$qq\bar{q}\bar{q}$ system in a potential model

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We have examined the $qq\bar{q}\bar{q}$ system in a nonrelativistic potential model with color-dependent confinement forces and hyperfine interactions by solving the four-particle Schrödinger equation variationally. We find that normally the ground state of this system consists of two free mesons, but that exceptions to this rule probably occur for $K\bar{K}$ systems, where we find weakly bound 0^{++} states with a meson-meson structure reminiscent of the nucleon-nucleon structure of the deuteron. We show that these states may be identified with the S^* and δ just below $K\bar{K}$ threshold. We further argue that the $qq\bar{q}\bar{q}$ system is not only nearly barren of bound states, but that it may not support any resonances. Finally, independent of their identification with observed states, we note that the $qq\bar{q}\bar{q}$ bound states are a model for the weak binding and color-singlet clustering observed in nuclei.

I. INTRODUCTION

We have examined the $qq\bar{q}\bar{q}$ system[1] since it is the simplest possible candidate for a multiquark hadron: its existence is neither forbidden by color confinement (like q, qq, and $qq\bar{q}$), nor required by it (like $q\bar{q}$ and qqq). This system has previously been studied quite extensively in the frameworks of the bag model,[2,3] of nonrelativistic potential models,[4] and from many other points of view.[5,6] In general, it has been concluded that a very dense discrete spectrum of such states exists, interlacing the normal $q\bar{q}$ spectrum. Our conclusion is that $qq\bar{q}\bar{q}$ bound states—with the probable exception of two S-wave deuteronlike states—do not exist, nor, probably, do any $qq\bar{q}\bar{q}$ resonances. We are at variance with previous potential-model studies because they have failed to properly take into account an important long-range color-mixing effect. That our conclusions differ from those of bag-model studies indicates that at the theoretical level the existence of multiquark hadrons is model dependent. However, if the analogs of the dynamics we are seeing in our model exist in the bag model, they are part of the as yet unknown properties of the surface of the bag; for that reason we believe that potential models are more realistic tools for studying these systems. Not coincidentally, our conclusions are more consonant with recent attempts within the bag-model approach to take into account bag fission via the P matrix.[7]

II. A SIMPLIFIED MODEL FOR THE $qq\bar{q}\bar{q}$ SYSTEM

The solution of the four-body $qq\bar{q}\bar{q}$ problem, in view of the nontrivial color degree of freedom this system possesses, is difficult in the simplest of models: even in the case of a pure harmonic color dependent potential the problem is nonseparable. In view of this complexity, we have chosen to solve the simplest model that we can devise which remains a realistic description of this system, namely,

$$H=\sum_{i=1}^{4}\left[m_i+\frac{p_i^2}{2m_i}\right]+\sum_{i<j}(H^{ij}_{\rm conf}+H^{ij}_{\rm hyp}) , \tag{1}$$

where with $\vec{r}_{ij}\equiv\vec{r}_i-\vec{r}_j$,

$$H^{ij}_{\rm conf}=-(e_0+\tfrac{1}{2}kr_{ij}^2)\frac{\vec{\lambda}_i}{2}\cdot\frac{\vec{\lambda}_j}{2} \tag{2}$$

and

$$H^{ij}_{\rm hyp}=-\frac{8\pi\alpha_s}{3m_im_j}\tilde{\delta}^3(\vec{r}_{ij})\vec{S}_i\cdot\vec{S}_j\frac{\vec{\lambda}_i}{2}\cdot\frac{\vec{\lambda}_j}{2} , \tag{3}$$

in which m_i, $\vec{r}_i$, $\vec{p}_i$, $\vec{S}_i$, and $\vec{\lambda}_i$ ($\vec{\lambda}_i\to-\vec{\lambda}_i^*$ for antiquarks) are the mass, position, momentum, spin, and color matrix of the ith particle, and where $H_{\rm conf}$ is the harmonic confinement potential and $H_{\rm hyp}$ the (contact part of) the color hyperfine interaction (in which we have used a smeared δ function $\tilde{\delta}$, as discussed in Appendix A). We note that such a Hamiltonian forbids the formation of isolated color nonsinglets, while confining both $q\bar{q}$ and qqq, and that it gives a reasonably adequate description of low-lying mesons and baryons.[8] Of course the Hamiltonian (1) is not complete: it should also include anharmonicities (which will arise when the harmonic potential is replaced by more realistic potentials including linear and Coulombic pieces),[9] the effects of possible $q\bar{q}$ annihilations via gluons,[10] and some relatively small spin-orbit and tensor effects.[10] It would also be useful to consider possible multibody forces[10] in (1), although the success of two-body forces in baryons at least partially indicates that it may be reasonable to neglect them. In any event, since the main issue we are addressing here is a qualitative one, we believe it is reasonable to hope that our simplified Hamiltonian, which encompasses the dominant effects of confinement and strong spin-spin interactions, will not be completely misleading.

In this regard, a comment on long-range color van der Waals—type forces is in order.[11] It has been pointed out that confining potentials of the type considered here lead to long-range power-law residual forces between color singlets; in our case, the potential (2) indeed leads to an r^{-2} potential between two mesons. This is not a fundamental difficulty with the picture: pair creation from the confining field energy would, if implemented, produce an exponential cutoff of this residual attraction. One might be concerned, however, that this spurious potential could produce misleading effects in our calculations. We believe that this is not the case: our variational wave functions, which are Gaussian-type, will not be strongly perturbed by an r^{-2} effect of the predicted strength (see Appendix D).

Of course the existence of the process of pair creation raises the issue of the importance of meson exchange. Studies of the nucleon-nucleon potential[11] along lines similar to the one being followed here, while very promising, in our opinion remain inconclusive on both the question of whether the short-range part of the nucleon-nucleon potential can be derived from the quark model and the question of the importance of pion exchange. Nevertheless, there are indications in these studies that, apart from a possible long-range pion-exchange contribution, these programs may succeed. In this problem, we can at least expect that the analogous pseudoscalar-meson-pseudoscalar-meson potential will be less influenced by meson exchange since the ρ meson is the lightest allowed exchange particle.

A. A simplifying case: Equal-mass quarks and no hyperfine interactions

We turn now to the solution of our Hamiltonian problem by beginning, in order to expose its complexities and introduce our method, with the Hamiltonian H_0 of four equal-mass quarks with no hyperfine interactions. Using the labels of Fig. 1, discarding the center-of-mass variables, and writing

$$|\psi\rangle = \psi_{\bar{3}3}\,|\bar{3}_{12}3_{34}\rangle + \psi_{6\bar{6}}\,|6_{12}\bar{6}_{34}\rangle \tag{4}$$

(see Appendix B for explicit color wave functions; here C_{ij} means that particles i and j are in a color state C of an overall color singlet) the Hamiltonian H_0 in the $(\,|\bar{3}_{12}3_{34}\rangle, |6_{12}\bar{6}_{34}\rangle)$ basis becomes

$$H_0 = \frac{1}{2m}(p_\sigma{}^2 + p_{\bar{\sigma}}{}^2 + p_\lambda{}^2) + \tfrac{8}{3}e_0 + \tfrac{1}{2}k \begin{bmatrix} 2\sigma^2 + 2\bar{\sigma}^2 + \frac{4}{3}\lambda^2 & -2\sqrt{2}\,\vec{\sigma}\cdot\vec{\bar{\sigma}} \\ -2\sqrt{2}\,\vec{\sigma}\cdot\vec{\bar{\sigma}} & \sigma^2 + \bar{\sigma}^2 + \frac{10}{3}\lambda^2 \end{bmatrix}, \tag{5}$$

where $\vec{\sigma} = 1/\sqrt{2}(\vec{r}_2 - \vec{r}_2)$, $\vec{\bar{\sigma}} = 1/\sqrt{2}(\vec{r}_3 - \vec{r}_4)$, and $\vec{\lambda} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2 - \vec{r}_3 - \vec{r}_4)$ or, in terms of the more mesonlike variables

$$\vec{x} = \frac{1}{\sqrt{2}}(\vec{\sigma} + \vec{\bar{\sigma}}) \tag{6}$$

and

$$\vec{y} = \frac{1}{\sqrt{2}}(\vec{\sigma} - \vec{\bar{\sigma}})\,, \tag{7}$$

$$H_0 = \frac{1}{2m}(p_x{}^2 + p_y{}^2 + p_\lambda{}^2) + \tfrac{8}{3}e_0 + \tfrac{1}{2}k \begin{bmatrix} 2x^2 + 2y^2 + \frac{4}{3}\lambda^2 & -\sqrt{2}(x^2 - y^2) \\ -\sqrt{2}(x^2 - y^2) & x^2 + y^2 + \frac{10}{3}\lambda^2 \end{bmatrix}. \tag{8}$$

Note that if we neglect the off-diagonal effect, H_0 appears to be confining in all three relative coordinates in both the $\bar{3}3$ and $6\bar{6}$ sectors [in the $\bar{3}3$ case

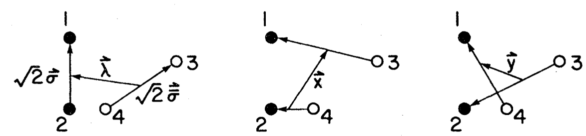

FIG. 1. The relative coordinates of the $qq\bar{q}\bar{q}$ system; darkened circles represent quarks and open circles represent antiquarks.

each pairwise interaction is confining, but in the $6\bar{6}$ case the (12) and (34) interactions are repulsive and the net confinement arises from the remaining four pairwise attractions]. It is on the basis of analyses of the $\bar{3}3$ (called *T*-baryonium) and $6\bar{6}$ (called *M*-baryonium) sectors either in isolation or with mixing treated perturbatively that previous studies of this system have proposed the existence of a rich discrete spectrum of $qq\bar{q}\bar{q}$ states. In the context of a bag model constrained to allow no bag fission, at this same level of approximation the Hamiltonian consists only of quark kinetic energies and bag terms so that such a treatment is justifiable. Nevertheless, the interpretation of the resulting states in the bag model is problematical: the very issues we are trying to address here are obscured by the constraints imposed on the bag surface.[7] In a model like ours, where confinement arises via a potential, this treatment not only should be, but also can be questioned: the mixing terms may produce a strong $\bar{3}3$-$6\bar{6}$ mixture that corresponds to two free mesons when viewed in a $q\bar{q}$-$q\bar{q}$ basis. This last contention can easily be verified qualitatively by taking the low-lying states of the isolated $\bar{3}3$ and $6\bar{6}$ systems and then calculating, for example, the mixing term between the ground state of the $\bar{3}3$ system and the $6\bar{6}$ state with $l=1$ in both the σ and $\bar{\sigma}$ coordinates.[12]

The essential flaw in the view that the $\bar{3}3$ and $6\bar{6}$ sectors can be treated in isolation—or that their mixing can be treated perturbatively—is perhaps now obvious. We know that in $q\bar{q}$-$q\bar{q}$ coordinates with x (or y) large this same Hamiltonian must describe two free mesons; in these circumstances, at least, the $\bar{3}3$-$6\bar{6}$ mixing term is important enough to completely destroy the apparent confinement of the isolated $\bar{3}3$ and $6\bar{6}$ sectors and lead to a meson-meson continuum, a nonperturbative effect. This observation means that we should subject the assumption that this system has, in any reasonable approximation, a tower of discrete bound states to close scrutiny. It moreover seems clear that if a discrete spectrum remains, it must correspond (in $q\bar{q}$-$q\bar{q}$ language) either to states below threshold for decay into two free mesons or to meson-meson resonances.

The complete solution of this system is obviously a complex problem. We will proceed here by first searching variationally for its ground state, which has itself been the subject of intense interest. We will later generalize to other sectors. Our main observation will be that every sector of this system supports a continuum of free, and probably weakly interacting, ground state and excited mesons; we will argue that in such circumstances it can support at most a single weakly bound state and a few broad resonances (which would, in any event, have no special relation to the $B\bar{B}$ channel).

In seeking a variational solution for the ground state of the $qq\bar{q}\bar{q}$ system we note the following:

(1) Since H_0 is invariant under independent rotations in $\vec{x}$, $\vec{y}$, and $\vec{\lambda}$, its solutions must be eigenfunctions of $\vec{L}_x$, $\vec{L}_y$, and $\vec{L}_\lambda$; for the ground state we naturally assume that $l_x=l_y=l_\lambda=0$. (These separate conservation laws are of course peculiar to the harmonic limit and should be used warily.)

(2) Since H_0 is invariant under $1\leftrightarrow 2$ and $3\leftrightarrow 4$ interchanges, its solutions must be either symmetric or antisymmetric under these operations. Proceeding in parallel to related problems in molecular bonding, it is natural (since as $x\to\infty$ or $y\to\infty$ the system must go over into the two degenerate color configurations $|1_{13}1_{24}\rangle$ or $|1_{14}1_{23}\rangle$) to satisfy the above requirements by taking for the trial ground states

$$|\psi_\pm\rangle=\frac{N_\pm}{\sqrt{2}}[\psi(x,y,\lambda)\,|\,1_{13}1_{24}\rangle\pm\psi(y,x,\lambda)\,|\,1_{14}1_{23}\rangle]\ , \tag{9}$$

where $N_\pm$ is a normalization factor. (See Appendix B once again for explicit color wave functions.)

Since the harmonic-oscillator Gaussian parameter $(km)^{1/4}$ is the only scale in H_0, if a bound state exists its wave function should be a smooth, slowly varying function on this scale. To search for bound states of this pure confinement problem we therefore took

$$\psi_\alpha(x,y,\lambda)=\sum_{j=1}^{j_{\max}}\prod_{i=1}^{3}\sum_{k=1}^{k_{\max}}c_{\alpha ijk}\exp(-\tfrac{1}{2}\beta_{\alpha ijk}{}^2\xi_i{}^2)\ , \tag{10}$$

where $(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3)\equiv(\vec{x},\vec{y},\vec{\lambda})$ and α is an index we will encounter below. Such a parametrization not only allows a quite general form for the dependence of ψ on each of the three variables separately, but also allows for correlations. In particular, if the system does not bind, this wave function will lead to the collapse of (9) into a state representing a "scattering state" of two free mesons infinitely separated with zero relative momentum. Using up to 31 parameters in ψ, we find no bound states; that is, the energy of (9) was always minimized with $\langle\,|\vec{x}|\,\rangle$ very large. We conclude that a distortion of the color and space wave functions of the mesons to take advantage of color recouplings (analogous to the distortion of the electron cloud in a molecule) is not sufficient to overcome the kinetic energy induced by bringing the mesons into proximity.

Before leaving this pure harmonic system, we comment on what we believe to be a spurious feature of our model when it is extended to higher angular momenta in the $\vec{\lambda}$ variable. We have remarked earlier that the separate conservation of $\vec{L}_x$, $\vec{L}_y$, and $\vec{L}_\lambda$ is a peculiarity of the harmonic limit; we know furthermore that strong anharmonicities are required to explain the spectrum of excited baryons and mesons even though the harmonic potential is adequate for the lowest-lying states.[8] If, however, we were to take the harmonic limit seriously, the decoupling of $\vec{x}$, $\vec{y}$, and $\vec{\lambda}$ orbital excitations would, for example, decouple the pure $\vec{\lambda}$ excitations with orbital angular momentum l_λ from states of free mesons with nonzero relative orbital angular momentum (since meson orbital angular momenta are carried by the $\vec{x}$ and $\vec{y}$ variables). This decoupling could then lead to bound $\vec{\lambda}$ excitations since even though they would be heavier than free mesons with relative orbital angular momentum, they could be lighter than the corresponding free mesons with l_λ in their internal coordinates.[4] Since the large anharmonicity known to be present in this system (as well as the hyperfine interactions: see below) will certainly strongly couple $\vec{x}$, $\vec{y}$, and $\vec{\lambda}$, we believe that it will continue to drive strong $\bar{3}3$-$6\bar{6}$ mixing (like that which occurs in the ground state), thereby destroying these unstable artifacts of the harmonic limit. We accordingly discount this possibility, and expect that in every sector of this system strong nonperturbative mixing effects must be in operation to produce free mesons from the discrete $\bar{3}3$ and $6\bar{6}$ spectra.

B. Turning on hyperfine interactions

We now proceed to a more realistic model by adding the hyperfine interaction (3), taking as trial ground states

$$|(SS)_+\rangle=\frac{N_{SS+}}{\sqrt{2}}[\psi_{SS+}(x,y,\lambda)\,|\,1_{13}1_{24}\rangle-\psi_{SS+}(y,x,\lambda)\,|\,1_{14}1_{23}\rangle]\,|\,S_{12}S_{34}\rangle\ , \tag{11a}$$

$$|(AA)_+\rangle=\frac{N_{AA+}}{\sqrt{2}}[\psi_{AA+}(x,y,\lambda)\,|\,1_{13}1_{24}\rangle+\psi_{AA+}(y,x,\lambda)\,|\,1_{14}1_{23}\rangle]\,|\,\vec{A}_{12}\cdot\vec{A}_{34}\rangle\ , \tag{11b}$$

and

$$|(SS)_-\rangle=\frac{N_{SS-}}{\sqrt{2}}[\psi_{SS-}(x,y,\lambda)\,|\,1_{13}1_{24}\rangle+\psi_{SS-}(y,x,\lambda)\,|\,1_{14}1_{23}\rangle]\,|\,S_{12}S_{34}\rangle\ , \tag{12a}$$

$$|(AA)_-\rangle=\frac{N_{AA-}}{\sqrt{2}}[\psi_{AA-}(x,y,\lambda)\,|\,1_{13}1_{24}\rangle-\psi_{AA-}(y,x,\lambda)\,|\,1_{14}1_{23}\rangle]\,|\,\vec{A}_{12}\cdot\vec{A}_{34}\rangle\ , \tag{12b}$$

where $(XX)_\sigma$ means spin states X [where $X=S$ for spin zero, leaving qq or $\bar{q}\bar{q}$ in a scalar state, or $X=\vec{A}$ for spin one, leaving qq or $\bar{q}\bar{q}$ in an axial-vector state, and in each case the overall spins are coupled to zero (see Appendix C for details)], and overall quark or antiquark color-space-spin symmetry σ (where $\sigma=+$ or $\sigma=-$ for symmetric or antisymmetric). The states (11) must consequently have the diquark and the antidiquark in the antisymmetric $\bar{3}$ and 3 of SU(3) flavor and so form ordinary flavor nonets (for this reason these states have been dubbed "cryptoexotic"[2]), while the states (12) have their diquarks and antidiquarks in a 6 and $\bar{6}$ of SU(3) flavor and so can form exotic SU(3) flavor multiplets. The states (11a), (11b), and the states (12a), (12b) are separately mixed by the hyperfine interaction; it is this mixing that can create an additional attractive force beyond that produced by confinement. It should be noted that with hyperfine interactions turned on, it is no longer true that $\vec{L}_x$, $\vec{L}_y$, and $\vec{L}_\lambda$ are separately conserved so that the form of the variational wave functions (10) cannot be fully justified. The same effect occurs in the ground-state baryons[13] where the hyperfine interaction can (and does) cause mixing between the $l_\rho=l_\lambda=0$ $(56,0^+)$ and the $l_\rho=l_\lambda=1$ $(70,0^+)$ supermultiplets. We assume for now that in $qq\bar{q}\bar{q}$, as in qqq, such effects do not strongly mix $l=0$ and $l>0$ wave functions [for example, the nucleon is approximately 6% $(70,0^+)$; such mixing would, in any event, only slightly increase the attraction in these states, thereby strengthening the conclusions we draw below].

To make contact with known physics within the context of our simplified Hamiltonian in the equal-mass limit, we fixed its parameters by performing fits in the $q\bar{q}$ sector to $\pi(140)$, $\rho(770)$, $K(495)$, $K^*(890)$, and $\phi(1020)$, both by taking the limit $x \to \infty$ and by using (1) directly in the $q\bar{q}$ sector. To do this we chose $m_u = m_d$ and m_s and then fit the π and ρ, K and K^*, and ϕ with the masses m_d, $\frac{1}{2}(m_d + m_s)$, and m_s, respectively. Since our fits always gave $m_u = m_d \simeq 0.33$ GeV, but with a shallow minimum, we imposed this "standard"[8] value so that $\omega_0 \equiv (k/m_d)^{1/2}$, e_0, m_s, and α_s were our only free parameters. Since our Hamiltonian contains no anharmonicity, it is unclear whether it is more appropriate to choose ω_0 so that the harmonic term has its "true" strength or so that it interpolates the full spin-independent potential; we therefore considered a reasonable range of values in this parameter. For a given choice of ω_0 our other parameters are tightly constrained. The resulting meson fits are shown in the first half of Table I (μ_P and μ_V are the masses of the pseudoscalar and vector mesons, respectively).

Using pairs of the wave functions of the form (10) and an angle $\theta_\pm$ (defined more fully in Appendix C) to describe the mixing within the pairs of states (11a), (11b) and (12a), (12b), we then searched for minima of (1) with up to 62-parameter variational wave functions explored sequentially. With our smeared hyperfine interaction, we found that the correlations introduced by the j summation in (10) were of minor importance. We also found that going beyond two-term Gaussian wave functions in x, y, and λ offered only marginal improvements in our energies. Thus we were able to do the bulk of our calculations in a (relatively small) 18-parameter space. The result of these searches is that we find only free meson pairs in the exotic sector (12a), (12b), and we normally find free mesons in the cryptoexotic sector (11a), (11b). However, for a range of masses around the masses of the SU(3) quarks we find that one nonet of states in the cryptoexotic sector can be weakly bound by the combination of color, space, and spin recouplings allowed by (11). Table I shows some properties of these states in various circumstances.

On examining this table, we see that the binding is always weak and that the output wave functions support a nuclear-physics-type interpretation: the $qq\bar{q}\bar{q}$ system has clustered into (approximately) ground-state color singlets which are considerably smaller than the intercluster separation ($x \gg r_{(q\bar{q})_1}$). So far as we know this is the first demonstration from "first principles" of such a phenomenon and we interpret this result as justifying attempts to understand the nucleus in terms of color-singlet qqq clusters.

III. SOME FLAWS IN THE MODEL PLUS SOME OTHER PHENOMENOLOGICAL CONSEQUENCES

Up to this point we believe that our treatment of this model for the $qq\bar{q}\bar{q}$ system has been reliable, and that the qualitative conclusions we have drawn as to the existence of a discrete $qq\bar{q}\bar{q}$ spectrum, and on the structure of those weakly bound states that can occur, are sound. If we wish to proceed further with this picture, however, we must face a rather severe flaw of the model which arises from its nonrelativistic character. It is gradually becoming understood how many relativistic effects in the nonrelativistic quark model can be subsumed by the choice of the model's parameters[14]: the constituent quark mass m, for example, is undoubtedly a repository of such effects. The problem here is related:

TABLE I. The cryptoexotic $qq\bar{q}\bar{q}$ system in various circumstances.

ω_0 (MeV)	α_s	e_0 (MeV)	m (MeV)	ω_{meson} (MeV)	μ_P (MeV)	μ_V (MeV)	$E_{qq\bar{q}\bar{q}} - 2\mu_P$ (MeV)	$\dfrac{\langle x^2 \rangle^{1/2}}{\langle r_{q\bar{q}}{}^2 \rangle^{1/2}}$	θ_+ (degrees)
200	2.7	-352	330	327	137	770	-81	2.8	2.5
			430	286	511	883	-17	3.8	1.1
			530	258		1024	-2	5.3	0.2
250	2.4	-455	330	408	141	771	-52	3.2	1.9
			440	353	508	874	-7	4.7	0.4
			550	316		1019	0	∞	0.0
300	2.2	-555	330	490	138	770	-35	3.6	1.3
			450	420	503	864	-2	5.9	~ 0.1
			570	373		1011	0	∞	0.0

we can on the basis of the model conclude that it is reasonable to view the $qq\bar{q}\bar{q}$ bound states as weakly bound meson-meson systems analogous to the deuteron. However, our Hamiltonian (because it is nonrelativistic) treats these mesons as though they had mass $\mu=2m$ [see the kinetic energy term for the coordinate x in (8)]. For heavier quarks this is of course an increasingly reasonable approximation, but for the light quarks it is a very poor one, mainly because of the strong spin-spin attraction in the pseudoscalars. The pion is, of course, the extreme example where $2m=660$ MeV while $\mu_\pi=140$ MeV.

In order to proceed, we therefore isolate as well as we can the contribution of the "meson" kinetic energies to our weakly bound states. To do this we define an effective meson-meson potential $V_{\rm eff}(x)$ by first defining

$$\psi_{\rm eff}(x)=\int\int d^3y\, d^3\lambda\, \psi^*_{PP}(y,\lambda)[\tfrac{3}{4}\psi_{AA^+}(x,y,\lambda)+\tfrac{1}{4}\psi_{SS^+}(x,y,\lambda)]\ , \tag{13}$$

where $\psi_{PP}(y,\lambda)$ is the internal wave function of the free pseudoscalar-pseudoscalar system which we obtain from the same Hamiltonian by holding x large. It should be noted that this definition neglects interferences between the configurations $|1_{13}1_{24}\rangle|P_{13}P_{24}\rangle$ and $|1_{14}1_{23}\rangle|P_{14}P_{23}\rangle$. This would be automatic if the mesons were pointlike objects; here, this step shows that the idea of $\psi_{\rm eff}(x)$ can only be qualitatively sound. In any event, with $\psi_{\rm eff}(x)$ so defined, we can take

$$V_{\rm eff}(x)\equiv E+\frac{\nabla_x{}^2\psi_{\rm eff}(x)}{2m\psi_{\rm eff}(x)}\ . \tag{14}$$

We can then use $V_{\rm eff}(x)$ in a Schrödinger equation with more realistic meson masses inserted. Some wave functions $\psi_{\rm eff}(x)$ and potentials $V_{\rm eff}(x)$ deduced in this way are shown in Fig. 2(a). Figure 2(b) shows the insensitivity of $V_{\rm eff}(x)$ to variations in ω_0 and σ (defined in Appendix A).

On examining the binding of mesons by the potentials $V_{\rm eff}$, we draw the following conclusions.

(1) Although $V_{\rm eff}$ for the "$\pi\pi$" system is stronger than that for heavier systems, it is far too weak to overcome the large kinetic energy of the pions and no bound state occurs. Thus very light quarks do not bind because the intrameson hyperfine interaction is too *strong;* in contrast, very heavy quarks did not bind because their hyperfine interactions were too *weak*.

(2) A similar analysis shows that for the $K\bar{K}$ system binding persists down to realistic values of the meson mass μ_K so that the original conclusions of the model are plausible: isoscalar and isovector $K\bar{K}$ bound states probably exist just below $K\bar{K}$ threshold. It is natural to associate these states with the peculiar states $S^*(980)$ and $\delta(980)$ as was done originally by Jaffe[2]; in the following we shall adduce further evidence in favor of this assignment.

(3) Although none of our results address very precisely the binding of the asymmetric $K\pi$ system, since the $K\pi$ reduced mass is much closer to that of the $\pi\pi$ system than that of the $K\bar{K}$ system, and since $K\bar{K}$ is itself just barely bound, it seems very unlikely that $K\pi$ binds. This guess could be supported by letting the $m_s\neq m_d$ mass asymmetry perturb the weakly bound $K\bar{K}$ system via (1).

(4) Finally, as mentioned previously, we can use these potentials in a Schrödinger equation with $l_x>0$ to conclude that, in this ground-state–ground-state channel at least, the forces are too weak to produce meson-meson resonances in the centrifugal barrier. We further presume (pending further investigation) that sectors describing the possible states of excited mesons will once again display a relatively weak effective potential. These sectors would, of course, be without true bound states since they could by rearrangement always fall apart into two ground-state mesons; with the anticipated weak effective potential they would also not support any centrifugal barrier resonances. While these features are only presumptive, we note that even if there were some rearrangement or centrifugal barrier resonances in these sectors, (1) they would be expected to be very broad since in addition to their decays by rearrangement and (or) via their natural resonance widths, they would also decay through the channels open to their constituent excited mesons and (2) they would have no special relationship to the $B\bar{B}$ channel. Such states would not then be interpretable as baryonia.

We now turn to another quarter entirely for further support for the interpretation of the S^* and δ as $K\bar{K}$ bound states. Jaffe[2] has already stressed that the peculiar masses and decay patterns of this approximately degenerate pair of states and their prospective nonet-mates make nearly untenable the usual $q\bar{q}$ analogy to the (ρ,ω) system. We will now show that, in strong contrast to both the $q\bar{q}$ and bag-model interpretations, the decay widths of the S^* and δ can be calculated quantitatively in terms

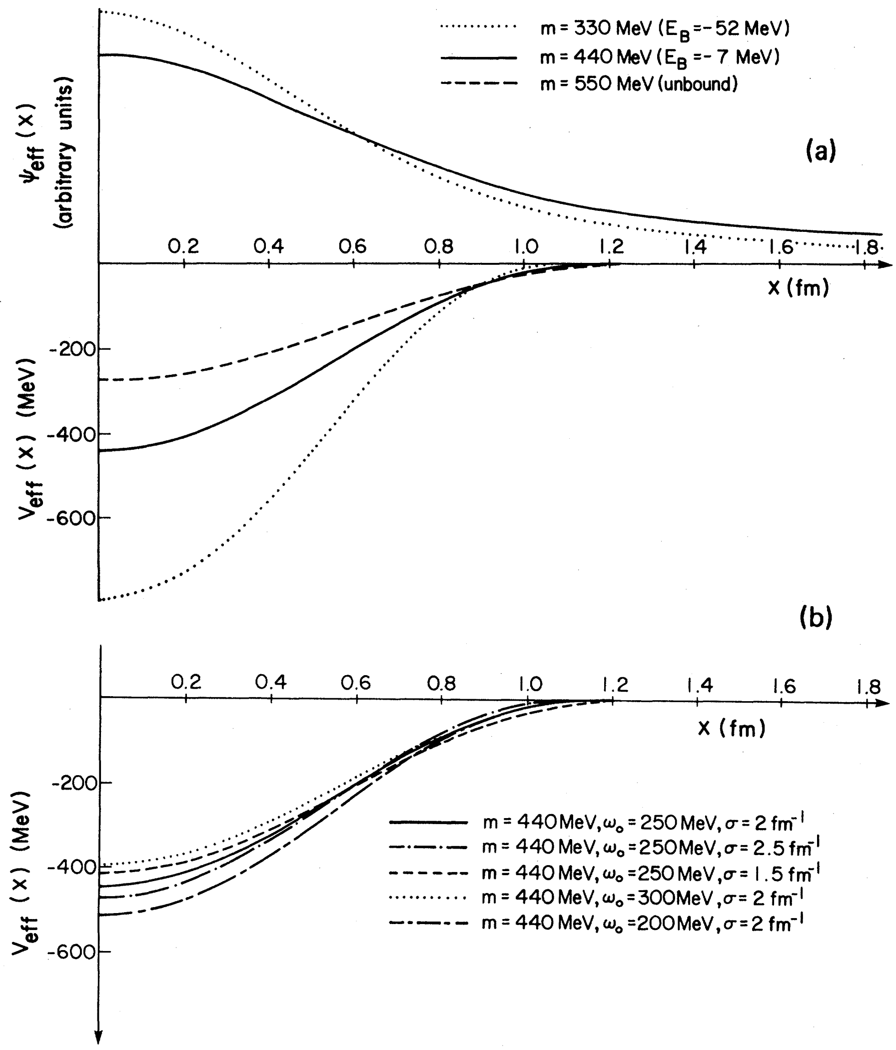

FIG. 2. (a) The effective wave functions and potentials deduced from Eqs. (13) and (14) with $\omega_0=250$ MeV, $\alpha_s=2.4$, $e_0=-455$ MeV, and $\sigma=2$ fm^{-1}. (b) The sensitivity of $V_{\rm eff}$ to ω_0 and σ.

of our $qq\bar{q}\bar{q}$ wave functions and a few simple assumptions.

Consider first the isoscalar $S^*(980)$. In our picture it can be viewed as a weakly bound $K\bar{K}$ system (as such, it is of course quite distinct from the unclustered state of the bag model). If the $\pi\pi$ channel were not open, our S^* would be strong-interaction stable, so it is clear that we may view its decay as occurring principally via inelastic $K\bar{K}\to\pi\pi$ scattering "inside" the S^*, and we should be able to calculate its width in terms of the observed S-wave phase shifts and inelasticities in the regions just below and just above the S^*. Using the approximately nonrelativistic character of this $K\bar{K}$ bound state we have for an S^* at rest

$$\langle\pi\pi\,|\,T\,|\,S^*\rangle = \frac{\sqrt{2M_{S^*}}}{2\mu_K}\int d^3p\,\phi_{S^*}(\vec{p})\langle\pi\pi\,|\,T\,|\,K(\vec{p})\bar{K}(-\vec{p})\rangle\ . \tag{15}$$

If we now parametrize the "background" under the S^* in terms of a very broad ϵ with couplings $g_{\epsilon\pi\pi}$ and $g_{\epsilon K\bar{K}}$ to the $\pi\pi$ and $K\bar{K}$ channels, normalized so that

$$\Gamma(\epsilon\to\pi\pi/K\bar{K}) = \frac{g_{\epsilon\pi\pi/\epsilon K\bar{K}}{}^2}{16\pi M_\epsilon}\left[\frac{p_{\pi/K}}{E_{\pi/K}}\right], \tag{16}$$

then we find in the approximation $\Gamma_\epsilon \gg |M_\epsilon - M_{S^*}|$ that

$$\Gamma(S^* \to \pi\pi) = \left[\frac{g_{\epsilon K\bar{K}}^2}{4\pi M_\epsilon^2}\right]\left[\frac{g_{\epsilon\pi\pi}^2}{4\pi M_\epsilon^2}\right]\frac{\pi M_\epsilon^2}{2\mu_K{}^2\Gamma_\epsilon{}^2}\,|\psi_{S^*}(0)|^2\,, \tag{17}$$

where $\psi_{S^*}(x)$ is the $K\bar{K}$ spatial wave function. Using the theoretical result that

$$\frac{g_{\epsilon K\bar{K}}^2}{g_{\epsilon\pi\pi}^2} = \frac{1}{3} \tag{18}$$

for $\epsilon = (1/\sqrt{2})\,(u\bar{u} + d\bar{d})$ and the empirically required mass and width $M_\epsilon \simeq 1100$ MeV, $\Gamma_\epsilon \simeq 1000$ MeV (values which agree with those which emerge from model calcuations[15]), we find, using our variational value for $\psi_{S^*}(0)$ that

$$\Gamma(S^* \to \pi\pi) \simeq 15 \text{ MeV} \tag{19}$$

with an uncertainty of about a factor of two due to the uncertainties in our calculated values for $\psi_{S^*}(0)$. This completely calculated quantity is in good agreement with measured values. Note that in this picture the $K\bar{K}$ threshold enhancement seen in this channel is to be principally attributed to an effect of the low-energy $K\bar{K}$ potential, since this enhancement would exist even if $\Gamma_{S^*} = 0$.

Similar results to these are easily obtained for the $\delta(980)$ which can, in this picture, decay only to $\eta\pi$. In this case we assume that the inelastic reaction $K\bar{K} \to \eta\pi$ proceeds through the degenerate $q\bar{q}$ isovector partner of the ϵ (which we call δ_2, denoting a δ-like two-quark state) and use the theoretical relations

$$g_{\delta_2\eta\pi}{}^2 : g_{\delta_2 K\bar{K}}{}^2 : g_{\epsilon\pi\pi}{}^2 = 1:1:3\ . \tag{20}$$

We can then calculate that

$$\Gamma(\delta \to \eta\pi) = \left[\frac{g_{\delta_2 \to \eta\pi}{}^2}{4\pi M_{\delta_2}{}^2}\right]\left[\frac{g_{\delta_2 \to K\bar{K}}{}^2}{4\pi M_{\delta_2}{}^2}\right]\frac{\pi M_{\delta_2}{}^2}{2\mu_K{}^2\Gamma_{\delta_2}{}^2}\,|\psi_\delta(0)|^2 \tag{21}$$

$$\simeq 40 \text{ MeV} \tag{22}$$

once again in good agreement with measurements. Although $S^* \to \pi\pi$ might be expected to behave similarly here and in the bag model (since in each case an $s\bar{s}$ annihilation is required), the δ has a "fall-apart" mode to $\eta\pi$ in the bag and would therefore, *a priori*, be expected to have a much larger width than the S^*. The ability of our picture to explain the narrow widths of these two states—an effect essentially of the weak binding which makes $\psi(0)$ small—must be counted as a success of this picture. These narrow widths can also be contrasted with a $q\bar{q}$ picture of these states where, for example, one would expect $\Gamma_{S^*} \simeq 1000$ MeV.

IV. CONCLUSIONS

Despite the several weaknesses of our model of the $qq\bar{q}\bar{q}$ system, we think it may provide a qualitatively valid picture of this simplest multiquark system. To summarize, we recall the following.

(1) The model gives no evidence for a rich discrete spectrum of $qq\bar{q}\bar{q}$ states and in particular does not seem to support the existence of baryonium-like states. This conclusion, in contrast with previous studies of this model, arises from properly taking into account $\bar{3}3$-$6\bar{6}$ mixing by the confinement potential and thereby discovering that at least in the sector we have explored, the tower of discrete states previously thought to exist does not persist. While studies of the forces between excited mesons will be required for confirmation, we suspect on this basis that only $q\bar{q}$ and qqq hadrons exist.

(2) The model points to the existence of weakly bound 0^{++} $qq\bar{q}\bar{q}$ states for some light quarks. When supplemented by a method for extracting from it an effective meson-meson potential, the model indicates that of a possible nonet of such bound states, probably only the $K\bar{K}$-like isoscalar and isovector states survive SU(3) breaking. We identify these states as the $S^*(980)$ and $\delta(980)$ and note that, in contrast to the bag model, their positions just below $K\bar{K}$ threshold are not accidental. We further strengthen these identifications by successfully calculating the widths of these states for decay into $\pi\pi$ and $\eta\pi$, respectively.

(3) Even if the identification of these bound states with the S^* and δ is incorrect, the model provides a picture of possible meson-meson bound states which

is remarkably like the nucleon-nucleon picture of the deuteron. The clustering of the $qq\bar{q}\bar{q}$ state into color singlets of dimensions considerably smaller than the intercluster separation emerged from first principles. A weak binding comparable to that in the deuteron was a further automatic feature of the model. These results lend support to the hope that nuclear properties can be deduced from the quark model.

While these various features of the model are very encouraging, we would like to close by mentioning some outstanding problems. We have already stressed that the model is badly flawed by its inability to take into account relativistic binding effects unless supplemented by something like our effective-potential approach. We see little hope of correcting this deficiency, but several avenues are open for correcting less fundamental problems with the model: the effects of anharmonic, tensor, spin-orbit, multibody, and $q\bar{q}$ annihilation forces could be investigated, and the unequal mass problem (especially as it applies to the possibility of stable charm-strange exotics[16]) could be considered. On the phenomenological side, this study leaves many questions unanswered: Do any weak resonances of excited mesons exist? Do the mesonic analogs of nuclei beyond deuterium exist? Why do meson-baryon bound states not (seem to) exist? Can we in fact apply this model to the $qqqqqq$ system to begin to derive some of the properties of nuclei from the quark model? It is especially appealing to hope to make some progress on this last question. After all, particle physics was born in an attempt to understand the nucleus, and it would be very satisfying if this goal could finally be realized.

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APPENDIX A: THE SMEARED δ FUNCTION: $\tilde{\delta}$

The hyperfine interaction $H_{\rm hyp} \propto \vec{S}_1 \cdot \vec{S}_2 \delta^3(\vec{r}_{12})$ is an illegal operator in the Schrödinger equation since for $s=0$ states it is more attractive than r^{-2} and so overpowers the kinetic energy $p^2/2m$. This problem can be overcome in principle by calculating corrections to $H_{\rm hyp}$ of higher order in (v/c) and α_s, but this is impractical. We evade this difficulty by identifying the physical origin of the resulting smoothing of $\delta^3(\vec{r}_{12})$ with the relativistic smearing[14] of the position operator of an effective two-component theory over distances of order m^{-1}. We therefore take

$$\delta^3(\vec{r}_{12}) \to \tilde{\delta}^3(\vec{r}_{12}) \equiv \frac{\sigma^3}{\pi^{3/2}} e^{-\sigma^2 r_{12}{}^2} \tag{A1}$$

and have in the text normally chosen $\sigma^{-1}=0.5$ fm$\sim 1/m$. We have checked that for reasonable variations in σ our results are not dramatically affected [see Fig. 2(b)]: a change in σ affects our fitted value of α_s with the result that the net change in the physics of the system is small.

APPENDIX B: COLOR WAVE FUNCTIONS

The calculations of the text rely in several instances on the use of explicit color wave functions and on transformations between various color bases. We reproduce some of the more useful results here, both for completeness and to simplify comparisons with other related calculations.

The $qq\bar{q}\bar{q}$ system can be coupled to a color singlet in several ways: we use in this paper (see Fig. 1 for notation) the bases $(|\bar{3}_{12}3_{34}\rangle, |6_{12}\bar{6}_{34}\rangle)$, $(|1_{13}1_{24}\rangle, |8_{13}8_{24}\rangle)$, and $(|1_{14}1_{23}\rangle, |8_{14}8_{23}\rangle)$, where in each case we have an overall color singlet made up of subunits C_{ij} of particles i and j in the color states $C=1, 3, \bar{3}, 6, \bar{6}$, or 8. Explicitly, we use

$$|\bar{3}^{\alpha}_{q_iq_j}\rangle = \frac{1}{\sqrt{2}} \epsilon^{\alpha\beta\gamma} q_i^{\beta} q_j^{\gamma} , \tag{B1}$$

$$|6^{\alpha}_{q_iq_j}\rangle = \frac{1}{\sqrt{2}} d^{\alpha\beta\gamma} q_i^{\beta} q_j^{\gamma} , \tag{B2}$$

$$|3^{\alpha}_{\bar{q}_i\bar{q}_j}\rangle = \frac{1}{\sqrt{2}} \epsilon^{\alpha\beta\gamma} \bar{q}_i^{\beta} \bar{q}_j^{\gamma} , \tag{B3}$$

$$|\bar{6}^{\alpha}_{\bar{q}_i\bar{q}_j}\rangle = \frac{1}{\sqrt{2}} d^{\alpha\beta\gamma} \bar{q}_i^{\beta} \bar{q}_j^{\gamma} , \tag{B4}$$

where $\epsilon^{\alpha\beta\gamma}$ is the standard alternating symbol with $\epsilon^{123}=1$, and where the nonzero d's are

$$d^{111}=d^{322}=d^{633}=\sqrt{2} , \tag{B5}$$

$$d^{212}=d^{221}=d^{413}=d^{431}=d^{523}=d^{532}=1 , \tag{B6}$$

from which it follows that

$$|\bar{3}_{12}3_{34}\rangle=\frac{1}{\sqrt{12}}\epsilon^{\alpha\beta\gamma}\epsilon^{\alpha\rho\sigma}|q_1^{\beta}q_2^{\gamma}\bar{q}_3^{\rho}\bar{q}_4^{\sigma}\rangle\ , \tag{B7}$$

$$|6_{12}\bar{6}_{34}\rangle=\frac{1}{\sqrt{24}}d^{\alpha\beta\gamma}d^{\alpha\rho\sigma}|q_1^{\beta}q_2^{\gamma}\bar{q}_3^{\rho}\bar{q}_4^{\sigma}\rangle\ . \tag{B8}$$

It is clear that the constraints imposed by the symmetries of H under $1\leftrightarrow2$ and $3\leftrightarrow4$ interchanges will be most easily implemented in this basis: the $\bar{3}$ and 3 are antisymmetric, while the 6 and $\bar{6}$ are symmetric, under these interchanges.

To transform to the other useful basis systems we use the results

$$|1_{13}1_{24}\rangle=(\tfrac{1}{3})^{1/2}|\bar{3}_{12}3_{34}\rangle+(\tfrac{2}{3})^{1/2}|6_{12}\bar{6}_{34}\rangle\ , \tag{B9}$$

$$|8_{13}8_{24}\rangle=-(\tfrac{2}{3})^{1/2}|\bar{3}_{12}3_{34}\rangle+(\tfrac{1}{3})^{1/2}|6_{12}\bar{6}_{34}\rangle\ , \tag{B10}$$

and

$$|1_{14}1_{23}\rangle=\tfrac{1}{3}|1_{13}1_{24}\rangle+\frac{2\sqrt{2}}{3}|8_{13}8_{24}\rangle\ , \tag{B11}$$

$$|8_{14}8_{23}\rangle=\frac{2\sqrt{2}}{3}|1_{13}1_{24}\rangle-\tfrac{1}{3}|8_{13}8_{24}\rangle\ . \tag{B12}$$

From this last, one can see that $|1_{13}1_{24}\rangle$ and $|1_{14}1_{23}\rangle$ span the space of internal color states; it is this observation that leads to the wave function (9) of the text.

APPENDIX C: SPIN WAVE FUNCTIONS

The calculations of the text rely on the use of explicit spin wave functions and on transformations between them which we catalog here. Since the space-color wave functions $|\psi_{\pm}\rangle$ of Eq. (9) are symmetric (+) or antisymmetric (−) under $1\leftrightarrow2$ or $3\leftrightarrow4$, we want to construct spin wave functions with these same symmetries. We therefore begin in the qq and $\bar{q}\bar{q}$ spaces and write

$$S_{ij}\equiv\frac{1}{\sqrt{2}}(\uparrow_i\downarrow_j-\downarrow_i\uparrow_j)\ , \tag{C1}$$

$$\vec{\mathrm{A}}_{ij}\equiv(A_{ij}^{+},A_{ij}^{0},A_{ij}^{-})=\left(\uparrow_i\downarrow_j,\frac{1}{\sqrt{2}}(\uparrow_i\downarrow_j+\downarrow_i\uparrow_j),\downarrow_i\downarrow_i\right)\ , \tag{C2}$$

where S stands for scalar and $\vec{\mathrm{A}}$ for axial vector. Then the combined spin states with total spin zero are

$$|S_{12}S_{34}\rangle=S_{12}S_{34}\ , \tag{C3}$$

$$|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{34}\rangle=\frac{1}{\sqrt{3}}(A_{12}^{+}A_{34}^{-}-A_{12}^{0}A_{34}^{0}+A_{12}^{-}A_{34}^{+})\ . \tag{C4}$$

To accomplish transformations to the meson bases we use the analogous states P_{ij} and $\vec{\mathrm{V}}_{ij}$ (standing for pseudoscalar and vector) to get

$$|P_{13}P_{24}\rangle=\tfrac{1}{2}|S_{12}S_{34}\rangle+\frac{\sqrt{3}}{2}|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{34}\rangle\ , \tag{C5}$$

$$|\vec{\mathrm{V}}_{13}\cdot\vec{\mathrm{V}}_{24}\rangle=\frac{\sqrt{3}}{2}|S_{12}S_{34}\rangle-\tfrac{1}{2}|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{34}\rangle\ , \tag{C6}$$

and

$$|P_{14}P_{23}\rangle=-\tfrac{1}{2}|S_{12}S_{34}\rangle+\frac{\sqrt{3}}{2}|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{24}\rangle\ , \tag{C7}$$

$$|\vec{\mathrm{V}}_{14}\cdot\vec{\mathrm{V}}_{23}\rangle=-\frac{\sqrt{3}}{2}|S_{12}S_{34}\rangle-\tfrac{1}{2}|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{34}\rangle\ . \tag{C8}$$

When coupled with the $|\psi_{\pm}\rangle$ to make the wave functions (11) and (12) these space-color-spin states admit, as discussed in the text, only certain flavor symmetries.

The hyperfine interactions mix the (cryptoexotic) states (11a) and (11b) among themselves and the (exotic) states (12a) and (12b) among themselves with mixing angles we call $30°+\theta_+$ and $30°+\theta_-$, respectively. In the two-free-meson limit, it is not difficult to show that the eigenstates of (1) in the sector (11) are

$$|\psi_{PP}\rangle=+\frac{\sqrt{3}}{2}|(AA)_+\rangle+\tfrac{1}{2}|(SS)_+\rangle\ , \tag{C9}$$

$$|\psi_{VV}\rangle=+\frac{\sqrt{3}}{2}|(SS)_+\rangle-\tfrac{1}{2}|(AA)_+\rangle\ . \tag{C10}$$

The coefficient of $|1_{13}1_{24}\rangle$ in (C9) is

$$\tfrac{1}{2}|S_{12}S_{34}\rangle+\frac{\sqrt{3}}{2}|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{34}\rangle\equiv|P_{13}P_{24}\rangle$$

while the coefficient of $|1_{14}1_{23}\rangle$ is

$$-\tfrac{1}{2}|S_{12}S_{34}\rangle+\frac{\sqrt{3}}{2}|\vec{\mathrm{A}}_{12}\cdot\vec{\mathrm{A}}_{34}\rangle\equiv|P_{14}P_{23}\rangle\ ,$$

i.e., $\theta_{\pm}=0$ corresponds to two free pseudoscalar (and two free vector) mesons.

APPENDIX D: LONG-RANGE COLOR POTENTIAL

We would like to estimate the effect of the long-range color van der Waals—type potential on our conclusions. To derive this potential we imagine two mesons held a distance r apart in their relative coordinate x by a stiff spring of spring constant K. In the $(|1_{13}1_{24}\rangle, |8_{13}8_{24}\rangle)$ basis the Hamiltonian is

$$H=\frac{p_x{}^2}{2m}+\begin{bmatrix}\frac{1}{2}K(\vec{x}-\vec{r})^2 & 0\\ 0 & c+\frac{1}{2}(K+3k)(\vec{x}-\vec{r}\,')^2\end{bmatrix}+\frac{p_y{}^2}{2m}+\frac{1}{2}k\begin{bmatrix}\frac{8}{3}y^2 & 0\\ 0 & \frac{1}{3}y^2\end{bmatrix}+\frac{p_\lambda{}^2}{2m}$$

$$+\frac{1}{2}k\begin{bmatrix}\frac{8}{3}\lambda^2 & 0\\ 0 & 2\lambda^2\end{bmatrix}+\frac{8}{3}e_0+H_{\text{pert}}\,, \tag{D1}$$

where

$$H_{\text{pert}}=\frac{1}{2}k\begin{bmatrix}0 & \frac{2\sqrt{2}}{3}(\lambda^2-y^2)\\ \frac{2\sqrt{2}}{3}(\lambda^2-y^2) & 0\end{bmatrix}, \tag{D2}$$

$$\vec{r}\,'=\left[\frac{K}{K+3k}\right]\vec{r}\to\vec{r},\text{ as } K\to\infty\,, \tag{D3}$$

and

$$c=\left[\frac{K}{K+3k}\right]\frac{3}{2}kr^2\to\frac{3}{2}kr^2 \text{ as } K\to\infty\,. \tag{D4}$$

The solution of (D1) with $H_{\text{pert}}=0$ is straightforward: it gives disjoint spectra of singlet and octet mesons localized at the position $\vec{r}$ with ground-state energies $2M_1+E_0$ and $2M_8+E_0+\frac{3}{2}kr^2$ in the limit $K\to\infty$, where $M_{1(8)}$ is the "internal" mass (from the quark masses and the eignevalues of the y and λ Hamiltonians) of a singlet (octet) meson and E_0 the ground-state energy of the oscillator centered at $\vec{r}$.

If there were a weak potential $V(\vec{r})$ between two ground-state mesons, it would manifest itself in these circumstances by producing a shift in the ground-state energy of $\Delta E=V(r)$. Thus we can find the effective residual potential between two color-singlet mesons by turning on H_{pert} and interpreting the shift in the ground-state energy as $V_{\text{vdW}}(r)$. If we were to just consider mixing between the ground states of the $|1_{13}1_{24}\rangle$ and $|8_{13}8_{24}\rangle$ systems, we would find for $m\simeq 0.33$ GeV

$$V_{\text{vdW}}^{(\text{gs-gs})}(r)=-\frac{1}{3mr^2}\,\frac{(\frac{8}{3})^{3/2}(\frac{1}{3})^{3/4}2^{3/4}\left[\dfrac{2}{(\frac{8}{3})^{1/2}+2^{1/2}}-\dfrac{2}{(\frac{8}{3})^{1/2}+(\frac{1}{3})^{1/2}}\right]^2}{\left[\dfrac{(\frac{8}{3})^{1/2}+(\frac{1}{3})^{1/2}}{2}\right]^3\left[\dfrac{(\frac{8}{3})^{1/2}+2^{1/2}}{2}\right]^3} \tag{D5}$$

$$\simeq-\frac{1.6\text{ MeV}}{[r\text{ (fm)}]^2}\,. \tag{D6}$$

Since, however, the excited states of the $|8_{13}8_{24}\rangle$ system will, as $r\to\infty$, all have virtually the same energy denominator $(\frac{3}{2}Kr^2)$, a variational calculation is more appropriate. We can actually find the exact energy shift as $r\to\infty$ if we notice that an $|8_{13}8_{24}\rangle$ wave function of the form $(\lambda^2-y^2)\times\exp[\frac{1}{2}\beta_1{}^2(\lambda^2+y^2)]$, with $\beta_1=(\frac{8}{3}km)^{1/4}$, will maximize the mixing matrix element with the ground-state wave function. This exact result gives the considerably stronger potential

$$V_{\text{vdW}}(r)=-\frac{1}{6mr^2}\simeq-\frac{20\text{ MeV}}{[r\text{ (fm)}]^2}\,. \tag{D7}$$

The effects of such a potential in the region $r>2$ fm, where, since it would be cut off by pair creation, it represents a spurious contribution to our effective potential, are not serious. In view of the fact that our wave functions die out very rapidly in this region we expect the error from this source to be less than 1 MeV.

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