PHYSICAL REVIEW D VOLUME 27, NUMBER 2 15 JANUARY 1983

Hadronization and its representation dependence

Mark Gross
The Enrico Fermi Institute and the Department of Physics, The University of Chicago, Chicago, Illinois 60637
(Received 14 June 1982)

A recently developed formalism for calculating the statistical mechanics of a one-dimensional quark gas is applied to a gas of arbitrary SU(2) representation quarks.

I. INTRODUCTION

Recently, a formalism was developed to solve the statistical mechanics of a one-dimensional gas of interacting quarks and antiquarks with classical positions and momenta.[1,2] The main innovation is that the group theory is treated exactly. As reviewed below, this formalism permits one to calculate thermodynamic quantities for a desired ensemble and internal symmetry group by either analytic techniques or Monte Carlo methods. Such quantities as the mean number of quarks per hadron then give some insight into the process of hadronization of quarks.

In this paper the system is chosen to consist of a fixed number n of SU(2) quarks at pressure p and temperature T. These choices are for simplicity, only. Certainly an SU(3) grand canonical ensemble would be of a more direct physical relevance and is in fact being employed by Nambu in a study of high-energy jets. But here we choose, instead, to take the representation of the quarks to be arbitrary so as to investigate the representation dependence of the hadronization process in the simplified, but more accessible model of an SU(2) n,p,T ensemble.

In Sec. II the quark-gas formalism is reviewed and applied to the chosen ensemble. In Sec. III the results of analytic and Monte Carlo calculations of thermodynamic quantities are presented. In Sec. IV these results are interpreted and discussed.

II. THE FORMALISM

What follows is a review of the one-dimensional quark-gas formalism. Some generality is sacrificed for the sake of brevity. A more comprehensive treatment can be found in Ref. 2.

The starting point is the exact Coulomb-gauge Hamiltonian for n quarks and antiquarks, each with mass m, in one space dimension:

$$H=-\frac{\alpha}{4}\sum_{i>j}^{n}\lambda_i\cdot\lambda_j\,|x_i-x_j|+\sum_{i=1}^{n}(m^2+k_i^2)^{1/2}\;. \tag{1}$$

Here $\lambda_i/2$ is the appropriate SU(N) charge of the ith particle, x_i is its position, and k_i is its momentum. Without loss of generality we will work in units where $\alpha=1$ in order to simplify some formulas.

Let $x_i\le x_2\le\cdots\le x_n$. If the entire system is in the singlet representation, H can be diagonalized with eigenvalues

$$E=\sum_{i=1}^{n-1}C_ir_i+\sum_{i=1}^{n}(m^2+k_i^2)^{1/2}\;,\quad r_i\equiv x_{i+1}-x_i\;. \tag{2}$$

Here C_i is the eigenvalue of the quadratic Casimir operator for the first i quarks and antiquarks. $[C_i,C_j]=0$, so each constructible sequence of irreducible representations (IR's) and each sequence of r_i and k_i gives an eigenstate of H with eigenvalue E.

The statistical mechanics of the n-particle system is now accessible. Specializing to the fixed p,T ensemble,

$$\begin{aligned}Z_p^{(n)}&\equiv e^{-\beta G}\\&=\sum_{\text{configurations}}\exp\left[-\beta\left[E+p\sum_i r_i\right]\right]\\&=z^nZ^{(n)}\;,\end{aligned} \tag{3}$$

where G is the Gibbs free energy,

$$z\equiv\int_{-\infty}^{\infty}dk\,\exp[-\beta(m^2+k^2)^{1/2}]=2mK_1(m\beta) \tag{4}$$

and

$$Z^{(n)}=\sum_{\text{IR}}\prod_{i=1}^{n-1}\int_0^\infty dr_i \exp[-\beta(C_i+p)r_i]$$
$$=\sum_{\text{IR}}\prod_{i=1}^{n-1}\frac{T}{C_i+p} . \tag{5}$$

We sum over all IR sequences which can be formed from the singlet state by successive addition of quarks and antiquarks, and which end in the singlet state. Note that the coordinates and momenta have been treated in a classical manner.

Further progress can be made with the help of a branching operator t, defined by its matrix elements,

$$\langle \text{IR2} \mid t \mid \text{IR1}\rangle=\begin{cases}1 & \text{if IR2 can be formed from IR1 by addition of a quark or antiquark ,}\\ 0 & \text{otherwise .}\end{cases} \tag{6}$$

We now may rewrite Eq. (5) in terms of a Hermitian transfer matrix Q:

$$Q\equiv\gamma^{-1}t\gamma^{-1} \quad \text{where } \gamma\equiv(C+p)^{1/2} ,$$
$$Z^{(n)}=T^{n-1}\langle 0 \mid \gamma Q^n\gamma \mid 0\rangle . \tag{7}$$

Here $\mid 0\rangle$ is the singlet IR. The eigenvectors of Q are states in "IR space" and they satisfy

$$Q \mid q_j\rangle=q_j \mid q_j\rangle\equiv q_j\gamma \mid \lambda_j\rangle ,$$
$$(C+p-q_j{}^{-1}t) \mid \lambda_j\rangle=0 . \tag{8}$$

Inserting $1=\sum_j \mid q_j\rangle\langle q_j \mid$ into Eq. (7), we obtain

$$Z^{(n)}=T^{n-1}\sum_j q_j{}^n \mid \langle 0 \mid \gamma \mid q_j\rangle \mid^2 . \tag{9}$$

In the thermodynamic limit ($n\rightarrow\infty$), only the largest eigenvalue $q_0(p)$ will contribute (if $\langle 0 \mid q_0\rangle\neq 0$), so

$$Z^{(n)}\rightarrow T^{n-1}q_0{}^np \mid \langle 0 \mid q_0\rangle \mid^2 \quad \text{as } n\rightarrow\infty , \tag{10}$$

$$\Longrightarrow \mu=-\frac{T}{n}\ln Z_p^{(n)}=-T\ln(zTq_0) \quad (n\rightarrow\infty) \tag{11}$$

is the Gibbs free energy per particle.

The average of any operator R over the ensemble (in the thermodynamic limit) is given by

$$\langle R\rangle=\frac{1}{Z_p^{(n)}}\sum_{\text{configurations}} R \exp\left[-\beta\left[E+p\sum_i r_i\right]\right] . \tag{12}$$

If R is a function of only the Casimir operator, it is easy to see that

$$\langle R(C)\rangle=\langle q_0 \mid R \mid q_0\rangle . \tag{13}$$

Later we will calculate $\langle C\rangle$, the average Casimir eigenvalue. For now we continue on with the thermodynamics of the quark-gas system.

Equation (11) implies

$$\frac{\partial\mu}{\partial p}=-T\frac{d}{dp}\ln q_0 . \tag{14}$$

But using Eqs. (3), (11), and (12),

$$\frac{\partial\mu}{\partial p}=-\frac{T}{n}\left\langle\sum_i(-\beta r_i)\right\rangle=\langle r\rangle\equiv l . \tag{15}$$

Thus $pl=\rho T$ where

$$\rho(p)=\frac{d}{d\ln p}(\ln q_0{}^{-1})=\left\langle\frac{p}{C+p}\right\rangle \tag{16}$$

can be interpreted as the effective number of degrees of freedom per quark. For example, if the quarks paired off into tightly bound mesons, ρ would be $\frac{1}{2}$.

Another simple calculation gives

$$\langle Cr\rangle+p\langle r\rangle=-\frac{1}{n}\frac{\partial}{\partial\beta}\ln Z^{(n)}=T$$
$$\Longrightarrow\langle Cr\rangle=T(1-\rho)=pl\left[\frac{1}{\rho}-1\right] . \tag{17}$$

Also we have the standard formulas for the heat capacity per quark at constant pressure and at constant length:

$$C_p=-T\mu_{TT} \quad \text{and} \quad C_l=C_p+T\frac{(\mu_{pT})^2}{\mu_{pp}} . \tag{18}$$

By Eqs. (15) and (16),

$$C_p-C_l=\left[\frac{d}{dp}(p/\rho)\right]^{-1} . \tag{19}$$

In order to exploit the above we must solve Eq. (8) for q_0 and $\mid q_0\rangle$ as functions of p. The general method of solution[2] will be applied to the problem of present interest—a system of SU(2) "I-quarks," i.e., particles which are each assigned to the color SU(2) IR which has

$$\text{dimension}=2I+1\equiv d$$
and (20)
$$\text{Casimir eigenvalue}=I(I+1)\equiv c \ .$$

The eigenvalue equation for Q can be mapped into an ordinary, homogeneous, linear differential equation as follows. Let the function $z^{m+1}\equiv e^{i(m+1)\theta}$ be associated with the IR of dimension $m+1$. C operating on the state z^{m+1} corresponds to multiplication by $\frac{1}{2}m(\frac{1}{2}m+1)$, or equivalently,

$$Ce^{i(m+1)\theta}=-\frac{1}{4}\left[\frac{d^2}{d\theta^2}+1\right]e^{i(m+1)\theta} \ . \tag{21}$$

To obtain the action of t on z^{m+1} we recall the rule for addition of spins:

$$I\otimes\frac{m}{2}=\left|\frac{m}{2}-I\right| \oplus\left|\frac{m}{2}-I+1\right|\oplus\cdots\oplus\frac{m}{2}+I \ ,$$
$$tz^{m+1}=(z^{-2I}+z^{-2I+2}+\cdots+z^{2I})z^{m+1} =\left[\frac{\sin d\theta}{\sin\theta}\right]e^{i(m+1)\theta} \ , \tag{22}$$

if $I\leq m/2$.

Thus it might seem that the eigenvalue equation, Eq. (8), can be mapped into the differential equation

$$\left[-\frac{1}{4}\left[\frac{d^2}{d\theta^2}+1\right]+p-q^{-1}\frac{\sin d\theta}{\sin\theta}\right]|\lambda\rangle=0 \ ,$$
with (23)
$$|\lambda\rangle=\sum_{m=0}^{\infty}A_m e^{i(m+1)\theta} \ .$$

However, we have cheated in writing down Eq. (23) because we used an expression for t which is valid only when acting on IR's $m\geq 2I$. How can we treat t properly without sacrificing the simplicity of Eq. (23)? The general solution to this kind of problem, discussed in Ref. 2, is to impose particular boundary conditions on the differential equation. For the I-quark problem this means we may keep Eq. (23) provided we extend the Fourier series for $|\lambda\rangle$ to all m and impose the constraint that $|\lambda\rangle$ be odd in θ. The coefficients A_m with $m<0$ are superfluous and are to be ignored in doing the thermodynamics.

Note that Eq. (23) may be written in the Schrödinger-equation form

$$\left[-\frac{1}{2}\frac{d^2}{d\theta^2}+V(\theta)\right]|\lambda\rangle=E\,|\lambda\rangle \ ,$$
where
$$V(\theta)=-\frac{2}{q}\frac{\sin d\theta}{\sin\theta}$$
and (24)
$$E=\tfrac{1}{2}-2p \ .$$

If q is very large, we may carry out standard perturbation theory in q^{-1}, using either of Eqs. (8) or (24). The appropriate periodic solution with largest eigenvalue q_0 for fixed p turns out to correspond to $p\ll 1$ with

$$q_0{}^{-1}=\sqrt{cp}\left[1-\frac{\eta}{2}\left[\frac{p}{c}\right]^{1/2} -\left[1-\frac{1}{d}-\frac{1+5\eta/4}{c}\right]\frac{p}{2} +\cdots\right], \quad p\ll 1 \tag{25}$$

where

$$\eta\equiv\begin{cases}1 & \text{if } 2I \text{ is even ,}\\ 0 & \text{if } 2I \text{ is odd .}\end{cases}$$

The corresponding eigenfunction $|\lambda_0(p)\rangle$ involves only the IR's $m\leq 4I$ to this order in p.

If q is very small we may perform a saddle-point approximation about the minima of $V(\theta)$. Near $\theta=0$, Eq. (24) becomes the harmonic-oscillator equation

$$\left[-\frac{1}{2}\frac{d^2}{d\theta^2}+\tfrac{1}{2}\omega^2\theta^2\right]|\lambda\rangle=E\,|\lambda\rangle \ , \tag{26}$$

where

$$\omega^2=\frac{8cd}{3q}$$

and

$$E=\tfrac{1}{2}-2p+\frac{2d}{q} \ .$$

We must maximize q at fixed p which corresponds to minimizing $E(q)$, but subject to the constraint that $|\lambda\rangle$ be odd in θ. So,

$$|\lambda_0\rangle \sim \theta e^{-\omega\theta^2/2} \text{ near } \theta=0$$

and

$$4d/q_0 = 4p + 3\omega - 1 \text{ for } q_0 \text{ small}$$

$$\Longrightarrow q_0{}^{-1} = \frac{p}{d}\left[1+\left[\frac{3c}{2p}\right]^{1/2}+\cdots\right], \quad p \gg c \,. \tag{27}$$

The Fourier series for $|\lambda_0\rangle$ must be odd in θ for any I, and also the series for integral I cannot contain any half-integral IR's (odd-m coefficients) because integral isospins cannot combine into half-integral isospins. Hence,

$$\begin{aligned} |\lambda_0(\theta+2\pi)\rangle &= |\lambda_0(\theta)\rangle \\ &= -|\lambda_0(-\theta)\rangle \text{ for all } I\,, \end{aligned} \tag{28}$$

$$|\lambda_0(\theta+\pi)\rangle = -|\lambda_0(\theta)\rangle \text{ for } I \text{ an integer}\,.$$

Note that

$$|\lambda\rangle \to 0 \text{ for } \theta \sim \left[\frac{2}{\omega}\right]^{1/2} \ll \pi$$

for $p \gg c$, as must be the case for an approximation of well-separated potential wells to make sense. The Fourier representation coefficients are easily obtained:

$$\begin{aligned} A_m &\propto \int_{-\infty}^{\infty} \theta e^{-\omega\theta^2/2} e^{-i(m+1)\theta} d\theta\,, \quad p \gg c \\ &\propto (m+1)\exp\left[-\frac{1}{4}\frac{(m+1)^2}{(2cp/3)^{1/2}}\right], \quad p \gg c \end{aligned} \tag{29}$$

except for the integral-I, odd-m coefficients, which are zero.

In the next section explicit low- and high-pressure expansions of thermodynamic quantities will be presented. But first we shall derive one more formula. From Eq. (13) it is clear that the relative probability of the IR m is

$$A_m{}^2(C_m+p)\,, \quad C_m \equiv \frac{m}{2}\left[\frac{m}{2}+1\right].$$

The average number of I quarks per hadron (QPH) is the same as the inverse of the probability for the singlet state $m=0$. Thus,

$$\text{QPH} = \frac{\sum_{m=0}^{\infty} A_m{}^2(C_m+p)}{A_0{}^2 p}\,. \tag{30}$$

III. THE THERMODYNAMICS

It is a straightforward matter to apply the formalism discussed in the previous section and calculate the I-quark thermodynamics. The results are given below and interpreted in the next section.

By Eqs. (16), (25), and (27), the effective number of degrees of freedom per quark is

$$\rho(p) = \begin{cases} \dfrac{1}{2}\left[1-\dfrac{\eta}{2}\left[\dfrac{p}{c}\right]^{1/2} - \left[\dfrac{2Ic}{d}-1-\eta\right]\dfrac{p}{c}+\cdots\right], & p \ll 1 \\ 1-\dfrac{1}{2}\left[\dfrac{3c}{2p}\right]^{1/2}+\cdots\,, & p \gg c \end{cases}$$

$$= \frac{p}{T}\langle r\rangle\,. \tag{31}$$

From Eqs. (11), (18), and (19),

$$C_p = \begin{cases} \dfrac{3}{2}+\dfrac{3T}{4m}+\cdots\,, & \dfrac{T}{m} \ll 1 \\ 2-\left[\dfrac{m}{T}\right]^2\left[\ln\dfrac{2T}{m}-\Gamma-1\right]+\cdots\,, & \dfrac{T}{m} \gg 1\,, \end{cases} \tag{32}$$

where Γ is Euler's constant, and

$$C_p - C_l = \begin{cases} \left[\dfrac{1}{2} - \dfrac{3\eta}{8}\right]\left[\dfrac{p}{c}\right]^{1/2} - \left[\dfrac{2Ic}{d} - 1 - \dfrac{33\eta}{32}\right]\dfrac{p}{c} + \cdots , & p \ll 1 \\ 1 - \dfrac{1}{4}\left[\dfrac{3c}{2p}\right]^{1/2} + \cdots , & p \gg c . \end{cases} \tag{33}$$

From Eq. (13), the average of the Casimir operator is given by

$$\frac{\langle C \rangle}{c} = \begin{cases} \dfrac{1}{2}\left[1 + \dfrac{\eta}{2}\left[\dfrac{p}{c}\right]^{1/2} + (2I - \eta)\dfrac{p}{c} + \cdots \right] , & p \ll 1 \\ (\tfrac{3}{8} p/c)^{1/2} + \cdots , & p \gg c . \end{cases} \tag{34}$$

Using Eq. (17),

$$\frac{\langle Cr \rangle}{\langle C \rangle \langle r \rangle} = \begin{cases} 2\dfrac{p}{c}\left\{1 + \dfrac{\eta}{2}\left[\dfrac{p}{c}\right]^{1/2} + \left[2I\left[\dfrac{2c}{d} - 1\right] - 2 - \dfrac{3\eta}{4}\right]\dfrac{p}{c}\right\} , & p \ll 1 \\ 1 + \cdots , & p \gg c . \end{cases} \tag{35}$$

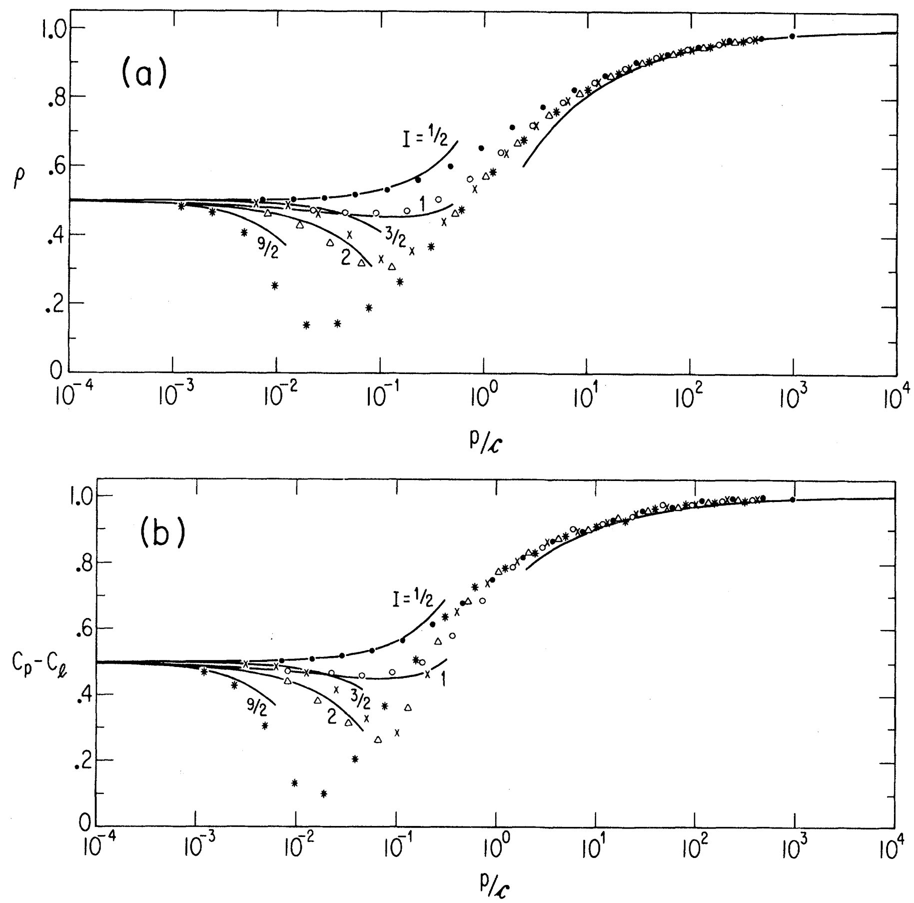

FIG. 1. Monte Carlo data for (a) ρ, (b) $C_p - C_l$, (c) $\langle C \rangle / c$, and (d) QPH vs p/c are plotted for the SU(2) representations $I = \frac{1}{2}$ (●), 1 (○), $\frac{3}{2}$ (×), 2 (△), 3 (□), and $\frac{9}{2}$ (∗) along with the corresponding asymptotic predictions of Sec. III. Note that the asymptotic behavior of the first three quantities is universal in p/c.

Finally, by Eq. (30), the average number of I quarks per hadron is

$$\mathrm{QPH}=\begin{cases} 2\left[1+\dfrac{\eta}{2}\left[\dfrac{p}{c}\right]^{1/2}+\left[\dfrac{2Ic}{d}-\dfrac{3\eta}{4}\right]\dfrac{p}{c}+\cdots\right], & p\ll 1 \\ \dfrac{\sqrt{\pi}}{4(1+\eta)}(8cp/3)^{3/4}+\cdots, & p\gg c\ . \end{cases} \tag{36}$$

Monte Carlo computations were also carried out for the IR's $I=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, 3, and $\frac{9}{2}$. Some of the results for ρ, C_p-C_I, $\langle C\rangle/c$, and QPH versus p/c are plotted in Fig. 1 along with the corresponding asymptotic predictions of Eqs. (31), (33), (34), and (36). p/c was used as the independent variable in order to point out some universal asymptotic behavior of the different IR's. However, since the low-pressure expansions given above are only valid for $p\ll 1$ (rather than $p\ll c$), the larger I Monte Carlo data diverge from the asymptotic curves at smaller values of p/c.

IV. INTERPRETATION AND DISCUSSION

It is time to offer a qualitative interpretation of the quantitative behavior of the I-quark system given in the last section. As can be seen from Eqs. (3), (5), and (12), the temperature dependence drops out of all quantities which are independent of the r_i. Thus we will concentrate on the pressure dependence of this system at fixed finite temperature.

From (5) we see that the dominant configurations of the system are those with small $\sum_i (C_i+p)r_i$ and large multiplicity or entropy. That is, G, the Gibbs

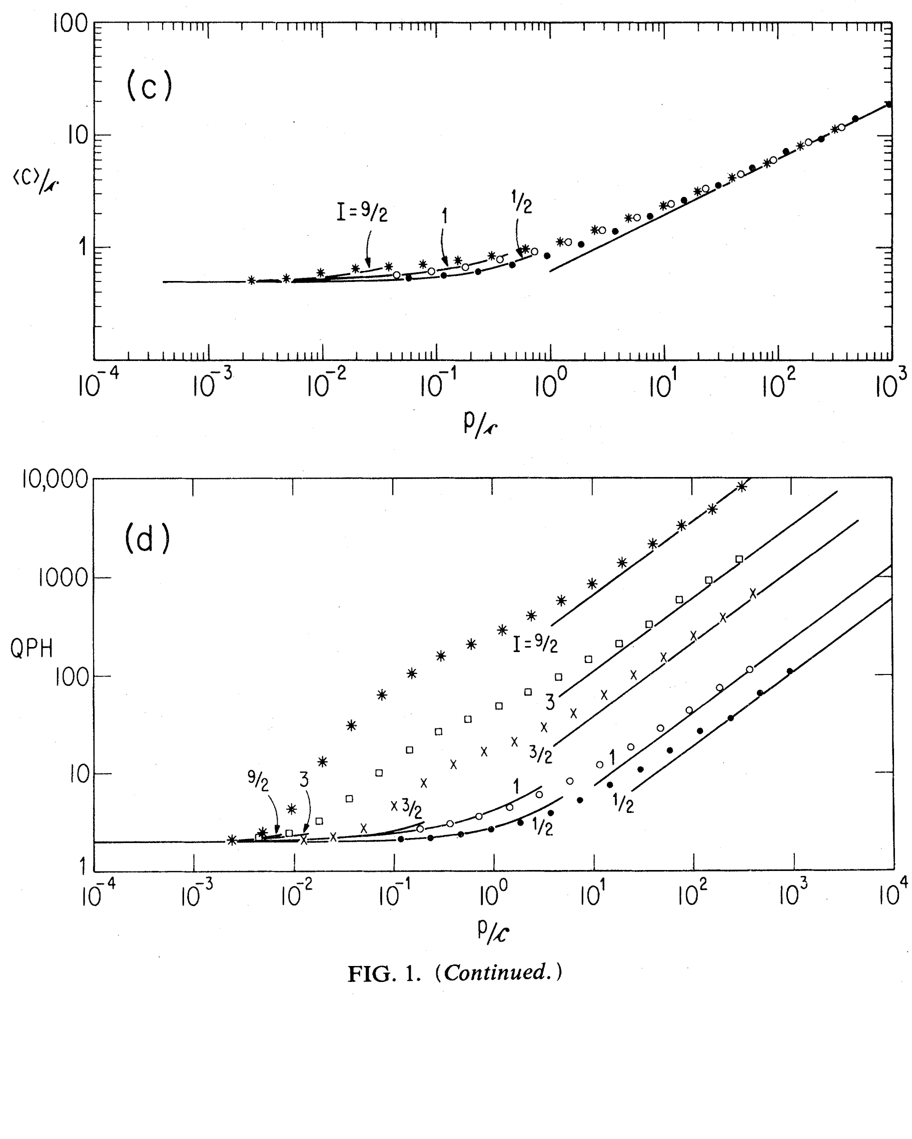

FIG. 1. (*Continued.*)

free energy, is minimized. At $p\to 0$ the system is made up of tightly bound mesons with intermeson separation $\to\infty$ and intrameson size $\to 0$. The corresponding isospin sequence is $0,I,0,I,\ldots$ with $\sum_i (C_i+p)r_i \to 0$. (A hadron made up of n I quarks is defined to be a singlet state followed by $n-1$ nonsinglet states followed by a singlet state. Thus a meson has the isospin sequence $0,I,0$.) Thus $\langle r\rangle\to\infty$, $\rho\to\frac{1}{2}$, $C_p-C_l\to\frac{1}{2}$, $\langle C\rangle/c\to\frac{1}{2}$, $\langle Cr\rangle/\langle C\rangle\langle r\rangle\to 0$, and QPH$\to 2$, in agreement with the results of Sec. III.

As the pressure is increased from zero, entropy considerations become important and other configurations begin to influence the thermodynamics. The first perturbations to the all-meson configuration come from baryon production which can occur only if I is integral. This is why only the integral-I, low-p expansions have $\sqrt{p}$ terms. More alternative configurations are available to the systems with larger I, so they tend to deviate more from the $p\to 0$ asymptotic behavior with increasing p. Since the singlet state is becoming more unlikely, QPH and $\langle C\rangle$ increase with p. This is more extreme for the higher-I systems, since as many as $2I+1$ IR's are available to the system following a nonsinglet IR, but only one is available following $|0\rangle$. Thus entropy makes the singlet state very unlikely in the large-I system.

As p is increased from zero so that the mesons get closer and can interact, the quarks not only reorient themselves into hadrons with more quarks, but these hadrons tend to be forced into excited states. The latter effect causes ρ to be greater than the average number of hadrons per quark$=(\text{QPH})^{-1}$, since the more a hadron is excited the more degrees of freedom there will be per quark. As we see from Fig. 1(a), ρ decreases more in the higher-I systems because they have more quarks per hadron, but eventually the hadrons become so highly excited that ρ begins to increase with p. In the $I=\frac{1}{2}$ system only, ρ never decreases.

As $p\to\infty$, arbitrarily large IR's are employed [as can be seen from Eq. (5)] so that $\langle C\rangle\to\infty$. Also the singlet probability (and the probability for any particular state) goes to 0, so QPH$\to\infty$. Thus the system becomes essentially one big hadron. $\rho\to 1$, however, so the quarks become independent particles $\Longrightarrow C_p-C_l\to 1$ and $\langle Cr\rangle/\langle C\rangle\langle r\rangle\to 1$. The system has passed from a hadron-gas phase to a quark-gas phase without a phase transition (in one dimension).

The larger-I systems continue to have larger QPH as $p\to\infty$, as explained above. Also, since the integral-I systems have only half as many IR's available as the half-integral systems, Eq. (30) predicts a relative factor of $\frac{1}{2}$ in their QPH as was found in Eq. (36).

Finally, an explanation for the universal behavior at large p/c is offered. When p is very large, the probability for the IR, m, should be a smooth function of m with a peak at some large m and little variation over the range $\Delta m\sim 2I$. Thus, to within a multiplicative constant we can rewrite Eq. (5) as

$$Z^{(n)}\sim\sum_{\text{IR}}{}'\prod_{i=1}^{n-1}\frac{T}{C_i/c+p/c}\,,\quad p\gg c \tag{37}$$

where now we allow only IR's whose isospins are approximate integral multiples of $\sqrt{c}$. For large enough p, $Z^{(n)}$ depends only on p/c. Thus, ρ, C_p-C_l, and many other thermodynamic functions depend only on p/c as $p\to\infty$.

It is hoped that this study of the SU(2) I-quark-gas system will be of some use in more physically realistic inquiries.

ACKNOWLEDGMENTS

Help from Y. Nambu was greatly appreciated. The author would also like to thank Sumit Das for a useful discussion. This work was supported in part by the National Science Foundation, Grant No. PHY79-23669, and by the Department of Energy, Grant No. AE-AC02-81ER 10957A.

[1]Y. Nambu and B. Bambah, Phys. Rev. D 26, 2871 (1982).

[2]Y. Nambu, B. Bambah, and M. Gross, Phys. Rev. D 26, 2875 (1982).