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Baryon and lepton numbers as broken local symmetries

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We investigate what kind of linear combination of baryon (B) and lepton (L) numbers ($B-\alpha L$) (α is a numerical constant) can be a generator of a grand unified group G, and consequently a broken local symmetry.

I. INTRODUCTION

Many years have passed since Lee and Yang raised the question whether baryon-number conservation resulted from an unbroken local symmetry—like electric-charge conservation.[1] The answer was that if it is a local symmetry it must be broken. On the other hand, baryon number can also be a global symmetry, accidentally or following automatically from the structure of the theory. A similar statement can be made about lepton number. Are baryon (B) and lepton (L) numbers local symmetries or just global symmetries? We still do not know the answer to this basic question despite the many successes of the standard $SU(3)_c \times SU(2)_L \times U(1)$ group and the effort that has gone into developing a satisfactory grand unified theory (GUT).

The standard Lagrangian that is taken to be invariant under $SU(3)_c \times SU(2)_L \times U(1)$ conserves global B and L separately. However, as 't Hooft has pointed out,[2] separate global conservation of B and L is broken by the triangle anomalies.[3,4] Interestingly, the difference, $(B-L)$, is not broken by the anomalies, i.e., exact conservation of $(B-L)$ at the level of the renormalizable $SU(3)_c \times SU(2)_L \times U(1)$ theory holds to all orders of perturbation.

In grand unified groups, i.e., groups larger than $SU(3)_c \times SU(2)_L \times U(1)$ or $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, separate baryon- and lepton-number conservation is usually explicitly broken. Strangely, the dim 6 operators, which are the lowest-dimensional operators responsible for proton decay and are invariant under $SU(3)_L \times SU(2)_L \times U(1)$, have the property of global $(B-L)$ conservation, i.e., $\Delta(B-L)=0$.[5] Higher-dimensional operators have different properties[5]: $\Delta(B+L)=0$ for dim 7, $\Delta(3B-L)=0$ for dim 9, $\Delta(3B+L)=0$ for dim 10, $\Delta B=2$ for dim 9, and $\Delta L=2$ for dim 5.

In minimal SU(5) GUT (Ref. 6), $\Delta(B-L)=0$ holds to all orders as a global symmetry,[7] even though $(B-L)$ is not a generator of the group. For $SU(4)\times SU(2)_L \times SU(2)_R$ (Ref. 8) or SO(10) (Ref. 9), $(B-L)$ is a generator of the group.[10] When local $(B-L)$ symmetry is broken and this is combined with the "weak" Gell-Mann—Nishijima relation, one can understand in a natural way the origin of neutron oscillation and Majorana neutrinos.[10] In $SU(8)_L \times SU(8)_R$,[11] the fermion number $F=3B+L$ is a global symmetry whereas $(B-L)$ is a local symmetry. If both symmetries are not broken, the proton never decays. In SU(16),[12] B, L, and F all become local symmetries.

One can go on enumerating different GUT groups and specify whether various linear combinations of B and L are global or local symmetries. In view of the importance of knowing whether $(B-\alpha L)$ (α is a numerical constant) can be broken local symmetry, we seek the grand unified groups that have $(B-\alpha L)$ as a generator, making some general assumptions about the color and electroweak classification schemes for the quarks and leptons. In Sec. II, we discuss the assumptions we make. Section III deals with the GUPA (grand unification of particles and antiparticles) all particles and antiparticles are placed in the same multiplet, while Sec. IV deals with the GUP approach where only particles are placed in the same multiplet. The last section is devoted to discussions and conclusions.

II. ASSUMPTIONS

We make the following assumptions. First, we assume that a grand unified group G contains $SU(2)_L$ and $(B-\alpha L)$ as local symmetries. Second, as has been done before,[13] we assume the quantum numbers of quarks and leptons as follows:

	G_c	I_{3L}	$B-\alpha L$	No.
N_L	1	$\frac{1}{2}$	$-\alpha$	n_l
E_L	1	$-\frac{1}{2}$	$-\alpha$	n_l
N_L^c	1	0	α	n_0
E_L^c	1	0	α	n_l
U_L	m	$\frac{1}{2}$	$\frac{1}{3}$	mn_q
D_L	m	$-\frac{1}{2}$	$\frac{1}{3}$	mn_q
U_L^c	$\overline{m}$	0	$-\frac{1}{3}$	mn_q
D_L^c	$\overline{m}$	0	$-\frac{1}{3}$	mn_q

(2.1)

where G_c denotes the color group [which is not fixed as SU(3)] and I_{3L} denote the third component of $SU(2)_L$. We assume that $m\neq 1$, i.e., quarks are not singlets under G_c. Note that we do not assume any particular values for the electric charge or weak hypercharges. We do not fix the number of quarks and lepton families. The $(B-\alpha L)$ quantum numbers are not enough to fix the group G, as we will see.

The number n_0 of N_L^c must be equal to the number n_l of E_L^c, as long as we assume G to be semisimple and $\alpha\neq 0$. This is because $0=\mathrm{Tr}(B-\alpha L)=\alpha(n_0-n_l)$. This is the simplest reason why SU(5) GUT cannot have $(B-\alpha L)$ $(\alpha\neq 0)$ as a broken local symmetry. Hereafter, we assume $n_0=n_l$.

III. GUPA APPROACH

We now become more specific and consider the case where all the particles and antiparticles in (2.1) are in the same irreducible representation (GUPA approach[14]). We denote a minimal semisimple factor subgroup of G, which contain $(B-\alpha L)$ as a generator, by $G(BL)$. We show in Appendix B that $G(BL)$ must be a simple group. Then $G(BL)$ must contain SU(2) as its subgroup. The reason is that if $G\supset SU(2)_L\otimes G(BL)$, we would have $(V+A)$ particles, since we then have various combinations of eigenvalues of $SU(2)_L$ and $G(BL)$. Similarly, $G(BL)$ must contain G_c, otherwise we would have "colored" leptons. Hence, we can regard $G(BL)$ as a grand unified group G. What is $G(BL)$ then?

First of all, we can eliminate the possibility of exceptional groups as $G(BL)$ since they do not satisfy the quartic trace identity $\mathrm{Tr}X^4=K(R)(\mathrm{Tr}X^2)^2$.[15] The best way of distinguishing SU(n) ($n\geq 3$) from other simple groups is to look at the trace, $\mathrm{Tr}X^3$.[16] We have in this case

$$\mathrm{Tr}I_{3L}{}^2(B-\alpha L)=-\tfrac{1}{2}(\alpha n_l-\tfrac{1}{3}mn_q)\ . \qquad (3.1)$$

Hence, *if* $3\alpha n_l\neq mn_q$, *G must be SU(n) with* $n=4(n_l+mn_q)$, since we must use a basic representation because of the $SU(2)_L$ and $(B-\alpha L)$ quantum numbers. It is easy to modify the argument in Ref. 17 to prove this. Thus, in particular, in order to have, e.g., B, $B+L$, or $F=3B+L$ (all have $\alpha\leq 0$) as a generator of the GUT group, we must use SU(n). Of course, one single totally antisymmetric representaion cannot make the theory anomaly free[4] and, thus, we are forced to introduce mirror fermions. The GUT group with $m=3$, $n_l=n_q=1$ is SU(16).[12]

For the case where $3\alpha n_l=mn_q$, we calculate $\mathrm{Tr}X^5$:

$$\mathrm{Tr}I_{3L}{}^2(B-\alpha L)^3=\tfrac{1}{2}\alpha n_l(-\alpha^2+\tfrac{1}{9})\ . \qquad (3.2)$$

Unless $\alpha=\frac{1}{3}$ (which we will discuss later), we have $\mathrm{Tr}X^5\neq 0$, which leads to G as SO(10) or SU(n) ($n\geq 3$).[13,15] We first investigate SO(10). Using the trace identity $\mathrm{Tr}X^7=D(R)\mathrm{Tr}X^2\mathrm{Tr}X^5$ for SO(10) (Ref. 15) and the fact that $\alpha\neq\frac{1}{3}$ and $Y=B-\alpha L$, we have

$$D(R)=\frac{7\,\mathrm{Tr}I_{3L}{}^4Y^3}{2\,\mathrm{Tr}I_{3L}{}^2\mathrm{Tr}I_{3L}{}^2Y^3}=\frac{21\,\mathrm{Tr}I_{3L}{}^2Y^5}{10\,\mathrm{Tr}Y^2\mathrm{Tr}I_{3L}{}^2Y^3} \qquad (3.3)$$

which yields

$$\alpha=1 \text{ or } \tfrac{1}{9}\ . \qquad (3.4)$$

In SO(10), representations with only $I_{3L}=\pm\frac{1}{2}$ and 0 are spinors or a vector.[17] Since a vector representation satisfies $\mathrm{Tr}X^5=0$, we use a spinor where $D(\text{spinor})=\frac{7}{16}$, which yields

$$n_l=n_q=1,\ m=3 \text{ or } n_l=1,\ n_q=3\ \ m=1 \ \text{ for } \alpha=1\ ,$$
$$n_l=3,\ n_q=1,\ m=1 \ \text{ for } \alpha=\tfrac{1}{9}\ . \qquad (3.5)$$

Thus, since $m\neq 1$ and SO(10) contains SU(3) as a subgroup, we have found that $G=SO(10)$, $G_c=SU(3)$, $(B-L)$ is a generator of G, and the representation is a spinor with $n_l=n_q=1, m=3$. For SU(n), we must use the basic representation, because of the reason given before. Hence, $n=4(n_l+mn_q)=4n_l(3\alpha+1)$.

Now, we discuss the case where $\alpha=\frac{1}{3}$ (i.e., the possibility of having $(3B-L)$ as a generator). In this case, $(3B-L)$ has only two eigenvalues, ± 1. Thus, we cannot have G as an exceptional group.[18] However, we cannot use odd-order trace identities for I_{3L} and $(3B-L)$, since $\mathrm{Tr}X^{\mathrm{odd}}=0$ for $X=I_{3L}+t(3B-L)$ where t is an arbitrary constant. By the two-eigenvalue condition, G is one of the classical groups and its corresponding representation is fixed as follows[18]: for SU(n), the totally antisymmetric representation; for SO($2n+1$), the spinor; for Sp($2n$), the basis; for SO($2n$), the basic or the spinor.

So far, we have used the information that $SU(2)_L$ and $(B-\alpha L)$ are contained in G. If we use the fact that G_c is also contained in G, then we have

$$\mathrm{Tr}[C_2(I_{3L})^2(B-\alpha L)]=\tfrac{1}{6}\langle C_2\rangle mn_q , \tag{3.6}$$

where C_2 denotes the second-order Casimir invariant of G_c and $\langle C_2\rangle$ is its expectation value in the quark sector. Since $m\neq 1$, we have $\langle C_2\rangle\neq 0$, i.e., $\mathrm{Tr}X^5\neq 0$, *independent* of α. Thus, for $\alpha=\frac{1}{3}$, only SU(n) ($n\geq 3$) and SO(10) can be used as G. However, for SO(10), we get

$$n_l=2,\quad mn_q=2 \quad \text{for } \alpha=\tfrac{1}{3} . \tag{3.7}$$

The choice, $m=2$, is unacceptable, in view of the quark line rule.[19] Thus, only SU(n) ($n\geq 3$) is left for $\alpha=\frac{1}{3}$.

IV. GUP APPROACH

We next discuss the other basic option (to the GUPA approach), namely, the case where only the particles are in the same irreducible representation (GUP approach[14]). The particles are as follows:

	G_c	I_{3L}	$B-\alpha L$	No.
N_L	1	$\frac{1}{2}$	$-\alpha$	n_l
E_L	1	$-\frac{1}{2}$	$-\alpha$	n_l
U_L	m	$\frac{1}{2}$	$\frac{1}{3}$	mn_q
D_L	m	$-\frac{1}{2}$	$\frac{1}{3}$	mn_q

(4.1)

First, $G(BL)$ must be a simple group, if we assume the absence of complicating U(1) factors. The reason is that $(B-\alpha L)$ has only two eigenvalues.[18] Then, we have $\mathrm{Tr}(B-\alpha L)=0$, which yields

$$3\alpha n_l=mn_q \text{ and } \alpha>0 . \tag{4.2}$$

In particular, $(B-\alpha L)$ $(\alpha\leq 0)$ *cannot be a local symmetry.* The dimension of the multiplet is

$$d=2n_l=2mn_q=2n_l(3\alpha+1) . \tag{4.3}$$

Second, we note that $G(BL)$ must contain G_c, since otherwise we would have "colored" leptons. Thus, we obtain [same notation as in Eq. (3.6)]

$$\mathrm{Tr}[C_2(B-\alpha L)]=\tfrac{2}{3}mn_q\langle C_2\rangle . \tag{4.4}$$

Unless quarks are singlets under G_c, we have $\mathrm{Tr}X^3\neq 0$. Hence, $G(BL)$ must be SU(n) ($n\geq 3$). Using the method in Ref. 18, the representation should be totally antisymmetric and n is given by

$$n=p(3\alpha+1)/(3\alpha)=p\frac{mn_q+n_l}{mn_q} , \tag{4.5}$$

where p is a positive integer. Furthermore, if $G(BL)$ contains SU(2), the representation must be the basic one.[18] We examine the smaller rank groups and the result is that $G=SU(8)_L\times SU(8)_R$ *with* $(B-L)$, $G_c=SU(3)$ *and* $n_l=n_q=1$. If $G(BL)$ does not contain $SU(2)_L$, then $G(BL)$ could be SU(3) with $(3B-2L)$, SU(4) with $(B-L)$,.... The smallest choice is SU(4), if we use the U(1) weak-hypercharge information.[14] The case where $G(BL)=SU(3)$ is also forbidden, if $G_c=SU(3)$. The GUT is then $SU(4)\times SU(2)_L\times SU(2)_R$.[8,10,14]

V. CONCLUSIONS

We summarize our results as follows. In the GUPA approach, if $3\alpha n_l\neq mn_q$, G can only be SU(n) with $n=4(n_l+mn_q)$ and the basic representation, in order to have $(B-\alpha L)$ as a generator. We need mirror fermions to cancel anomalies. In particular, if B, L, or $F=3B+L$ are generators, G is SU(n) [e.g., SU(16) with $m=3$, $n_l=n_q=1$ (Ref. 12)]. If $3\alpha n_l=mn_q$, then G is *either* SO(10) with $(B-L)$, $G_c=SU(3)$, and $n_l=n_q=1$, *or* SU(n) with $(B-\alpha L)$, $n=4n_l(3\alpha+1)$ (α arbitrary, except n must be an integer), and the basic representation. Thus, in the GUPA approach, only SU(n) [$n=4(n_l+mn_q)$] with the basic representation can have $(B-\alpha L)$ as a generator, except for SO(10), which has $(B-L)$ as a generator. Note that we must have *right-handed neutrinos,* in order to have $(B-\alpha L)$ $(\alpha\neq 0)$ as a broken local symmetry. In particular, SU(5) GUT cannot have $(B-\alpha L)$ $(\alpha\neq 0)$ as a local symmetry, but it does have $(B-L)$ as a *global* symmetry. In the GUP approach, we cannot have $(B-\alpha L)$ $(\alpha\leq 0)$ (e.g., B, L, or $F=3B+L$) as a generator. Only the difference $(B-L)$ can be a generator for $SU(8)_L\times SU(8)_R$ (Ref. 11) or $SU(4)\times SU(2)_L\times SU(2)_R$.[8,10]

Since GUPA groups usually contain GUP groups as subgroups, the relationship between $(B-\alpha L)$ generators of GUPA and GUP groups is interesting to examine. The GUPA group, SO(10), contains the GUP group, $SU(4)\times SU(2)_L\times SU(2)_R$, as a subgroup and both groups have $(B-L)$ and only this combination as a generator. The other permissible GUPA group, SU(16), contains the GUP group, $SU(8)_L\times SU(8)_R$, as a subgroup; however, while

SU(16) has B, L, and $F=3B+L$ as generators, $SU(8)_L \times SU(8)_R$ has only $(B-L)$ as a generator.

One should note that the only assumptions we have made to prove the results above are the existence of a multiplet consisting of left-handed $SU(2)_L$ doublets and right-handed $SU(2)_L$ singlets, with baryon number $\frac{1}{3}$ for quarks and lepton number 1 for leptons. We did not allow $(V+A)$ particles to mix with $(V-A)$ particles in the same multiplet. It is intriguing that we end up with SO(10)- or SU(n)-type grand unification groups, just from the condition that $(B-\alpha L)$ be a broken local symmetry, without knowing the strong-interaction group and the electric charges of quarks and leptons. Implications of having $(B-\alpha L)$ conserved for cosmology have been investigated.[20]

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APPENDIX A

In this appendix, we summarize results in our previous papers,[13–19,22] which have been used in this paper and may be useful for other purposes.

1. Trace identities

The possible orders of independent symmetric Casimir invariants for simple Lie algebras are only the following[21]:

$$\begin{aligned}
&SU(n){:}2,3,\ldots,n\ ,\\
&SO(2n+1){:}2,4,\ldots,2n\ ,\\
&Sp(2n){:}2,4,\ldots,2n\ ,\\
&SO(2n){:}2,4,\ldots,2(n-1),\ \text{and}\ n\ ,\\
&E_6{:}2,5,6,8,9,12\ ,\\
&E_7{:}2,6,8,10,12,14,18\ ,\\
&E_8{:}2,8,12,14,18,20,24,30\ ,\\
&F_4{:}2,6,8,12\ ,\\
&G_2{:}2,6\ .
\end{aligned} \tag{A1}$$

Odd-order trace identities

The absence of odd-order symmetric Casimir invariants for groups $SO(2n+1)$, $SO(4n)$, $Sp(2n)$, E_7, E_8, E_4, and G_2 is explained by the fact that these algebras have only self-contragredient representations (i.e., $X \sim -X^t$) or these groups have only real or pseudoreal representations. Thus, for these groups, we have

$$\mathrm{Tr}X^p = 0\ (p\text{:odd integer})\ . \tag{A2}$$

For groups SU(n) ($n \geq 3$), E_6, SO($4n+2$) ($n \geq 2$), we can classify them by nonvanishing traces:

$$\begin{aligned}
&\text{If } \mathrm{Tr}X^3 \neq 0, \text{ then } G=SU(n)\ (n\geq 3)\ ,\\
&\text{If } \mathrm{Tr}X^5 \neq 0, \text{ then } G=SU(n)\ (n\geq 3),\ E_6, \text{ or } SO(10)\ ,\\
&\text{If } \mathrm{Tr}X^7 \neq 0, \text{ then } G=SU(n)\ (n\geq 3),\ E_6,\ SO(10), \text{ or } SO(14)\ ,\\
&\text{If } \mathrm{Tr}X^9 \neq 0, \text{ then } G=SU(n)\ (n\geq 3),\ E_6,\ SO(10),\ SO(14), \text{ or } SO(18)\ .
\end{aligned} \tag{A3}$$

Futhermore, for SU(3) and SU(4), we have[15]

$$\mathrm{Tr}X^5 = A(R)\mathrm{Tr}X^2\mathrm{Tr}X^3\ , \tag{A4}$$

$$A(R) = \frac{5\,d(R_{\mathrm{adj}})}{2[6+d(R_{\mathrm{adj}})]d(R)}\left[4-\frac{C_2(R_{\mathrm{adj}})}{C_2(R)}\right]\ , \tag{A5}$$

where $d(R)$ and $C_2(R)$ denote the dimension and the eigenvalue of the second-order Casimir invariant for the irreducible representation R. For E_6 and SO(10), we have[13,15]

$$\mathrm{Tr}X^7 = D(R)\mathrm{Tr}X^2\mathrm{Tr}X^5\ , \tag{A6}$$

$$D(R) = \frac{35\,d(R_{\mathrm{adj}})}{4[10+d(R_{\mathrm{adj}})]d(R)}\left[\frac{12}{5}-\frac{C_2(R_{\mathrm{adj}})}{C_2(R)}\right]\ . \tag{A7}$$

We can go on as follows:

$$\mathrm{Tr}X^9 = F(R)\mathrm{Tr}X^2\mathrm{Tr}X^7 \quad \text{for } SO(14)\ , \tag{A8}$$

$$\mathrm{Tr}X^{11} = G(R)\mathrm{Tr}X^2\mathrm{Tr}X^9 \quad \text{for } SO(18) \tag{A9}$$

although we have not yet calculated explicit forms of constants, $F(R)$ and $G(R)$, which depend only on R, but not on X.

Even-order trace identities

For even-order trace identities, we have[22]

$$\mathrm{Tr}X^4-K(R)(\mathrm{Tr}X^2)^2=C_4(t)D^{(4)}(R)\,, \tag{A10}$$

$$K(R)=\frac{d(R_{\mathrm{adj}})}{2[2+d(R_{\mathrm{adj}})]d(R)}\left[6-\frac{C_2(R_{\mathrm{adj}})}{C_2(R)}\right]\,, \tag{A11}$$

where $D^{(4)}(R)$ denotes the modified Dynkin indices of fourth order.[22] The numerical constant $C_4(t)$ does not depend on R. This formula holds for any simple Lie algebras, except D_4 [$=$SO(8)], because D_4 has two independent fourth-order Casimir invariants. In particular, for SU(2), SU(3), G_2, F_4, E_6, E_7, and E_8, we have

$$\mathrm{Tr}X^4-K(R)(\mathrm{Tr}X^2)^2=0 \tag{A12}$$

since these groups do not have genuine fourth-order Casimir invariants. We can distinguish E_6, E_7, and E_8 by trace identities: E_6 differs from E_7 and E_8 by having $\mathrm{Tr}X^5\neq 0$. E_8 satisfies

$$\mathrm{Tr}X^6=H(R)(\mathrm{Tr}X^2)^3\,, \tag{A13}$$

$$H(R)=\frac{15}{[2+d(R_{\mathrm{adj}})][4+d(R_{\mathrm{adj}})]}\left[\frac{d(R_{\mathrm{adj}})}{d(R)}\right]^2 \times\left[1-\frac{1}{2}\frac{C_2(R_{\mathrm{adj}})}{C_2(R)}+\frac{1}{12}\left[\frac{C_2(R_{\mathrm{adj}})}{C_2(R)}\right]^2\right]. \tag{A14}$$

2. Representations

From the number of eigenvalues of an irreducible representation, we can limit the types of representations of a simple Lie algebra.

Two-eigenvalue representations

If an irreducible representation, R, has only two eigenvalues, then it is one of the following[18]:

$$\begin{aligned}
&R=\lambda_j\ (1\le j\le n)\ \text{ for SU}(n+1)\ (n\ge 1)\,,\\
&R=\lambda_n(\text{spinor})\ \text{ for SO}(2n+1)\ (n\ge 3)\,,\\
&R=\lambda_1(\text{basic})\ \text{ for Sp}(2n)\ (n\ge 2)\,,\\
&R=\lambda_1(\text{basic}),\ \lambda_{n-1}\text{ or }\lambda_n(\text{spinor})\\
&\quad\text{for SO}(2n)\ (n\ge 4)\,,
\end{aligned} \tag{A15}$$

where λ_j denotes the fundamental weight. Note that none of the exceptional Lie algebras, E_6, E_7, E_8, F_4, or G_2, is allowed. The form of the operator which has two eigenvalues can be specified[18].

Representations with $\pm\frac{1}{2}$ and 0 eigenvalues of SU(2) subalgebra

If the irreducible representation R has the eigenvalues, $\pm\frac{1}{2}$ and 0, of the SU(2) subalgebra of G, then it is one of the following[17]:

$$\begin{aligned}
&R=\lambda_j\ (1\le j\le n)\ \text{ for SU}(n+1)\ (n\ge 1)\,,\\
&R=\lambda_1\text{ or }\lambda_n\ \text{ for SO}(2n+1)\ (n\ge 3)\,,\\
&R=\lambda_j\ (1\le j\le n)\ \text{ for Sp}(2n)\ (n\ge 2)\,,\\
&R=\lambda_1,\lambda_{n-1},\text{ or }\lambda_n\ \text{ for SO}(2n)\ (n\ge 4)\,,\\
&R=\lambda_2\ \text{ for } G_2\,,\\
&R=\lambda_4\ \text{ for } F_4\,,\\
&R=\lambda_1\text{ or }\lambda_5\ \text{ for } E_6\,,\\
&R=\lambda_1\ \text{ for } E_7\,,\\
&\text{None for } E_8\,.
\end{aligned} \tag{A16}$$

Totally antisymmetric representations in SU(n)

The representation, $R=\lambda_j$ $(1\le j\le n-1)$, corresponds to the totally antisymmetric tensor with j indices, $\phi_{\mu_1\mu_2\cdots\mu_j}$. If we denote the generator of SU(n) $(n\ge 3)$ as

$$[B_\nu{}^\mu,B_\beta{}^\alpha]=\delta^\mu_\beta B_\nu{}^\alpha-\delta^\alpha_\nu B_\beta{}^\mu\,,\qquad \sum_{\mu=1}^{n}B_\mu{}^\mu=0\,, \tag{A17}$$

then, the action of $B_\nu{}^\mu$ on $\phi_{\mu_1\cdots\mu_j}$ is given by

$$B_\nu{}^\mu\phi_{\mu_1\cdots\mu_j}=\sum_{p=1}^{j}\delta^\mu_{\mu_p}\phi_{\mu_1\cdots\hat{\nu}\cdots\mu_j}-\frac{j}{n}\delta^\mu_\nu\phi_{\mu_1\cdots\mu_j}\,, \tag{A18}$$

where $\hat{\nu}$ implies the replacement of μ_p by ν. The element of the Cartan subalgebra (diagonal generator) is given by

$$A=\sum_{\lambda=1}^{n}\xi_\lambda B^\lambda_\lambda\,, \tag{A19}$$

where ξ_λ are numerical constants and they satisfy $\sum_{\lambda=1}^{n}\xi_\lambda=0$. The action of A on $\phi_{\mu_1\cdots\mu_j}$ is then

$$A\phi_{\mu_1\cdots\mu_j}=(\xi_{\mu_1}+\xi_{\mu_2}+\cdots+\xi_{\mu_j})\phi_{\mu_1\cdots\mu_j}\,; \tag{A20}$$

i.e., the eigenvalues of A are given by

$$\xi_{\mu_1}+\xi_{\mu_2}+\xi_{\mu_3}+\cdots\ \xi_{\mu_j}\,. \tag{A21}$$

3. Eigenvalues for semisimple groups

The requirement of finite-dimensional unitary representations of G leads essentially to

$$G = G_1 \otimes G_2 \otimes \cdots \otimes G_N \otimes U(1) \otimes U(1) \otimes \cdots \otimes U(1) , \quad (A22)$$

where G_j are simple Lie groups. Thus, any physical diagonal operator is expressed as

$$Q = Q_1 + Q_2 + \cdots + Q_N + C , \quad (A23)$$

where Q_j belongs to the Cartan subalgebra of G_j and C denotes the contribution from U(1) factors. In order to have m distinct eigenvalues for Q, we must satisfy[14]

$$N+1 \le \sum_{j=1}^{N} n_j - N + 1 \le m \le \prod_{j=1}^{N} n_j , \quad (A24)$$

where n_j denotes the number of eigenvalues of Q_j. If $m = \sum_{j=1}^{N} n_j - N + 1$, then eigenvalues of Q are equally spaced. For example, if $m = 2$, i.e., if Q has only two eigenvalues, then we have $N+1 \le 2$, i.e., $N = 1$. That is, Q must belong to only one simple group G and possible U(1) constant.

APPENDIX B

We show that $G(BL)$ must be a simple group, if it is semisimple. For the case where $\alpha \ne \pm\frac{1}{3}$ or 0, then $G(BL)$ has four distinct eigenvalues. Using Eq. (A24) of Appendix A, $G(BL)$ can be one of the following:

$$\begin{aligned} &G(BL) = G_1 \otimes G_2 \otimes G_3 \ \text{ with } n_1 = n_2 = n_3 = 2 , \\ &G(BL) = G_1 \otimes G_2 \ \text{ with } n_1 = 3,\ n_2 = 2 , \\ &G(BL) = G_1 \otimes G_2 \ \text{ with } n_1 = n_2 = 2 , \\ &G(BL) = G_1 \ \text{ with } n_1 = 4 , \end{aligned} \quad (B1)$$

where G_j are simple groups and we assumed the absence of U(1) factors. We note that $SU(2)_L$ must be contained in one of G_j, since the rank of SU(2) is 1. We denote four distinct eigenvalues of $G(BL)$ as a, b, c, d where $a > b > c > d$. The corresponding eigenvalues of I_{3L} are as follows:

a	b	c	d	
0	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	for $\alpha > \frac{1}{3}$
$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	for $0 < \alpha < \frac{1}{3}$
$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	0	for $-\frac{1}{3} < \alpha < 0$
$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	for $\alpha < -\frac{1}{3}$

(B2)

We denote the generator $(B - \alpha L)$ as

$$B - \alpha L = Y_1 + Y_2 + \cdots , \quad (B3)$$

where Y_j is a generator of G_j. The eigenvalue of Y_j is denoted by b_j^k $(k = 1, 2, \ldots, m)$ where m is the number of distinct eigenvalues and n_j^k is the number of the eigenvalue b_j^k.

Now we proceed with proof. For the case where $G(BL) = G_1 \otimes G_2 \otimes G_3$ with $n_1 = n_2 = n_3 = 2$, we have

$$a = b_1^1 + b_2^1 + b_3^1 , \quad (B4)$$

$$b = b_1^1 + b_2^1 + b_3^2 = b_1^1 + b_2^2 + b_3^1 = b_1^2 + b_2^1 + b_3^1 , \quad (B5)$$

$$c = b_1^1 + b_2^2 + b_3^2 = b_1^2 + b_2^1 + b_3^2 = b_1^2 + b_2^2 + b_3^1 , \quad (B6)$$

$$d = b_1^2 + b_2^2 + b_3^2 . \quad (B7)$$

If G_1 contains $SU(2)_L$ as a subgroup, then the b_1^1 sector of Y_1 must have both $\pm\frac{1}{2}$ and 0 of $SU(2)_L$, because of Eqs. (B2), (B5), and (B6) (b and c have $I_{3L} = \pm\frac{1}{2}$ and 0). However, this is in conflict with the fact that $a = b_1^1 + b_2^1 + b_3^1$ has either $\pm\frac{1}{2}$ or 0 of $SU(2)_L$, but not both. Hence, G_1 cannot have $SU(2)_L$ as a subgroup. Similarly, we can show that G_2 and G_3 cannot have $SU(2)_L$ as a subgroup.

For the case where $G(BL) = G_1 \otimes G_2$ with $n_1 = 3$ $n_2 = 2$, we have

$$a = b_1^1 + b_2^1 , \quad (B8)$$

$$b = b_1^1 + b_2^2 = b_1^2 + b_2^1 , \quad (B9)$$

$$c = b_1^2 + b_2^2 = b_1^3 + b_2^1 , \quad (B10)$$

$$d = b_1^3 + b_2^2 . \quad (B11)$$

We can show that G_2 cannot have $SU(2)_L$ as a subgroup, by looking at the b_2^1 sector of Eqs. (B8)—(B10). We can also show that G_1 cannot have $SU(2)_L$ as a subgroup, because of the b_1^2 sector in Eqs. (B9) and (B10).

For the case where $G(BL) = G_1 \times G_2$ with $n_1 = n_2 = 2$, we have

$$\begin{aligned} &a = b_1^1 + b_2^1,\ \ b = b_1^1 + b_2^2 , \\ &c = b_1^2 + b_2^1,\ \ d = b_1^2 + b_2^2 \end{aligned} \quad (B12)$$

or

$$\begin{aligned} &a = b_1^1 + b_2^1,\ \ b = b_1^2 + b_2^1 , \\ &c = b_1^1 + b_2^2,\ \ d = b_1^2 + b_2^2 . \end{aligned} \quad (B13)$$

We cannot use the same method as before. We use the fact that one of G_j must contain the color group G_c as a subgroup, since we assume that G_c is a simple group. If we assume that G_1 has $SU(2)_L$ as a subgroup, then we have the following:

For Eq. (B12), the b_1^1 sector has $I_{2L} = \pm\frac{1}{2}$ only, while the b_1^2 sector has $I_{3L} = 0$ only; i.e.,

$$-\tfrac{1}{3} < \alpha < 0 \text{ or } \alpha < -\tfrac{1}{3} . \quad (B14)$$

For Eq. (B13), the b_1^1 sector has $I_{3L} = 0$ only, while the b_1^2 sector has $I_{3L} = \pm\frac{1}{2}$ only; i.e.,

$$\alpha > \tfrac{1}{3} . \quad (B15)$$

For Eq. (B13), the b_1^1 sector has $I_{3L}=\pm\frac{1}{2}$ only, while the b_1^2 sector has $I_{3L}=0$ only; i.e.,

$$0<\alpha<\frac{1}{3}\ . \tag{B16}$$

We show that each case leads to inconsistency. For Eq. (B14), G_1 cannot have G_c as a subgroup, since we have either a for quarks and b for leptons or a for leptons and b for quarks. The group G_2 cannot have G_c either, since we have either a for quarks and c for leptons or a for leptons and c for quarks. For Eq. (B15), G_1 cannot have G_c, since then we have a for antileptons and c for antiquarks. The group G_2 cannot either, since then we have a for antilepton and b for quarks. Similarly, for Eq. (B16), we can show that none of G_1 and G_2 can have G_c as a subgroup. For the case where G_2 contains $SU(2)_L$ as a subgroup, we can proceed exactly in the same way.

For the case where $\alpha=0$, we can show the inconsistency as above. For the case where $\alpha=\pm\frac{1}{3}$, we have only two eigenvalues, and thus $G(BL)$ is a simple group.

Therefore, we have proved that $G(BL)$ must be a simple group, if it is semisimple.

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