden surface. Further, since $C_P\to\infty$ as $u\to 0$ and as $u\to\infty$, there exists a positive value of u, for $\Gamma<1$, at which $C_P(u)$ is a minimum. This is a property of the Γ metric, without any reference to the boundary constraint. Numerical calculation reveals that u_0 is small compared to unity (e.g., $u_0<0.01$) for any value of Γ less than 1. Thus at u_0, C_p behaves as $u^{\Gamma^2-\Gamma}$; in particular, C_P is decreasing with increasing u. Therefore the minimum value of C_P must occur at a u value greater than u_0, i.e., on a nonforbidden surface. The C_P values of the forbidden $u=$constant surfaces are bounded *below* by $C_P(u_0)$, given approximately by Eqs. (5.8) and (5.10); neglecting higher-order terms, I obtain

$$C_P(u_0)=(4M/\Gamma)[B(1-\Gamma^2/2,1-\Gamma^2/2)]^{1/\Gamma}\times(1-\Gamma)^{2-2/\Gamma} . \tag{5.13}$$

There exists one forbidden and one nonforbidden $u=$constant surface having C_P equal to any given value greater than $C_P(u_0)$. These results appear to rule out any interpretation of the constraint (3.43) as a lower bound on C_P for acceptable boundary surfaces.

The other simple size measures for the $u=$constant surfaces, C_E and A, vanish at $u=0$ for $\Gamma<1$ and are monotone increasing with u. The condition $u\geq u_0$ for nonforbidden surfaces does put lower bounds on C_E and A for such surfaces, namely, $C_E(u_0)$ and $A(u_0)$, respectively. Equations (5.8), (5.11), and (5.12), with higher-order terms neglected, give

$$C_E(u_0)=(4\pi M/\Gamma)[(1-\Gamma)^2/B(1-\Gamma^2/2,1-\Gamma^2/2)]^{(1-\Gamma)/\Gamma^2} , \tag{5.14}$$

$$A(u_0)=(16\pi M^2/\Gamma^2)B\left[\frac{3-\Gamma^2}{2},\frac{3-\Gamma^2}{2}\right][(1-\Gamma)^2/B(1-\Gamma^2/2,1-\Gamma^2/2)]^{(1-\Gamma)^2/\Gamma^2} . \tag{5.15}$$

However, these equations imply that both $C_E(u_0)\to 0$ and $A(u_0)\to 0$ as $\Gamma\to 0$. Thus given any positive lower bounds on C_E and A, there exists a prolate Γ geometry with nonforbidden surfaces, according to (3.43), having C_E and A values smaller than those bounds. This indicates that condition (3.43) cannot be interpreted in general as a lower bound on C_E or A for acceptable boundary surfaces either.

Condition (3.43), as illustrated in this example, also does not appear to bear upon the hoop conjecture. The conjecture would place a lower bound on the largest circumference of allowed boundary surfaces, but the constraints of Sec. III serve to identify forbidden rather than allowed surfaces. If the Sec. III constraints placed an upper bound on the largest circumference of forbidden boundary surfaces, the result would support, though not prove, the conjecture. But as shown above, forbidden $u=$constant surfaces in prolate Γ metrics can have arbitrarily large polar circumferences. In fact the example of $u=$constant surfaces in Γ metrics does not even test the boundary constraints obtained here against the hoop conjecture; none has arbitrarily small circumference in every direction. Since the integral in (5.10) exceeds $\pi\sinh^{1-\Gamma^2}u$ for all $\Gamma<1$, C_P exceeds C_E for all $u=$constant surfaces in a prolate Γ metric. Numerical calculations show that the minimum value of $C_P(u)$ decreases monotonically with increasing Γ, and therefore $C_P(u)$ for any prolate Γ metric is bounded below by the Schwarzschild minimum $4\pi M$ (a similar result holds for the oblate, $\Gamma>1$, geometries, with the roles of C_P and C_E reversed). Consequently, I have not been able to relate the $\mathscr{I}_1\geq 0$ boundary constraint to the hoop conjecture.

VI. AN EXAMPLE OF THE TOROIDAL-TOPOLOGY BOUNDARY CONSTRAINT

The toroidal boundary constraint (4.11) can also be examined by means of a simple example. One tractable toroidal exterior geometry has a ring source in the background space, given by $\rho_E=b$, $z_E=0$ in terms of the Weyl coordinates.[15] I shall call this exterior solution the "ring metric."

A. The ring metric

This geometry is most easily described using polar coordinates $(\tilde r,\tilde\theta)$ centered on the ring, related to the Weyl cylindrical coordinates by $\rho_E=b+\tilde r\cos\tilde\theta$, $z_E=\tilde r\sin\tilde\theta$. The range of $\tilde r$ is from 0 to ∞; $\tilde\theta$ ranges from 0 to 2π if $\tilde r\leq b$. The coordinate φ is common to both systems. The solution to (3.3) and (3.6) for this system is

$$\psi_E=-(M/\pi)[m/(\rho_E b)]^{1/2}K(m)\,, \tag{6.1}$$

where

$$m\equiv 4\rho_E b/[(\rho_E+b)^2+z_E^2]=4b(\tilde r\cos\tilde\theta+b)/[\tilde r^2+4b(\tilde r\cos\tilde\theta+b)] \tag{6.2}$$

and

$$K(m)=\int_0^{\pi/2}(1-m\sin^2\alpha)^{-1/2}d\alpha \tag{6.3}$$

is the complete elliptic integral of the first kind with parameter m. The solution to Eqs. (3.4) and (3.5) for this function ψ_E is given by

$$\gamma_E=-[M^2m^2/(4\pi^2\rho_E b)][K^2(m)-4(1-m)\dot K(m)\dot K(m)-4m(1-m)\dot K^2(m)]$$
$$-[M^2m^2/(4\pi^2b^2)][K^2(m)-4(1-m)K(m)\dot K(m)+4(1-m)(2-m)\dot K^2(m)]\,, \tag{6.4}$$

where

$$\dot K(m)\equiv dK(m)/dm=\tfrac{1}{2}\int_0^{\pi/2}(1-m\sin^2\alpha)^{-3/2}\sin^2\alpha\,d\alpha\,. \tag{6.5}$$

As before, M is the total mass of the system measured gravitationally at infinity.

The asymptotic forms of ψ_E and γ_E near the ring source ($\tilde r\to 0$) are useful for analyzing surfaces with condition (4.11). These forms are

$$\psi_E=-[M/(\pi b)]\{\ln(8b/\tilde r)-[\tilde r\cos\tilde\theta/(2b)]\ln(8b/\tilde r)+\tilde r\cos\tilde\theta/(2b)+O(\tilde r^2/b^2)\ln(b/\tilde r))\}\,, \tag{6.6}$$

$$\gamma_E=-[M^2/(4\pi^2b^2)]\{\cos\tilde\theta[4b/\tilde r-(\tilde r/b)\ln^2(8b/\tilde r)+(3\tilde r/b)\ln(8b/\tilde r)-5\tilde r/(2b)]$$
$$+2[\ln^2(8b/\tilde r)-2\ln(8b/\tilde r)+1]+O(\tilde r^2/b^2)\ln^2(b/\tilde r))\}\,, \tag{6.7}$$

and are valid for $\tilde{r} \ll b$.

The three-metric of the Weyl exterior geometry for this solution is given by (3.2). In terms of the coordinates $(\tilde{r},\tilde{\theta},\varphi)$, this metric has the form

$$d\sigma_E{}^2=\exp[2(\gamma_E-\psi_E)][d\tilde{r}^2+\tilde{r}^2 d\tilde{\theta}^2]+\exp(-2\psi_E)(b+\tilde{r}\cos\tilde{\theta})^2 d\varphi^2 \qquad (6.8)$$

with ψ_E and γ_E as given above.

B. Application of the boundary constraint

The surfaces of constant $\tilde{r}$, $\tilde{r}\in(0,b)$, are a convenient family of surfaces to test as possible boundaries; Eq. (6.6) shows that these surfaces approach equipotentials of ψ_E as $\tilde{r}/b\to 0$, although they are not such in general. These surfaces cannot be tested by constraint (3.43); the necessary condition (3.41) for the existence of the $\mathscr{I}_1$-maximizing coordinate derivative $\mathscr{Z}'_0$ translates in this example to

$$\sin^2\tilde{\theta}\le\exp[2\gamma_E(\tilde{r},\tilde{\theta})] \quad \forall\tilde{\theta}\in[0,2\pi] \,. \qquad (6.9)$$

Numerical calculations show this to be violated for every case I have examined. However, these surfaces are ideal for testing with the constraint (4.11). All the necessary quantities can be calculated from the equations of Sec. VI A. I label the left-hand side of (4.11) $\mathscr{I}_2$; it is given here by

$$\mathscr{I}_2(\tilde{r})=\int_0^{2\pi}\tilde{r}\,\partial(2\psi_E-\gamma_E)/\partial\tilde{r}\,d\tilde{\theta}+2\pi[(1-\tilde{r}^2/b^2)^{-1/2}-2] \,, \qquad (6.10)$$

where

$$\partial\psi_E/\partial\tilde{r}=[M/(2\pi)](m\rho_E b)^{-1/2}\{[(8\tilde{r}\rho_E b-4b\tilde{r}^2\cos\tilde{\theta})/(4\rho_E b+\tilde{r}^2)^2 + m\cos\tilde{\theta}/\rho_E]K(m)+2m\dot{K}(m)(8\tilde{r}\rho_E b-4b\tilde{r}^2\cos\tilde{\theta})/(4\rho_E b+\tilde{r}^2)^2\} \qquad (6.11)$$

and

$$\partial\gamma_E/\partial\tilde{r}=[M^2/(4\pi^2 b)]((m\cos\tilde{\theta}/\rho_E^2)[K^2(m)-(1-m)L^2(m)] + \{[2K(m)L(m)-(2-m)L^2(m)]/\rho_E+mL^2(m)/b\} \times(8\tilde{r}\rho_E b-4b\tilde{r}^2\cos\tilde{\theta})/(4\rho_E b+\tilde{r}^2)^2) \qquad (6.12)$$

with

$$L(m)\equiv K(m)+2m\dot{K}(m) \qquad (6.13)$$

and m as given by (6.2). Possible $\tilde{r}$=constant boundary surfaces must have $\mathscr{I}_2(\tilde{r})\ge -2\pi$; surfaces with $\mathscr{I}_2(\tilde{r})<-2\pi$ are forbidden.

Numerical calculations show that $\mathscr{I}_2(\tilde{r})$ increases monotonically with increasing $\tilde{r}/b$, with b fixed. A careful evaluation of Eqs. (6.11) and (6.12) at $\tilde{r}=b$ shows that both $\partial\psi_E/\partial\tilde{r}$ and $\partial\gamma_E/\partial\tilde{r}$ are bounded at that limit; consequently the integral in (6.10) remains finite as $\tilde{r}\to b$. The second term diverges; thus

$$\lim_{\tilde{r}\to b}\mathscr{I}_2(\tilde{r})=+\infty \,. \qquad (6.14)$$

The behavior of $\mathscr{I}_2$ in the limit $\tilde{r}\to 0$ can be determined using the asymptotic forms of ψ_E and γ_E, Eqs. (6.6) and (6.7), or by expanding (6.11) and (6.12) in that limit. The resulting integral can be evaluated explicitly, with the result

$$\mathscr{I}_2(\tilde{r})=2\pi\{2M/(\pi b)-[M^2/(\pi^2 b^2)][\ln(8b/\tilde{r})-1]-1+O(\tilde{r}^2/b^2) +O((M/b)(\tilde{r}^2/b^2)\ln(b/\tilde{r}))+O((M^2/b^2)(\tilde{r}^2/b^2)\ln^2(b/\tilde{r}))\} \qquad (6.15)$$

for $\tilde{r}\ll b$. The three separate error terms are given because different error contributions dominate in different ranges of b/M values. This result implies

$$\lim_{\tilde{r}\to 0}\mathscr{I}_2(\tilde{r})=-\infty \,, \qquad (6.16)$$

the limit taken at fixed b. Limits (6.14) and (6.16), plus the monotonicity of $\mathscr{I}_2(\tilde{r})$, show that for every ring geometry (every value of b), there exists a value $\tilde{r}_0\in(0,b)$ at which $\mathscr{I}_2(\tilde{r})=-2\pi$, with $\mathscr{I}_2<-2\pi$ for $\tilde{r}<\tilde{r}_0$ and $\mathscr{I}_2>-2\pi$ for $\tilde{r}$ between $\tilde{r}_0$ and b.

Hence every ring geometry has a range of $\tilde{r}$=constant surfaces, viz., those with $\tilde{r}\in(0,\tilde{r}_0)$, which are forbidden as matter/vacuum boundaries by (4.11), and a range of such surfaces, with $\tilde{r}\in[\tilde{r}_0,b)$, not so forbidden. The value of $\tilde{r}_0$ can be obtained in the appropriate limit from Eq. (6.15); neglecting the error terms, I find

$$\tilde{r}_0=8b/\exp(1+2\pi b/M) . \qquad (6.17)$$

The approximation involved here is accurate for $\tilde{r}_0 \ll b,M$; because of the exponential in the denominator, this approximation is very good for $b\geq M$ and becomes arbitrarily accurate as b/M increases.

As in the example of Sec. V, the boundary constraint obtained in this calculation assumes a simple form in the limit of a large background-space source, which here means $b/M\to\infty$ (and there meant $\Gamma^{-1}=a/M\to\infty$). In this limit $\tilde{r}_0/b$ becomes very small so the surface $\tilde{r}=\tilde{r}_0$ becomes an equipotential of ψ_E. Then Eqs. (6.6) and (6.17) imply

$$\lim_{b/M\to\infty} \psi_E \,|_{\tilde{r}=\tilde{r}_0}=-2+O(M/b) . \qquad (6.18)$$

The function ψ_E increases outward from the ring source, so in this limit the constraint $\tilde{r}\geq\tilde{r}_0$ on possible boundary surfaces becomes equivalent to (3.37). This is a somewhat serendipitous result, since (3.37) follows from the spherical-topology boundary constraint under the assumption that the interior coordinates are double matched. The toroidal-topology constraint, however, does not specify or restrict the interior coordinates at all.

C. Interpretation of the toroidal constraint: Sizes and the hoop conjecture

Like the results of Sec. V, this toroidal example indicates that the boundary constraint (4.11) cannot be simply interpreted as a bound on surface size. The $\tilde{r}$=constant surfaces in the ring metrics have three simple size measures of interest, the meridian circumference λ_{max}, the outer equatorial circumference C_E^+ (defined as the circumference at $\tilde{\theta}=0$), and the area A. These are given by

$$\lambda_{max}(\tilde{r})=\int_0^{2\pi}\exp[\gamma_E(\tilde{r},\tilde{\theta})-\psi_E(\tilde{r},\tilde{\theta})]\tilde{r}\,d\tilde{\theta} , \qquad (6.19)$$

$$C_E^+(\tilde{r})=2\pi(b+\tilde{r})\exp[-\psi_E(\tilde{r},\tilde{\theta}=0)] , \qquad (6.20)$$

$$A(\tilde{r})=2\pi\int_0^{2\pi}\exp[\gamma_E(\tilde{r},\tilde{\theta})-2\psi_E(\tilde{r},\tilde{\theta})]\times(b+\tilde{r}\cos\tilde{\theta})\tilde{r}\,d\tilde{\theta} , \qquad (6.21)$$

where ψ_E and γ_E are given by (6.1) and (6.4) or (6.6) and (6.7) in the appropriate limit. [The inner equatorial circumference C_E^-, that is, the equatorial circumference calculated at $\tilde{\theta}=\pi$, can also be obtained easily in a form like (6.20), but aside from the curious result that for certain values of b and $\tilde{r}$, $C_E^- > C_E^+$, it provides little additional information]. In the limit $\tilde{r}\ll b$, in which (6.6) and (6.7) are valid, the integrals in the above equations can be performed, giving

$$\begin{aligned}\lambda_{max}(\tilde{r})=&2\pi\tilde{r}\exp\{[M/(\pi b)]\ln(8b/\tilde{r})-[M^2/(2\pi^2b^2)][\ln(8b/\tilde{r})-1]^2\}\\&\times I_0([M^2/(4\pi^2b^2)][4b/\tilde{r}-(\tilde{r}/b)\ln^2(8b/\tilde{r})+(3\tilde{r}/b)\ln(8b/\tilde{r})-5\tilde{r}/(2b)]\\&+[M/(\pi b)][\tilde{r}/(2b)][\ln(8b/\tilde{r})-1])[1+O((\tilde{r}/b^2)\ln^2(b/\tilde{r}))] , \end{aligned}\qquad (6.22)$$

$$\begin{aligned}C_E^+(\tilde{r})=&2\pi(b+\tilde{r})\exp([M/(\pi b)]\{\ln(8b/\tilde{r})-[\tilde{r}/(2b)]\ln(8b/\tilde{r})+\tilde{r}/(2b)\})\\&\times[1+O((\tilde{r}^2/b^2)\ln(b/\tilde{r}))] , \end{aligned}\qquad (6.23)$$

$$\begin{aligned}A(\tilde{r})=&4\pi^2\tilde{r}\exp\{[2M/(\pi b)]\ln(8\tilde{b}/r)-[M^2/(2\pi^2b^2)][\ln(8b/\tilde{r})-1]^2\}\\&\times\{bI_0([M^2/(4\pi^2b^2)][4b/\tilde{r}-(\tilde{r}/b)\ln^2(8b/\tilde{r})+(3\tilde{r}/b)\ln(8b/\tilde{r})-5\tilde{r}/(2b)]\\&+[M/(\pi b)](\tilde{r}/b)[\ln(8b/\tilde{r})-1])\\&-\tilde{r}I_1([M^2/(4\pi^2b^2)][4b/\tilde{r}-(\tilde{r}/b)\ln^2(8b/\tilde{r})+(3\tilde{r}/b)\ln(8b/\tilde{r})-5\tilde{r}/(2b)]\\&+[M/(\pi b)](\tilde{r}/b)[\ln(8b/\tilde{r})-1])\}[1+O((\tilde{r}^2/b^2)\ln^2(b/\tilde{r}))] , \end{aligned}\qquad (6.24)$$

where I_0 and I_1 are hyperbolic Bessel functions. As $\tilde{r}\to 0$ (with b fixed), Eqs. (6.22) and (6.24) are dominated by the exponential behavior of I_0, while (6.23) is dominated by $\exp\{[M/(\pi b)]\ln(8b/\tilde{r})\}$, so in this limit λ_{max}, C_E^+, and A all diverge to $+\infty$.

This asymptotic behavior shows that the boundary constraint $\mathscr{I}_2\geq -2\pi$, which in this example takes the form $\tilde{r}\geq\tilde{r}_0$, cannot be interpreted in general as identifying as forbidden surfaces with $\lambda_{\max}$, C_E^+, or A values smaller than some lower bound. Every ring geometry contains forbidden $\tilde{r}$=constant surfaces with $\lambda_{\max}$, C_E^+, and A values greater than any given bounds, greater than those values for any given nonforbidden surface. Thus for a boundary surface to have a size—as measured by $\lambda_{\max}$, C_E^+, or A—greater than some fixed bound is not equivalent to satisfying the constraint (4.11). In fact, numerical calculations show that in a wide range of ring geometries, the $\tilde{r}$=constant surfaces with the smallest values of $\lambda_{\max}$, C_E^+, and A satisfy (4.11).

As $b/M\rightarrow\infty$, with $\tilde{r}/M$ fixed, $\lambda_{\max}(\tilde{r})$ approaches the limit $2\pi\tilde{r}$. Thus $\tilde{r}$=constant surfaces exist in ring geometries with arbitrarily small values of $\lambda_{\max}$, and since $\tilde{r}_0/M$ is dominated by the exponentially decreasing denominator of (6.17) in the $b/M\rightarrow\infty$ limit, ring geometries exist in which surfaces with arbitrarily small $\lambda_{\max}$ values are not forbidden by the toroidal boundary constraint. The constraint, therefore, does not imply a lower bound on $\lambda_{\max}$ for acceptable boundary surfaces. I cannot prove a similar result for the size measures C_E^+ and A using the ring geometries, however, because $\tilde{r}$=constant surfaces with arbitrarily small C_E^+ and A values do not exist in those geometries. Approximate and numerical calculations indicate that the A values for $\tilde{r}$=constant surfaces, for any b/M values, are bounded below by a value slightly in excess of $4M^2$. Similarly, C_E^+ values for these surfaces have a lower bound slightly greater than $18M$.

The above results indicate that the toroidal boundary constraint (4.11) is not directly connected with the hoop conjecture. Like the spherical-topology constraint, the inequality (4.11) serves to identify forbidden boundary surfaces rather than to specify allowed ones; the hoop conjecture characterizes allowed surfaces. Also, the ring-metric example shows that surfaces forbidden by (4.11) can have arbitrarily large circumferences, a result which, while not disproving the hoop conjecture, does not support it. Of course, like the Γ-metric calculation of Sec. V, the ring-metric example does not actually test the hoop conjecture, since the $\tilde{r}$=constant surfaces examined all have circumferences (C_E^+) greater than a fixed bound of order M.

VII. EXTENSION OF THESE RESULTS

Some of the results of Secs. III and IV can be applied to surfaces in a wider class of exterior geometries than the Weyl metrics. Specifically, the assumption of vacuum in the exterior can be relaxed. The spatial metric form (3.2) can be obtained in any axisymmetric, static exterior; the corresponding four-metric takes the form (3.1) only in special cases of this, including vacuum,[16] while the Weyl equations (3.3), (3.4), and (3.5) occur only for the vacuum case. The derivation of the spherical-topology boundary constraint, up to the inequality (3.24), depends only on the three-metric form (3.2); the Weyl vacuum field equations are used only in obtaining (3.30) from (3.24). Thus the spherical-topology constraint $\mathscr{I}_1[\mathscr{Z}']\geq 0$ can be applied to surfaces in any axisymmetric, static exterior geometry, provided $\mathscr{I}_1[\mathscr{Z}']$ is given by

$$\mathscr{I}_1[\mathscr{Z}']=\int_0^{\lambda_{\max}}d\lambda\{R\,[d(2\psi_E-\gamma_E)/dn-\alpha_E'+\alpha_I'] +Z'-\mathscr{Z}'(1+\gamma_I)\} \qquad (7.1)$$

with all quantities defined as before, rather than by (3.30). The results (3.36) and (3.37) for the case of double-matched coordinates do not apply for nonvacuum exteriors, since these are derived from (3.30). The maximization of $\mathscr{I}_1$ with respect to $\mathscr{Z}'$ carried out in Sec. III E can be performed in nonvacuum cases, however, since (3.24) has the same $\mathscr{Z}'$ dependence as (3.30). The derivation of the toroidal-topology boundary constraint in Sec. IV never utilizes the Weyl field equations, so the constraint (4.11) can be applied in nonvacuum exteriors with no change.

It is thus possible to identify surfaces in arbitrary axisymmetric, static exterior geometries which are forbidden as boundaries of momentarily static, nonsingular matter systems by means of the inequalities (3.24) or (4.11), for either spherical or toroidal topology, respectively. The usefulness of this generalization to nonvacuum exteriors is limited, however, by the fact that the exterior geometry, specifically the exterior coordinates and the potentials ψ_E and γ_E, must be known explicitly in order to evaluate the constraint inequalities. Consequently, these constraints are likely to prove most useful for special exteriors, such as the Weyl vacuum geometries or electrovacuum generalizations thereof.

VIII. SUMMARY

Approaching the problem of boundary constraints for noncollapsing axisymmetric systems via the time-symmetric initial-value formalism, I have here obtained the two geometric conditions $\mathscr{I}_1\geq 0$ and $\mathscr{I}_2\geq -2\pi$ for spherical and toroidal topologies, respectively, where $\mathscr{I}_1$ is given by the left side of (3.24) or (3.30), and $\mathscr{I}_2$ is given by the left side of (4.11). Both of these constraints identify surfaces in given exterior geometries which are forbidden as boundaries of momentarily static, nonsingular matter systems, i.e., surfaces for which the derived inequalities are violated are so forbidden. Applied to specific examples using as exterior geometries the prolate Γ metrics and the toroidal ring metrics, these constraints define neighborhoods of the Weyl singularities within which the surfaces examined are forbidden; i.e., the constraints require that the surfaces of momentarily static material bodies giving rise to these exterior geometries lie outside the delineated neighborhoods.

I have not been able to cast these boundary constraints in any form suggestive of bounds on surface size. Further, the results of the example calculations of Secs. V and VI are not in accord with any interpretation of the constraints as limits on boundary circumference or area. Consequently, these constraints do not in any obvious way embody or support the hoop conjecture. Neither do my results disprove the conjecture: since none of the source-surrounding surfaces in the geometries I have examined have circumferences much smaller than $4\pi M$ in all directions, no counterexample to the hoop conjecture could be found. Apparently the quantities $\mathscr{I}_1$ and $\mathscr{I}_2$ appearing in the constraint inequalities describe geometric properties of momentarily static axisymmetric systems, as constructed here, distinct from boundary size; my results show that these properties do impose limits on the construction of such systems. I have not succeeded in formulating an intuitive interpretation of these properties; perhaps their meaning can be clarified by further researches and by studying the application of these constraints to more examples of Weyl exterior geometries.

The initial-value formalism used here might prove useful in other investigations of size constraints and the hoop conjecture. Manipulations of the initial-value equation and junction conditions different from those I employed in Secs. III and IV might yield different results and constraints, perhaps ones readily related to system size and to the hoop conjecture. It is also possible that results germane to the hoop conjecture might be obtained directly from the vacuum, static Einstein field equations, or from the Weyl form of those equations given in Sec. III A. The fact that all source-surrounding surfaces in the Weyl geometries I have studied here have largest circumferences exceeding a limit of order M lends support to this possibility.

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APPENDIX A: EXISTENCE OF ISOTHERMAL INTERIOR COORDINATES

The uniformization theorem for simply connected Riemann surfaces states that a simply connected Riemann surface is conformally equivalent to (i.e., there exists a bijective function, analytic in the local complex coordinates on the Riemann surface, from that surface onto) exactly one of the following: the extended complex plane $C \cup \{\infty\}$, the complex plane C, or the unit disk $\Delta = \{z \in C : |z| < 1\}$, according as whether the surface is elliptic, parabolic, or hyperbolic, respectively.[17] A Riemann surface is elliptic if it is compact (closed), hyperbolic if it is noncompact and carries a negative nonconstant subharmonic function, and parabolic if it is noncompact but carries no such function. Further, any simply connected domain in the extended complex plane which omits two or more points of $C \cup \{\infty\}$ is hyperbolic.[18] Of interest here is the application of the uniformization theorem to the two-surface $\mathscr{M} = I\,|_{\varphi=\varphi_0}$, a φ=constant slice of the interior, to establish the existence of a single patch of isothermal coordinates covering all of $\mathscr{M}$, and thus a single patch of coordinates for I in which the metric has the form (3.12) or (4.1).

The nonsingularity assumption made here implies the existence of a C^1 two-metric[19] on $\mathscr{M}$. This, in turn, implies the existence of local C^1 isothermal coordinates[20] in neighborhoods of all points of $\mathscr{M}$. Where two such coordinate patches, say (x,y) and (u,v), overlap, they must be related by the Cauchy-Riemann equations, hence $u \pm iv$ must be an analytic function of $x+iy$; that is, the local isothermal coordinates impart a Riemann-surface structure to $\mathscr{M}$. Let $\mathscr{N} = \Sigma\,|_{\varphi=\varphi_0}$ be a constant-φ slice of the entire hypersurface of time symmetry (interior I and exterior E). By the above arguments, $\mathscr{N}$ is also a Riemann surface. I assume that $\mathscr{N}$, like, $\mathscr{M}$, is simply connected; the uniformization theorem then guarantees that $\mathscr{N}$ is conformally equivalent to some domain in $C \cup \{\infty\}$. The function establishing this conformal equivalence maps $\mathscr{M} \subset \mathscr{N}$ into some subdomain of this; since $\mathscr{M}$ omits more than two points of $\mathscr{N}$ and the function is bijective, the image of $\mathscr{M}$ omits more than two points of $C \cup \{\infty\}$ and is therefore hyperbolic. Consequently, $\mathscr{M}$ is hyperbolic and conformally equivalent to the unit disk Δ and to any hyperbolic domain of $C \cup \{\infty\}$. The functions on $\mathscr{M}$ establishing these equivalences are analytic in $x+iy$ for all local isothermal coordinates

(x,y), so they preserve the isothermal form of the metric and provide the desired global isothermal coordinate patches on $\mathscr{M}$.

The interior coordinates (ρ_I,z_I) can be taken to fill any hyperbolic domain in the extended complex plane. For example, for a toroidal interior a unit disk translated into the right-half plane, i.e.,

$$\{z\in C:\ z=\alpha+re^{i\theta},\alpha>1,0\leq r<1,\ 0\leq\theta\leq 2\pi\}\ ,$$

or simply the unit disk Δ [if the metric form (4.1) is used so that the restriction $\rho_I>0$ is not required], might be a convenient choice. Let f: $\mathscr{M}\rightarrow\Delta$ be the conformal map establishing these interior coordinates, neglecting the translation by α. Then f maps the boundary meridian $\partial\mathscr{M}$ onto the unit circle. Results from the mathematical theory of conformal representation serve to establish the regularity of this coordinate map at the boundary. Specifically, if $R\subset C$ is a simply connected domain and $P\in\partial R$ a frontier point of R such that it is possible to construct two circles through P, one entirely inside R and one entirely outside, and if g: $R\rightarrow\Delta$ is a conformal map of R into the unit disk, then as z approaches P in R, the derivative $g'(z)$ tends to a unique, finite, nonzero limit.[21] Here P is a boundary meridian point; let $N(P)$ be a local (isothermal) coordinate neighborhood of P on the φ=constant surface $\mathscr{N}$. I take R to be (the local-coordinate image of) $N(P)\cap\mathscr{M}$ and g to be the global coordinate map f composed with the local coordinate map on $N(P)$. The quoted theorem implies that the derivative of f is nonvanishing—the interior coordinates are regular—at any meridian point P at which the tangent vector $d/d\lambda$ is continuous. Most importantly, this means that $d\theta/d\lambda$, the derivative of coordinate polar angle with respect to meridian proper length, does not vanish anywhere on a C^1 or smoother meridian; the length λ and coordinate angle θ are monotonic functions of each other.

Similar conclusions obtain for spherical-topology interiors, except that for this case $\partial\mathscr{M}$ has "corners" where the meridian meets the symmetry axis. If a unit disk is chosen as the range of the interior coordinates, the coordinate map f must behave as $(z-z_0)^2$, in terms of a local complex coordinate z, in the vicinity of such a corner (at z_0), in order to convert the right angle of the corner into a straight angle on the unit circle. It is convenient to compose such a coordinate function with a function such as

$$\zeta(w)=-i\frac{[-i(w+1)/(w-1)]^{1/2}-1}{[-i(w+1)/(w-1)]^{1/2}+1}\ ,\quad w\in\Delta \tag{A1}$$

which, using the principal branch of the square root, maps the unit disk bijectively onto the right half of the unit disk. The points $w=\pm 1$ are mapped to the corners $\zeta=\mp i$, the square-root behavior of $\zeta(w)$ there converting straight angles to right angles. The composition of f as above, a Möbius transformation on the unit disk if needed to position the corner points, and the function ζ gives a conformal map of $\mathscr{M}$ onto the right half of the unit disk, providing isothermal interior coordinates appropriate to the spherical-topology case. These coordinates are regular at all boundary points, the square-root behavior of ζ at the corner points canceling the square behavior of f there.

APPENDIX B: INJECTIVITY OF MATCHED INTERIOR COORDINATES

The existence of isothermal interior coordinates, the first of which matches the corresponding exterior Weyl coordinate (ρ_E) at the boundary, hinges on the injectivity of the analytic function $\tilde{\rho}_I+i\tilde{z}_I$:$\mathscr{M}\rightarrow C$ defined by the stipulation that the real part $\tilde{\rho}_I$ is a harmonic function on $\mathscr{M}$ with boundary values $\tilde{\rho}_I=\rho_E$ on the boundary meridian and, in the spherical-topology case, $\tilde{\rho}_I=0$ on the symmetry axis. Alternatively, the function $\tilde{\rho}_I+i\tilde{z}_I$ can be regarded as an analytic function on the domain $D\subset C$ occupied by some global isothermal interior coordinates (ρ_I,z_I), with the boundary values for $\tilde{\rho}_I$ appropriately mapped from $\partial\mathscr{M}$ to ∂D. By the results of Appendix A, D may without loss of generality be taken to be the unit disk Δ for the toroidal-topology case, or the right half thereof $\Delta_+\equiv\{z\in\Delta:\text{Re}z>0\}$, for the spherical-topology case.

The number of times an analytic function takes on a given value in its range can be counted by examining its behavior on the boundary of its domain. Let $\mathscr{D}\subset C$ be a domain bounded by a simple closed curve $\mathscr{C}$, and let $w=f(z)$ be an analytic function regular in $\mathscr{D}$ and on $\mathscr{C}$; further let $\mathscr{C}'$ be the image of $\mathscr{C}$ under f, and suppose z_0 is a point in $\mathscr{D}$ with $f(z_0)$ not on $\mathscr{C}'$. Then the quantity

$$[1/(2\pi)]\Delta_{\mathscr{C}}\arg[f(z)-f(z_0)]=[1/(2\pi i)]\oint_{\mathscr{C}}d\ln[f(z)-f(z_0)]\ , \tag{B1}$$

where $\Delta_{\mathscr{C}}$ denotes variation around $\mathscr{C}$, equals the number of zeros of $f(z)-f(z_0)$ in $\mathscr{D}$, a positive integer since z_0 is in $\mathscr{D}$. But this integral is equal to

$$[1/(2\pi i)]\oint_{\mathscr{C}'}[1/(w-w_0)]dw=[1/(2\pi)]\Delta_{\mathscr{C}'}\arg(w-w_0)\ , \tag{B2}$$

which is just the number of times the curve $\mathscr{C}'$

encircles the point $w_0=f(z_0)$, i.e., the number of times the point $f(z)$ circles w_0 as z is moved once or around the curve $\mathscr{C}$. In particular, f is injective in $\mathscr{D}$ if $\mathscr{C}'$ is simple, i.e., unless the curve $\mathscr{C}'$ is traced multiple times by $f(z)$ as z circles $\mathscr{C}$ once or $\mathscr{C}'$ loops inside itself.[22]

Thus the image points of ∂D under the map $\tilde{\rho}_I+i\tilde{z}_I$ reveals whether the matched-interior-coordinate function is injective. The complete image depends on the interior geometry, but the first coordinates of the image points are given by the boundary conditions on $\tilde{\rho}_I$. If the exterior radial coordinate of the meridian, $R(\lambda)$, has a single local maximum, then the image curve under the matched-coordinate map cannot be traced multiple times, nor can it loop inside itself. Consequently for $R(\lambda)$ to have a single local maximum is a sufficient condition for the injectivity of the matched-coordinate map $\tilde{\rho}_I+i\tilde{z}_I$, and thus for the existence of the matched coordinates described in Sec. III C. It may be noted that this condition obtains in the examples discussed in Sec. V, where matched interior coordinates are assumed.

APPENDIX C: GLOBAL MAXIMALITY OF $\mathscr{I}_1[\mathscr{Z}'_0]$

In cases where the conditions for the existence and maximality of the $\mathscr{I}_1$-maximizing interior coordinate derivative $\mathscr{Z}'_0$, described in Sec. III E, are satisfied, the question whether $\mathscr{Z}'_0$ gives a global (in the space of functions $\mathscr{Z}'$) or just a local maximum of $\mathscr{I}_1$ can be examined by reducing the variational problem to a collection of maximization problems in one real variable. Let $\eta(\lambda)$ be a C^1 function on $[0,\lambda_{\max}]$ with appropriate end-point values. If I take $\mathscr{Z}'=\mathscr{Z}'_0+q\eta$, then the functional $\mathscr{I}_1[\mathscr{Z}']$ becomes a function of the variable q, a different function for each η, which I denote $\mathscr{I}_1(q;\eta)$. The variational problem with respect to $\mathscr{Z}'$ is equivalent to the one-variable maximization problem with respect to q, considered for all possible functions η. In particular, $\mathscr{Z}'_0$ gives a global maximum of $\mathscr{I}_1[\mathscr{Z}']$ if and only if $q=0$ gives a global (over the domain of q) maximum of $\mathscr{I}_1(q;\eta)$ for every possible η.

The derivatives of $\mathscr{I}_1(q;\eta)$ are given by expressions similar to (3.38) and (3.39),

$$d\mathscr{I}_1(q;\eta)/dq=\int_0^{\lambda_{\max}}\{\tfrac{1}{2}\ln[R'^2+(\mathscr{Z}'_0+q\eta)^2]-\psi_E\}\eta\, d\lambda\ , \tag{C1}$$

$$d^2\mathscr{I}_1(q;\eta)/dq^2=\int_0^{\lambda_{\max}}\{(\mathscr{Z}'_0+q\eta)/[R'^2+(\mathscr{Z}'_0+q\eta)^2]\}\eta^2 d\lambda\ . \tag{C2}$$

The choice of $\mathscr{Z}'_0$, Eq. (3.40), ensures $d\mathscr{I}_1/dq=0$ at $q=0$ for every η, and if the topology of the system imposes $\mathscr{Z}'_0\le 0$ on the entire meridian, then $d^2\mathscr{I}_1/dq^2<0$ holds at $q=0$ for every η, so $\mathscr{Z}'=\mathscr{Z}'_0$ or $q=0$ is a local maximum of $\mathscr{I}_1$. It remains to be shown whether, for any particular η, there exist any other local (in q) maxima with larger values of $\mathscr{I}_1$ than $\mathscr{I}_1(0;\eta)$ or $\mathscr{I}_1$ takes on values larger than $\mathscr{I}_1(0;\eta)$ at the boundaries of the domain of q.

I have not been able to resolve these questions. A principal difficulty is delineating the domain of q, given η, i.e., characterizing the set of functions $\mathscr{Z}'$ which are possible second-coordinate derivatives for some interior geometry. If, for example, $\mathscr{Z}'\le 0$ on the entire meridian [by (3.15), this is equivalent to $d\rho_I/dn\ge 0$ on the entire meridian] holds for any admissible interior second coordinate, with matched first coordinate, then $d^2\mathscr{I}_1/dq^2<0$ holds on the entire domain of q for any η. If the domain of q is connected, this implies that $q=0$ is a global maximum for any η, hence that $\mathscr{Z}'=\mathscr{Z}'_0$ is a global maximum. The sufficient condition $\mathscr{Z}'\le 0$ is guaranteed if α'_I, defined by (3.17), is required to be non-negative, provided $R'>0$, $\mathscr{Z}'=0$ at $\lambda=0$, $R'<0$, $\mathscr{Z}'=0$ at $\lambda=\lambda_{\max}$, and the matched interior coordinates are injective, so the interior coordinate image of the meridian does not loop, as discussed in Appendix B. For these conditions imply $\tan^{-1}(\mathscr{Z}'/R')\in[-\pi,0]$, which precludes $\mathscr{Z}'>0$. I have not succeeded, however, in establishing these or any other sufficient conditions for the global maximality of $\mathscr{Z}'_0$ in general, nor in finding any constraints on exterior quantities under which such conditions obtain. The question of global maximality remains open; I assume it in applying the results of Sec. III E.

APPENDIX D: THE INTEGRAL OF α'_I FOR TOROIDAL GEOMETRIES

It is easily shown that

$$\int_0^{\lambda_{\max}}\alpha'_I d\lambda=-\int_0^{\lambda_{\max}}\frac{P'\mathscr{Z}''-\mathscr{Z}'P''}{P'^2+\mathscr{Z}'^2}d\lambda=2\pi \tag{D1}$$

holds for any toroidal geometry, with arbitrary nonsingular interior and with $P(\lambda)$ and $\mathscr{Z}(\lambda)$ the meridian values of any choice of isothermal interior coordinates. First, rearranging Eq. (4.10) gives

$$\int_0^{\lambda_{\max}} \alpha_I' d\lambda = [1/(2\pi)] \int_I d^3V e^{-\beta}[8\pi\epsilon + (\nabla\beta)^2] - \int_0^{\lambda_{\max}} d\lambda\, \mathrm{Tr} S_E \,. \tag{D2}$$

Everything on the right side of this equation is gauge invariant, i.e., invariant under conformal transformations of the interior coordinates, so the integral of α_I' over the meridian is likewise invariant. It can therefore be evaluated directly by making a convenient choice of interior coordinates. By the results described in Appendix A, the interior coordinates may be chosen to fill the unit disk, so that $P(\lambda)=\cos[\theta(\lambda)]$, $\mathscr{Z}(\lambda)=\sin[\theta(\lambda)]$, with the polar angle θ a monotonic function of λ. Substituting these into the definition of α_I', I obtain

$$\int_0^{\lambda_{\max}} \alpha_I' d\lambda = -\int_{2\pi}^0 d\theta = +2\pi \,, \tag{D3}$$

which is the desired result. The orientation of the end-point values of $\theta(\lambda)$ is determined by the stipulation that (3.15) give the outward-directed normal vector, which requires $d\theta/d\lambda < 0$.

A similar result does not obtain for the integral of α_E'. In the ring-metric example of Sec. VI, the integral of α_E' over the meridian does equal 2π. In Thorne's[8] toroidal exterior solution, however, there are closed curves, candidates for boundary meridians, which are curves of constant ρ_E; the integral of α_E' over such curves is zero. Thus it is necessary to evaluate the integral of α_E' explicitly in each individual case.

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