Bound states of a relativistic quark confined by a vector potential

Chris Long and D. Robson
Physics Department, Florida State University, Tallahassee, Florida 32306
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The prevailing view that infinitely rising potentials which are used in the Dirac equation to confine quark-antiquark systems must not be Lorentz vectors is reexamined. By considering the infinite-mass limit of the Bethe-Salpeter equation, a relativistic single-particle equation which does allow vector confinement potentials to yield a discrete spectrum is obtained.

Mesons with one light quark and one heavy quark are known to exist and should be describable at least approximately by a single-particle relativistic equation. However, the Dirac equation with a pure Lorentz vector potential which is infinite at large separations is known[1] to have no bound states. For this reason it has been argued[2] that the long-range confining potential must be a Lorentz scalar. The purpose of the present work is to point out that a vector confining potential *does* give bound states when it is used in a one-particle reduction of the two-particle instantaneous Bethe-Salpeter equation.

Here the terms vector and scalar refer to the way the potential is added to the free-particle Dirac Hamiltonian. Specifically, vector potential means that the potential is multiplied by the same Dirac matrices as the energy-momentum four-vector, while scalar means that the potential is multiplied by the same Dirac matrix as the rest mass of the constituent particle. The particular matrices required depend on one's choice of representation.

We begin from a relativistic two-particle equation and then let the mass of one particle become very large thereby yielding a one-particle equation. Following the classic paper by Salpeter,[3] but using more modern notation we begin from the modified Bethe-Salpeter equation

$$F(p)\psi(p)=-(2\pi i)^{-1}\int d^4k\,\beta_a\beta_b\bar{G}(k)\psi(p+k) \quad (1)$$

in which the interaction is denoted by $\bar{G}(k)$ and

$$p=(p^0,\vec{p}),\quad k=(k^0,\vec{k})$$

are four-momenta. In Eq. (1), $F(p)$ is defined by

$$F(p)=[\mu_a E-H_a(\vec{p})+p^0][\mu_b E-H_b(\vec{p})-p^0]\;,$$

$$H_a(\vec{p})=m_a\beta_a+\vec{p}\cdot\vec{\alpha}_a,\quad H_b(\vec{p})=m_b\beta_b-\vec{p}\cdot\vec{\alpha}_b\;,$$

$$\mu_a=m_a/(m_a+m_b),\quad \mu_b=1-\mu_a\;.$$

In order to simplify Eq. (1), the instantaneous approximation is usually adopted. In this approximation k^0 is set equal to zero. The result $|k^0|\ll|\vec{k}|$ occurs automatically when the mass of one of the particle (m_b, say) becomes very large. This is due to the conservation of four-momentum at the massive-fermion interaction vertex, i.e.,

$$k^0=[m_b{}^2+(\vec{p}\,'_b)^2]^{1/2}-[m_b{}^2+(\vec{p}_b)^2]^{1/2}$$

$$\vec{k}=\vec{p}\,'_b-\vec{p}_b$$

so that

$$\frac{k^0}{|\vec{k}|}\to 0 \quad \text{as } m_b\to\infty\;.$$

This result is valid whenever one of the vertices contains a massive-fermion line.

Thus, in the limit of one of the masses very large, the instantaneous approximation becomes exact. Using $k^0=0$ Salpeter reduces Eq. (1) to a set of four coupled three-dimensional equations,

$$[E-E_a(\vec{p})-E_b(\vec{p})]\phi^{++}(\vec{p})=\Lambda_a^+(\vec{p})\Lambda_b^+(\vec{p})\int d^3k\,G'(\vec{k})\phi(\vec{p}+\vec{k})\;, \quad (2a)$$

$$[E+E_a(\vec{p})+E_b(\vec{p})]\phi^{--}(\vec{p})=-\Lambda_a^-(\vec{p})\Lambda_b^-(\vec{p})\int d^3k\,G'(\vec{k})\phi(\vec{p}+\vec{k})\;, \quad (2b)$$

$$\phi^{+-}(\vec{p})=0\;, \quad (2c)$$

$$\phi^{-+}(\vec{p})=0\;, \quad (2d)$$

where

$$E_a(\vec{p})=(m_a^2+\vec{p}^{\,2})^{1/2}\ ,\quad \Lambda_a^{\pm}(\vec{p})=\tfrac{1}{2}[1\pm H_a(\vec{p})/E_a(\vec{p})]\ ,$$

with similar definitions for particle b.

In Eqs. (2) the interaction is represented by

$$G'(k)=\beta_a\beta_b\bar{G}(k),\quad k=(0,\vec{k})\ ,$$

and

$$\phi^{++}(\vec{p})=\Lambda_a^{+}(\vec{p})\Lambda_b^{+}(\vec{p})\phi(\vec{p})\ ,\quad \phi^{+-}(\vec{p})=\Lambda_a^{+}(\vec{p})\Lambda_b^{-}(\vec{p})\phi(\vec{p}),\ \text{etc.}$$

Equations (2a) and (2b) are coupled, but the effects of this coupling on the eigenvalues and eigenfunctions are proportional to $(2m_b)^{-1}$ so that in the limit $m_b\to\infty$ we have the uncoupled equations

$$[E-E_a(\vec{p})-E_b(\vec{p})]\phi^{++}(\vec{p})=\Lambda_a^{+}(\vec{p})\Lambda_b^{+}(\vec{p})\int d^3k\, G'(\vec{k})\phi^{++}(\vec{p}+\vec{k}) \tag{3a}$$

and

$$[E+E_a(\vec{p})+E_b(\vec{p})]\phi^{--}(\vec{p})=-\Lambda_a^{-}(\vec{p})\Lambda_b^{-}(\vec{p})\int d^3k\, G'(\vec{k})\phi^{--}(\vec{p}+\vec{k})\ . \tag{3b}$$

In the same limit we have

$$E_b\to m_b$$

and

$$\Lambda_b^{\pm}(p)\to(1\pm\beta_b)/2\ ,$$

so Eq. (3a) becomes

$$[E'-E_a(\vec{p})]\Lambda_a^{+}(\vec{p})\chi(\vec{p})=\Lambda_a^{+}(\vec{p})\int d^3k\, g(\vec{k})\Lambda_a^{+}(\vec{p})\chi(\vec{p}+\vec{k})\ , \tag{4}$$

where $E'=E-m_b$ is the total energy of particle a,

$$\chi(\vec{p})=[(1+\beta_b)/2]\phi(\vec{p})$$

is the wave function of particle a, and

$$g(\vec{k})=[(1+\beta_b)/2]G'(\vec{k})[(1+\beta_b)/2]$$

is the interaction of particle a with the much more massive particle b.

Introducing $V(\vec{r})$ and $\chi(\vec{r})$ as the interaction and wave function in coordinate space, and noting that $E_a(\vec{p})\Lambda_a^{+}(\vec{p})=H_a(\vec{p})\Lambda_a^{+}(\vec{p})$, Eq. (4) can be written in operator notation as

$$(E'-H_a)\Lambda_a^{+}\chi(\vec{r})=\Lambda_a^{+}V(\vec{r})\Lambda_a^{+}\chi(\vec{r})\ , \tag{5}$$

which differs from the Dirac equation by the presence of the projection operators Λ_a^{+}.

Equation (5) can be converted to an ordinary eigenvalue problem by applying a single free-particle Foldy-Wouthuysen transformation which has the properties

$$UU^{\dagger}=U^{\dagger}U=1\ ,U\Lambda_a^{+}U^{\dagger}=(1+\beta_a)/2\ .$$

Equation (5) then gives

$$[(1+\beta_a)/2]\{E'-U[H_a+V(\vec{r})]U^{\dagger}\}[(1+\beta_a)/2]U\chi(\vec{r})=0\ . \tag{6}$$

In this representation the projections can be carried out by simply removing the negative-energy components. There is a similar reduction of Eq. (3b) which describes the antiparticle of the composite system. When applied to the hydrogen atom, Eq. (6) gives the usual relativistic energy corrections of order $(v/c)^2$ to the nonrelativistic theory.

The Hamiltonian of Eq. (6) with $V(\vec{r})=m_a^2|\vec{r}|$ was represented as a matrix in the spherical-harmonic-oscillator basis and diagonalized. Figure 1 shows the lowest-energy eigenvalues plotted as a function of the oscillator-basis length scale a_0, a variational parameter. As the number of basis states is increased, the low-lying eigenstates become stable

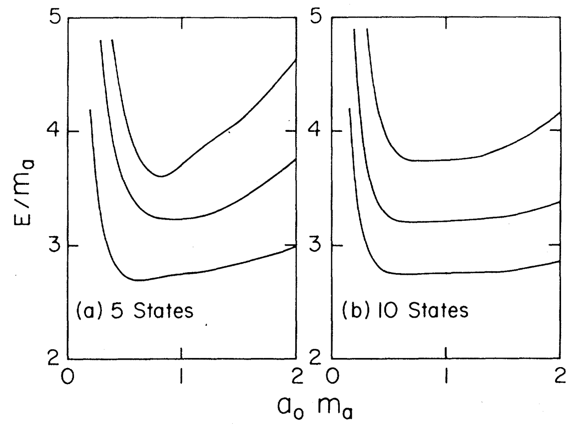

FIG. 1. Energy eigenvalues obtained by diagonalizing Eq. (6) with the potential $V(\vec{r})=m_a{}^2r$ showing convergence as the number of oscillator-basis states is increased from five states (a) to ten states (b), as a function of the oscillator length scale a_0, a variational parameter.

and more accurate in a similar way to their Schrödinger counterparts.

For comparison, when the Dirac equation [Eq. (6) with the operators $(1+\beta_a)/2$ replaced by unity] was diagonalized using the same potential, the numerical solutions did not converge. This is a numerical manifestation of the absence of bound states.

In summary we find that the $m_b \rightarrow \infty$ limit of the Bethe-Salpeter equation yields a projected equation which allows infinitely rising vector potentials to be used to describe bound states of quark-antiquark systems with one massive constituent. The lack of bound states in the Dirac equation is due to the coupling between the positive- and negative-energy components in the Foldy-Wouthuysen representation. While the above discussion leads to a workable relativistic equation for vector confining potentials it does not in any way prevent the existence of confinement interactions with different Lorentz properties.

Since many interesting quarkonium states involve $m_a = m_b$ we have also begun investigating the Bethe-Salpeter equation with m_a, m_b finite. In the instantaneous approximation and a vector confinement potential a bound state spectrum is obtained under most circumstances.[4] Detailed results on the more general two-particle equations will be reported at a later time.

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[1]M. S. Plesset, Phys. Rev. 41, 278 (1932); M. E. Rose, *ibid.* 82, 470 (1951).

[2]S. B. Gerasimov, in *Progress in Particle and Nuclear Physics,* edited by D. H. Wikinson (Pergamon, New York, 1982), Vol. 8, p. 209. R. H. Dalitz, *ibid.* p. 39.

[3]E. E. Salpeter, Phys. Rev. 87, 328 (1952).

[4]L. Hostler and W. Repko, Ann. Phys. (N.Y.) 130, 329 (1980), and references therein.