Casimir energy of confined massive quarks

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We calculate the zero-point energy of massive fermions confined in a spherical cavity with MIT bag boundary conditions. The result contains new ultraviolet-divergent terms in addition to those occurring in the massless case. We discuss the divergence structure of the Casimir energy in the framework of the field-theoretical bag model.

The Casimir effect[1] has recently attracted the interest of particle physicists[2-6] in the context of the MIT bag model.[7] The vacuum fluctuations of quark and gluon fields are changed by the bag boundary conditions and this affects the dynamics (and statics) of the boundary. The effect may even be relevant for the structure of the CQD vacuum.[6] We will consider here the Casimir energy of massive fermions in a spherical cavity. The calculation for the massless case was first performed by Bender and Hays[3] who found it to be divergent, as for gluons. The calculation was repeated recently by Milton[5] who argued for a special subtraction prescription and obtained a finite contribution proportional to the inverse bag radius R^{-1}.

Our result for the massive case will turn out to contain new divergences in addition to those of the massless case. The occurrence of such additional divergences was found already in Ref. 3 for massive fermions confined to an infinite slab. We have tried to understand the origin of the divergence structure of the Casimir energy on general grounds. We base this discussion on a previous consideration[8] by one of us (J.B.) on the role of fermionic vacuum fluctuations in the field-theoretical bag model of Creutz and of Friedberg and Lee[9] and on recent work by Candelas.[10] Since our result is at variance with a previous one, Eq. (2.24) in Ref. 3 which would not lead to additional divergences, this served us also as an independent check.

The Casimir energy can be obtained from the (Euclidean) Green's function of the confined fermion field which satisfies

$$\left(\gamma_0\frac{\partial}{\partial\tau}+i\vec{\gamma}\cdot\vec{\nabla}-m\right)S_F(\vec{x},\vec{x}',\tau)=-\delta(\tau)\delta^3(\vec{x}-\vec{x}') \tag{1}$$

and the boundary condition

$$(1+i\hat{x}\cdot\vec{\gamma})S_F(\vec{x},\vec{x}',\tau)\big|_{|\vec{x}|=R}=0 \ . \tag{2}$$

The Green's function S_F can be decomposed into a free-space part S_F^0 given by

$$S_F^0(\vec{x},\vec{x}',\tau)=\left(\gamma_0\frac{\partial}{\partial\tau}+i\vec{\gamma}\cdot\vec{\nabla}+m\right)G(\vec{x},\vec{x}',\tau) \ ,$$

$$G(x,x',\tau)=\frac{mK_1(m[(\vec{x}-\vec{x}')^2+\tau^2]^{1/2})}{4\pi^2[(\vec{x}-\vec{x}')^2+\tau^2]^{1/2}} \tag{3}$$

and a boundary part S_F^B which is a solution of the homogeneous version of Eq. (1). The Casimir energy is then related to S_F^B via

$$E_C=-\lim_{\tau\to 0}\frac{\partial}{\partial\tau}\int d^3x\,\mathrm{Tr}[S_F^B(\vec{x},\vec{x},\tau)\gamma_0] \ . \tag{4}$$

We have determined S_F^B for the case of massive fermions following the procedure described in Refs. 3 and 5. We are not going to present here the detailed calculation, but give just our final result for the Casimir energy,

$$E_C=-\frac{1}{2\pi R}\sum_{j=1/2}^{\infty}(2j+1)\int_{mR}^{\infty}dx\,[x^2-(mR)^2]^{1/2}\left[2(e_{j+1/2}s'_{j-1/2}+e_{j-1/2}s'_{j+1/2})-\frac{d}{dx}\ln\left(s_{j+1/2}{}^2+s_{j-1/2}{}^2+\frac{2mR}{x}s_{j-1/2}s_{j+1/2}\right)\right]\times\cos\{[x^2-(mR)^2]^{1/2}\delta\} \ , \tag{5}$$

where $\delta=\tau/R$ and

$$s_l(x)=\left(\frac{\pi x}{2}\right)^{1/2}I_{l+1/2}(x)\ ,\quad e_l(x)=\left(\frac{2x}{\pi}\right)^{1/2}K_{l+1/2}(x) \tag{6}$$

are the Riccati-Bessel functions. This is a generalization of the massless results of Bender and Hays[3] and of Milton.[5]

Using the uniform asymptotic (Debye) expansion of the Bessel functions[11] we obtain in the $m \to 0$ limit

$$E_C(m=0) = -\frac{1}{4\pi R}\sum_{j=1/2}^{\infty}(2j+1)\int_0^{\infty} dz\, z^2 t^5 \cos z\nu\delta \ , \tag{7}$$

where $t=(1+z^2)^{-1/2}$ and $\nu = j$. This agrees with Milton's result except that he assigns $\nu = j+1$. The evaluation of this expression can now be done in two ways: either one first performs the z integration and then the sum over j using the Euler-MacLaurin formula (as was done by Milton) or one does first the summation over j [which reduces to power series in $\exp(\pm iz\delta)$] and then the integration over z. Using both ways and with Milton's and our assignment for ν we find

$$E_C(m=0) = \frac{R}{3\pi\tau^2} - \frac{1}{144\pi R} + O(\tau^2) \ . \tag{8}$$

Here we differ from Milton who finds $-1/48\pi R$ for the second term.

For the massive case the Debye expansion leads to the result

$$E_C = -\frac{1}{4\pi R}\sum_{j=1/2}^{\infty}(2j+1)\int_{mR}^{\infty} dx\, [x^2-(mR)^2]^{1/2}\nu^{-2} z t^3 (t^2 - 2mR)\cos\{[2^2-(mR)^2]^{1/2}\delta\} \ , \tag{9}$$

where $x = \nu z$. Performing the j summation one obtains in the leading order to the ν^{-1} expansion a new contribution proportional to the mass, so that

$$E_C = E_C(m=0) - \frac{m}{\pi}\left[\frac{R^2}{\tau^2} - \frac{1}{24}\int_0^{\infty} dz\, z^2 t^3\right] \ . \tag{10}$$

The new contribution proportional to the mass contains not only the τ^{-2} divergence, which is a signal of a quadratic ultraviolet divergence, but also a logarithmically divergent term independent of τ. Milton's subtraction procedure (leaving out the τ^{-2} term) does therefore not lead to a finite result here.

The structure of these divergences can be discussed in the framework of the field-theoretical bag model of Creutz and Friedberg and Lee,[9] in which a fermion field interacts via a Yukawa coupling $f\bar{\psi}\psi\phi$ with a self-coupled scalar field. The cancellation of divergences occurring in this model in the vacuum and bag sector was discussed by one of us (J.B.) in Ref. 8. The result can be stated as follows: The energy of the fermionic vacuum fluctuations in a potential $f\,\sigma(x)$, which is equivalent to the Casimir energy, contains in the perturbative expansion (see Fig. 1)

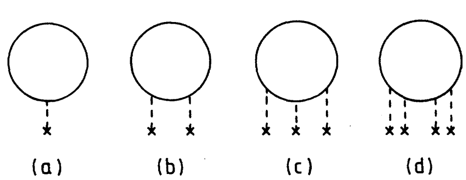

FIG. 1. (a)–(d) Divergent contributions to the Casimir energy in a perturbative expansion with respect to an external potential (× – – –). The graphs refer to old-fashioned perturbation theory.

divergent terms of the form

$$E_\sigma^{\rm div} = -\int d^3x\left[\sum_{n=0}^{4}\frac{\tilde{C}_n}{n!}\sigma^n(\vec{x}) + \tfrac{1}{2}[\vec{\nabla}\sigma(\vec{x})]^2\right] \ , \tag{11}$$

where the constants $\tilde{C}_n$ and D are ultraviolet divergent integrals given by

$$\tilde{C}_n = f^n\left[\frac{\partial}{\partial m}\right]^n\int\frac{d^3p}{(2\pi)^3}(p^2+m^2)^{1/2} \quad (n=0,\ldots,4) \ , \tag{12}$$

$$D = -\frac{f^2}{3(2\pi)^2}\int\frac{dp}{p}$$

up to finite terms. This can be verified by explicit calculation up to fourth-order perturbation theory for a fermion field in an external potential. If the potential σ is the expectation value of a quantized scalar field ϕ, as is the case in the field-theoretical bag model,[9] then there are counterterms of the form $C_n\phi^n(\vec{x})$ and $D[\vec{\nabla}\phi(\vec{x})]^2$ which are necessary for the renormalization of the self-couplings of the scalar field and its wave function. This implies that in such a model the divergences of the Casimir energy are compensated. Actually it was proposed in Ref. 8 to take the vacuum fluctuations of the fermion field into account in the determination of the confining potential $f\,\sigma(\vec{x})$, which is exactly equivalent to taking into account the Casimir effect.

Unfortunately, the divergence structure of the Casimir effect found in Ref. 8 cannot be compared in detail to that of Eqs. (8) and (10), since the regularization of the divergent integrals (e.g., by a momentum cutoff) was not considered at all in Ref. 8. But we can compare the leading divergences. The coefficient $\tilde{C}_0$ is just the fermionic vacuum energy before taking normal ordering, and is irrelevant here. The coefficient $\tilde{C}_1$, represented in Fig. 1(a), is proportional to the mass. Its leading divergence should be the same as for the new contribution proportional to the mass given in Eq. (10). In the calculation of this term an exponential energy cutoff $\exp(-E|\tau|)$ was introduced. Therefore the factor τ^{-2} implies quadratic ultraviolet divergence and this agrees with the UV divergence of $\tilde{C}_1$. The coefficient $\tilde{C}_2$ is also quadratically divergent. It is present even in the $m \rightarrow 0$ limit, and therefore its divergence should agree with the one of the Casimir energy for the massless case Eq. (10). The leading divergence of this term is indeed also quadratic as implied by the factor τ^{-2}. We cannot compare lower-order logarithmic divergences corresponding to the contributions from $\tilde{C}_{3,4}$ and from the nonleading parts of $\tilde{C}_{1,2}$, since, as stated above, the regularization has not been discussed in Ref. 8, and since a regularization of the fermion propagators would manifest itself in a different way in $\tilde{C}_{1-4}$ which contain different numbers of propagators. In addition we find that the Debye expansion used in the calculation of the Casimir effect is not reliable for the nonleading divergences.

In order to give concrete evidence for this statement we consider the divergent integral $\tilde{C}_1$. Since the contribution proportional to $\tilde{C}_1$ comes from a graph with only one fermion propagator, a regularization by an exponential cutoff can be considered as equivalent to the regularization used in the calculation of the Casimir effect. We write

$$\tilde{C}_1 = \lim_{\tau \rightarrow 0} 2\, f m G(\vec{x}, \vec{x}, \tau) \quad , \tag{13}$$

where $G(\vec{x}, \vec{x}, \tau)$ is defined in Eq. (3). When we integrate this expression over the bag volume, the result should be strictly proportional to R^3. However, if we expand the propagator with respect to Riccati-Bessel functions and use the Debye expansion we obtain

$$\frac{4\pi R^3}{3} \tilde{C}_1 \simeq \frac{4\pi R^3}{3} \frac{m}{\pi^2 \tau^2} + \frac{mR}{6\pi} \int_0^\infty dz \, \frac{1-t}{z^2 t} \quad . \tag{14}$$

The UV divergence structure of this term is the same as that of the term proportional to the mass in Eq. (10), which it should be. But the term proportional to R in Eq. (14) should not occur at all! It seems therefore that the results for the Casimir energy obtained by using the Debye expansion are unreliable except for the leading divergence. (A similar suspicion has been expressed by Milton.[4])

It is interesting to note that the leading divergence in the Casimir energy Eq. (10), is proportional to R^2: it should be canceled by the product of $\tilde{C}_1$ with the volume integral of $\sigma(x)$, which therefore must receive its main contribution from the surface region. This is in accordance with the qualitative behavior of $\sigma(x)$ in the field-theoretical bag model.[9] Similarly the leading mass-dependent divergence for massive fermions within an infinite slab calculated by Bender and Hays[3] is quadratic as that of $\tilde{C}_1$ and is proportional to the surface [compare their Eqs. (B9) and (B11), ϵ is equivalent to τ/L and a factor A is the area has been lost between Eqs. (B7) and (B8)]. Their result that no additional divergence is obtained when introducing a mass in the case of a scalar field in one space dimension agrees also with our reasoning: the graph of Fig. 1(a) is logarithmically divergent for a scalar loop in one dimension, but this divergence is already present in the absence of mass and the logarithmic divergent part of $\int dk/(k^2+m^2)^{1/2}$ is independent of m. While all this fits very nicely, we have to admit that it is less obvious to understand how the factor R in the quadratic divergence of the massless case [Eq. (8)] arises from an integral over $\sigma^2(\vec{x})$ in a limiting process.

So far we have understood qualitatively the origin of the divergences in the Casimir energy and the cancellation of these divergences in the framework of a field-theoretical model. It should be clear from this discussion that the calculation of "finite terms" cannot be performed without carrying through the subtraction procedure in detail and it will require detailed renormalization prescriptions. It will be—to say the least—very difficult to perform. A simpler practical cure may be abstracted from a recent paper by Candelas.[10] He discusses the Casimir effect with applications in solid-state physics. The ultraviolet-divergent integrals for idealized boundaries occur in a "geometrical expansion" as factors multiplying the surface tension and some leading curvature tensions. Candelas finds agreement with known results for the surface tension of metals if he introduces an UV cutoff at the atomic scale. One might therefore be tempted to use a similar cutoff procedure. Physically this seems plausible, since the creation of virtual fermion-antifermion pairs with high momentum will certainly lead to surface fluctuations. Alternatively one could introduce all those surface and curvature tensions which appear with divergent factors in the Casimir energy from the outset with finite coefficients and consider the divergent contributions as being absorbed into their renormalization. Unfortunately this would lead to a proliferation of free parameters in the bag model, since there is no special dynamical reason why the renormalized coefficients should be zero.

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