Electron-electron scattering at high energies and fixed angle

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In this paper, we sum all (leading and nonleading) logarithms in electron-electron scattering at high energy and fixed angle. By Pauli statistics, the *e-e* scattering amplitude is equal to the difference of two terms related to each other by interchanging the two incoming particles. Each of these terms is in the form of a product of an exponential function, the wave-function renormalization constants, and a scaling function. While the exponent of this exponential function is complicated, it is related to that of the vertex function. Thus, each of these terms is in the form of a product of the two-electron vertex functions, a scaling function, and an exponential function which is explicitly given. The method we use is obtained by modifying slightly the one of Yennie and collaborators, which proves to be a powerful tool to deal with scattering problems of large momentum transfers.

I. INTRODUCTION

We present here a study of the electron-electron scattering amplitude in QED at high energy and fixed angle, or $s\to\infty$ with t/s fixed. We have succeeded in summing all (leading as well as nonleading) logarithms of this amplitude. More precisely, let p_1 and p_2 (p_1' and p_2') be the incoming (outgoing) momenta of the electrons and m the electron mass; then the electron-electron scattering amplitude is asymptotically equal to

$$\frac{1}{t}[\bar{u}_1'\Gamma_\mu(t)u_1][\bar{u}_2'\Gamma^\mu(t)u_2]e^{K(t)}H(\theta,e(t))$$

$$-\text{the preceding term with particles 1 and 2 interchanged}\ . \qquad (1.1)$$

In (1.1) $\Gamma_\mu(t)$ is the vertex function for the electron, u_1 is the spinor wave function for electron 1, and similarly for u_1', u_2, and u_2'. Also,

$$K(t)=i\int\frac{d^4q}{(2\pi)^4}\frac{e^2(q^2)}{q^2-\lambda^2}\left[\frac{2p_1'-q}{2p_1'\cdot q-q^2}-\frac{2p_1+q}{2p_1\cdot q+q^2}\right]\cdot\left[\frac{2p_2'+q}{2p_2'\cdot q+q^2}-\frac{2p_2-q}{2p_2\cdot q-q^2}\right], \qquad (1.2)$$

where $e(t)$ is the running coupling constant evaluated at t and λ is the photon mass introduced for regulating infrared divergences. The function H is a perturbation series of the running coupling constant $e(t)$, with the coefficient scaling functions which depend on θ but not ω, its lowest-order term being simply $e^2(t)$. As usual, we have defined

$$s=4(\omega^2+m^2)\ ,$$
$$t=-2\omega^2(1-\cos\theta)\ ,$$
$$u=-2\omega^2(1+\cos\theta)\ ,$$

where ω is the magnitude of the incident momentum and θ is the scattering angle, both being measured in the center-of-mass system. The second term in (1.1) is obtained from the first term in (1.1) by making the change of $u_1\leftrightarrow u_2$, $t\to u$, and $\theta\to\pi-\theta$. The two terms in (1.1) together preserve the Pauli statistics. The asymptotic form of the vertex function, with all logarithmic factors included, has been given by a number of authors.[1-3] This asymptotic form is also derived naturally in our calculation of the *e-e* scattering amplitude.

It is appropriate to compare (1.1) with previous works in the literature. The electron-electron scattering amplitude in QED (Refs. 4 and 5) and the quark-quark scattering amplitudes in QCD (Refs. 6 and 7) in the limit $s\to\infty$ with θ fixed have been studied before. However, all of these works are restricted to summing leading terms. Certain processes in QED have been calculated to the same degree of accuracy as the present work. For example, we have shown[8] that the photon-photon scattering amplitude approximately scales; i.e., it is a function of θ and $e(t)$ only. It has been surmised that the electron-electron scattering amplitude has a more complicated asymptotic form. From (1.1), we see that the complications are entirely contained in the form factors and the function $e^{K(t)}$. Equation (1.1) has some resemblance to the Wu-Yang hypothesis,[9] which presumably holds for the scattering of colorless bound states of quarks, not that of quarks or electrons. We shall discuss this in a future paper.

Finally, we give one word of warning. Equation (1.1) includes all logarithmic factors, but it does not take into account the terms which are of the order of s^{-1} (logarithmic factors ignored). Although such terms are small individually, they may add up to be larger than the expression (1.1).

II. MULTIPHOTON EXCHANGE

We shall begin by studying the diagrams of multiphoton exchange. These diagrams are of course interesting in their own right. But the reason for us to study them first is that by so doing we may illustrate clearly the method we are going to use. Also, it turns out that these diagrams give the important factor e^K in (1.1).

One-photon exchange. The lowest-order diagrams are the two diagrams in Fig. 1. The amplitude corresponding to diagram 1(a) is equal to $\mathscr{M}^{(0)}$, where

$$\mathscr{M}^{(0)}=e^2\frac{(\bar{u}'_1\gamma_\mu u_1)(\bar{u}'_2\gamma^\mu u_2)}{t} . \tag{2.1}$$

The amplitude corresponding to diagram 1(b) is equal to

$$-e^2\frac{(\bar{u}'_2\gamma_\mu u_1)(\bar{u}'_1\gamma^\mu u_2)}{u} . \tag{2.2}$$

No approximation is made in (2.1) and (2.2). Notice that the expression in (2.2) is obtained from $\mathscr{M}^{(0)}$ by exchanging particle 1 with particle 2 and inserting a minus sign. Notice also that both the numerators and the denominators of these expressions are of the order of ω^2, as the components of a spinor is of the order of $\omega^{1/2}$. Thus, both expressions approach limits which depend on θ but are independent of ω as $\omega\to\infty$ with θ fixed.

Two-photon exchange. There are four diagrams of two-photon exchange. Two of them are illustrated in Fig. 2. The other two can be obtained from those in Fig. 2 by exchanging particle 1′ with particle 2′. We shall calculate all the logarithmic factors from these diagrams in the limit $\omega\to\infty$ with θ fixed.

It is appropriate to make a brief outline here of the method we shall use. Since the electron-electron scattering amplitude is dimensionless, it must be a function of the dimensionless variables θ, m/ω, and λ/ω. Therefore, the limit $\omega\to\infty$ with θ fixed is the same as $m\to 0$ and $\lambda\to 0$, with ω and θ fixed. We shall therefore study the scattering amplitude in the latter limit.

It is well known that if we set $\lambda=m=0$, there are two momentum regions in which the Feynman integrals may be divergent[10,11]: (i) The neighborhood of $q=0$, where q is the momentum of a virtual photon. This happens even if $m\neq 0$. For example, consider diagram 2(a). In the neighborhood of $q_1=0$, the propagator of photon 1 blows up like q_1^{-2}, while each of the electron propagators blows up like q_1^{-1}. Thus, d^4q_1 times these propagators diverges logarithmically, i.e., like d^4q_1/q_1^4. We shall refer to such a divergence as a soft divergence. (ii) The neighborhood of $q_\mu=xp_\mu$, where q is the momentum of any virtual line and p is any of the external momentum. (There is a restriction on x: $|x|\leq 1$, but this is not important for our present purpose.) Consider again diagram 2(a). Let $\vec{p}_1$ be in the positive z direction. Then if $\lambda=m=0$, the integrand of the Feynman integral has the following factors in the denominator:

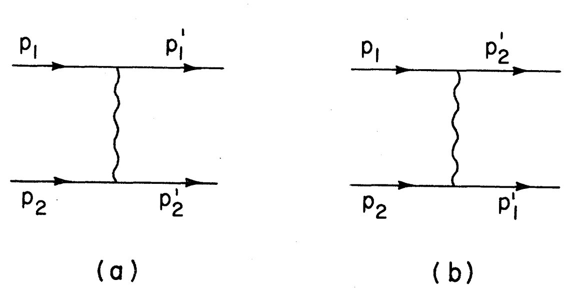

FIG. 1. The two lowest-order diagrams for *e-e* scattering.

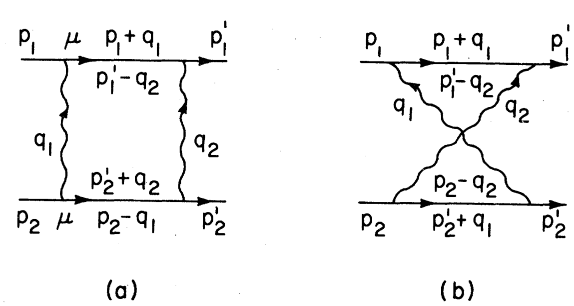

FIG. 2. Two of the fourth-order diagrams for *e-e* scattering.

$$q_1{}^2=q_{1+}q_{1-}-\vec{q}_{1\perp}{}^2=2\omega x q_{1-}-q_{1\perp}{}^2$$

and

$$(p_1+q_1)^2=2\omega(1+x)q_{1-}-q_{1\perp}{}^2 ,$$

where $q_\pm=q_0\pm q_3$. We have also defined $q_+\equiv 2\omega x$. Because of the vanishing of the two expressions above at $q_{1-}=q_{1\perp}{}^2=0$, the Feynman integral may diverge. Indeed, let $q_{1-}=O(\epsilon)$ and $q_{1\perp}{}^2=O(\epsilon)$, then d^4q is $O(\epsilon^2)$. Thus, d^4q divided by the product of these expressions is logarithmically divergent. We shall refer to such a divergence as a mass divergence.

It is important to observe that the mass divergences for the diagrams of multiphoton exchange cancel one another. Consider, for example, $q_1=xp_1$. Then the numerator for the first electron has the factors

$$(\not p_1+\not q_1)\gamma_\mu u(p_1)=2(p_1+q_1)_\mu u(p_1)$$
$$=2\left[\frac{1}{x}+1\right]q_{1\mu}u(p_1) . \qquad (2.3)$$

Equation (2.3) is obtained because $\not p_1 u(p_1)=\not q_1 u(p_1)=0$ as $m=0$. This means that the polarization of photon 1 is parallel to its momentum. Therefore, by gauge invariance, the sum of the integrands of the two diagrams in Fig. 2 vanishes at $q_1=xp_1$. Consequently, the sum of the Feynman integrals corresponding to the two diagrams is convergent in the neighborhood of $q_1=xp_1$. By these arguments, we easily prove that this sum has no mass divergences. More generally, we may show that the sum of multiphoton exchange amplitudes has no mass divergences.

It remains to handle the soft divergences. This has been done by Yennie and collaborators.[11] They rearrange each virtual-photon propagator according to

$$\frac{g_{\mu\nu}}{q^2}=\frac{G_{\mu\nu}}{q^2}+\frac{K_{\mu\nu}}{q^2} , \qquad (2.4)$$

where

$$G_{\mu\nu}=g_{\mu\nu}-bq_\mu q_\nu , \qquad (2.5)$$

$$K_{\mu\nu}=bq_\mu q_\nu , \qquad (2.6)$$

q is the photon momentum, and b is given by

$$b(p_i,p_f,q)=\frac{4(p_f-q)\cdot(p_i-q)}{(2q\cdot p_f-q^2)(2q\cdot p_i-q^2)} . \qquad (2.7)$$

(The numerator of our b function differs from that of Grammer and Yennie. However, since the difference vanishes as $q\rightarrow 0$, either definition will do.) We shall explain below how to choose p_i and p_f for each virtual photon. The point in rearranging the propagator according to (2.4) is that the term $G_{\mu\nu}/q^2$ gives no soft divergence at $q=0$, while the contribution of the term $K_{\mu\nu}/q^2$ can be calculated in closed form.

According to Grammer and Yennie,[11] one may, since $q_1+q_2=\Delta$, multiply the integrand of the Feynman integrals for the diagrams in Fig. 2 by

$$\frac{(q_1+q_2)^2}{\Delta^2}=\frac{q_1{}^2+q_2{}^2+2q_1\cdot q_2}{\Delta^2} . \qquad (2.8)$$

Let us first discuss the term $q_1{}^2/\Delta^2$ in (2.8). For this term the Feynman integrals for both diagrams in Fig. 2 have no infrared divergence at $q_1=0$. Thus, we only need to handle the infrared divergence at $q_2=0$. We break up the photon propagator for photon 2 according to (2.4), with p_i and p_f defined according to the positions of its vertices in relation to the positions of the vertices for photon 1. More specifically, we define p_i to be p_2 $(-p_2')$, if the lower end of photon 2 is attached to the incoming (outgoing) side relative to photon 1, and p_f to be p_1' $(-p_1)$, if the upper end of photon 2 is attached to the outgoing (incoming) side relative to photon 1. Thus, for diagram 2(a), we define

$$p_i=-p_2' ,\ \ p_f=p_1' ,$$

and for diagram 2(b), we define

$$p_i=p_2 ,\ \ p_f=p_1' .$$

If photon 2 is a K photon [i.e., if we use the second term in (2.4) as the propagator for photon 2], the amplitude for diagram 2(a) is

$$\mathscr{M}^{(0)}k(-p_2',p_1') , \qquad (2.9)$$

where $\mathscr{M}^{(0)}$ is given by (2.1) and

$$k(-p_2',p_1')\equiv ie^2\int\frac{d^4q}{(2\pi)^4}\frac{b(-p_2',p_1',q)}{q^2-\lambda^2} , \qquad (2.10)$$

where we have denoted q_2 by q. In deriving (2.10) we have manipulated the numerator factors of the upper electron line by writing $\not q_2$ as

$$\not q_2=(\not p_1'-m)-(\not p_1'-\not q_2-m) .$$

The term $(\not p_1'-m)$ after operating on $\bar u(p_1')$ vanishes. The term $(\not p_1'-\not q_2-m)$ cancels the propagator for the upper virtual electron in diagram 2(a); similarly for the lower electron line. In the same way, the term $q_1{}^2/\Delta^2$ for diagram 2(b), with photon 2 being a K photon, is

$$-\mathscr{M}^{(0)}k(p_2,p_1') . \qquad (2.11)$$

Adding to (2.9) and (2.11) the amplitudes from the term $q_2{}^2/\Delta^2$ in (2.8), with photon 1 a K photon, we get

$$\mathscr{M}^{(0)}k\ , \tag{2.12}$$

where

$$k=k(-p_2',p_1')-k(p_2,p_1')+k(p_2,-p_1) -k(-p_2',-p_1)\ . \tag{2.13}$$

More specifically,

$$k=4i\int\frac{d^4q}{(2\pi)^4}\frac{e^2}{q^2-\lambda^2}\left[\frac{(p_2'+q)\cdot(p_1'-q)}{(2q\cdot p_2'+q^2)(2q\cdot p_1'-q^2)}-\frac{(p_2-q)\cdot(p_1'-q)}{(2q\cdot p_2-q^2)(2q\cdot p_1'-q^2)} +\frac{(p_2-q)\cdot(p_1+q)}{(2q\cdot p_2-q^2)(2q\cdot p_1+q^2)}-\frac{(p_2'+q)\cdot(p_1+q)}{(2q\cdot p_2'+q^2)(2q\cdot p_1+q^2)}\right]. \tag{2.14}$$

The sum of the amplitudes corresponding to diagrams 2(a) and 2(b) is therefore equal to

$$k\mathscr{M}^{(0)}+\mathscr{M}_s^{(1)}+\mathscr{M}_c^{(1)}\ , \tag{2.15}$$

where $\mathscr{M}_s^{(1)}$ is equal to the sum of two terms: the amplitude corresponding to the two diagrams in Fig. 2, with the integrand multiplied by $q_1{}^2/\Delta^2$ and with photon 1 a g photon (i.e., its propagator is $g_{\mu\nu}/q^2$) and photon 2 a G photon (i.e., its propagator is $G_{\mu\nu}/q^2$), and the amplitude obtained from it by interchanging the roles of photon 1 and photon 2. Also, $\mathscr{M}_c^{(1)}$ is equal to the Feynman amplitude corresponding to the two diagrams in Fig. 2 with the integrand multiplied by $2q_1\cdot q_2/\Delta^2$ and with both photons g photons.

As was discussed by Grammer and Yennie, $\mathscr{M}_s^{(1)}$ and $\mathscr{M}_c^{(1)}$ have no soft divergences. We shall further show that they have no mass divergences either. To begin, let us recall that we have argued that the sum of two-photon exchange amplitudes has no mass divergences. Exactly the same arguments may be used to show that $\mathscr{M}_c^{(1)}$ has no mass divergences. Next we show that k has no mass divergences. For example, if $m=\lambda=0$, the third and the fourth terms in the brackets of (2.14) are both divergent at $q=xp_1$. However, it is easy to verify that these two terms cancel each other. Thus, the integral in (2.14) is convergent at $q=p_1x$. Similar considerations apply if q is parallel to other external momenta. Thus, k has no mass divergences. It then follows that $\mathscr{M}_s^{(1)}$, which, by (2.15), can be expressed by quantities with no mass divergences, is free of mass divergences as well.

Since $\mathscr{M}_c^{(1)}$ and $\mathscr{M}_s^{(1)}$ have neither soft divergences nor mass divergences, they are finite at $m=\lambda=0$, or at the limit $s\to\infty$ with θ fixed. Thus, the logarithmic factor of s is contained entirely in k.

As will be seen shortly, it is convenient to write $\mathscr{M}_c^{(1)}$ as

$$\mathscr{M}_c^{(1)}=\int d^4x e^{ix\cdot\Delta}M_c^{(1)}(x)\ , \tag{2.16}$$

where

$$M_c^{(1)}(x)=\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}e^{-i(q_1+q_2)\cdot x}\frac{2q_1\cdot q_2}{\Delta^2}$$
$$\times\ \text{integrand from Feynman rules corresponding to the two diagrams in Fig. 2}\ . \tag{2.17}$$

[The expression (2.16) is easily shown to be valid if we substitute (2.17) into (2.16) and carry out the integration over x.] We note from Fig. 2 that, as far as (2.17) is concerned, $q_1+q_2\neq\Delta$. Thus, the integrand from the Feynman rules corresponding to the two diagrams in Fig. 2 may be defined in two different ways, as the momenta of the virtual electron lines may be defined either in terms of q_1 or in terms of q_2 (see the figure). A symmetrical form for $M_c^{(1)}$ is obtained if we express the integrand in (2.17) as a sum of two terms, each of which corresponds to one of the definitions mentioned above, and divide by 2.

The two diagrams obtained from the ones in Fig. 2 by interchanging particle 1′ and particle 2′ can be treated in exactly the same way, and we shall not elaborate on such a treatment.

Exchange of three or more photons. The above treatment can be easily extended to the diagrams of n-photon exchanges. Let the momenta of the photons be $q_1,q_2,\dots,q_n$, respectively, and let us make the photons distinguishable. In other words, the same Feynman diagram with different labeling of the photon momenta is considered to be a different diagram. Thus, we overcount by a factor $n!$ and the resulting amplitude must be divided by $n!$

We multiply the integrands of the Feynman integrals by

$$1=\frac{q_1{}^2+q_2{}^2+\cdots+q_n{}^2+2q_1\cdot q_2+2q_1\cdot q_3+\cdots+2q_{n-1}\cdot q_n}{\Delta^2}\ . \tag{2.18}$$

For the term $q_1{}^2/\Delta^2$ above, there is no divergence at $q_1=0$. Thus, we only need to rearrange the propagators of photons $2,3,\ldots,n$. We define the p_1 leg as the set of all of the upper electron lines on the incoming side relative to photon 1. The p_1' leg is similarly defined for the outgoing side relative to photon 1. The p_2 and p_2' legs are similarly defined for the lower electron lines. The momentum p_i in (2.7) is defined to be p_2 $(-p_2')$ if the lower end of the photon in consideration is attached to the p_2 (p_2') leg, and the momentum p_f in (2.7) is defined to be $-p_1$ (p_1') if the upper end of the photon in consideration is attached to the p_1 (p_1') leg; similarly for the terms $q_2{}^2/\Delta^2$, etc. For the term $2q_1\cdot q_2$, we split it into the sum of two terms $q_1\cdot q_2+q_1\cdot q_2$, and for the first (second) term, we use photon 1 (photon 2) to separate the p leg from the p' leg. The Feynman integral is convergent both at $q_1=0$ and at $q_2=0$, and we need to rearrange the propagators of photons $3,4,\ldots,n$ only.

Consider first the term $q_1{}^2/\Delta^2$ in (2.18). We keep photon 1 as a g photon and split each of the other $(n-1)$ photons into a K photon plus a G photon. Thus, each diagram is expanded into a sum of diagrams with each of the $(n-1)$ photons being either a K photon or a G photon. Each of such diagrams occurs $(n-1)!/(n_K!n_G!)$ equivalent ways which differ from one another only in the labeling of the momenta, where n_K (n_G) is the number of K (G) photons in the diagram, with $n_K+n_G=n-1$. As we add up the diagrams of all different ways of inserting the K photons, but with the g photon and the n_G G photons in a fixed order, the contribution of the K photons merely gives a factor k^{n_K}. Thus, the sum of all diagrams with n_K K and n_G G photons is $k^{n_K}\mathcal{M}_{n_G}$, where $\mathcal{M}_{n_G}$ is the sum of amplitudes corresponding to all diagrams of (n_G+1)-photon exchange with a g photon and n_G G photons, and with the integrand of the Feynman integral multiplied by $q_1{}^2/\Delta^2$, where q_1 is the momentum of the g photon. Together with the combinatoric factors given above, we get

$$\frac{1}{n!}\frac{(n-1)!}{n_K!n_G!}k^{n_K}\mathcal{M}_{n_G}\ . \tag{2.19}$$

The amplitude in (2.19) must be multiplied by n to take care of the fact that there are n-squared terms in (2.18). Thus, summing the squared terms in (2.18) over n and n_K, we get

$$\sum_{n_K}\sum_{n_G}k^{n_K}\mathcal{M}_{n_G}/(n_K!n_G!)=e^k\mathcal{M}_s\ , \tag{2.20}$$

where

$$\mathcal{M}_s\equiv\sum_{n=0}^{\infty}\mathcal{M}_n/n!$$

We may similarly treat the crossed terms in (2.18), which are $n(n-1)$ in number. For the term $q_1\cdot q_2$, we keep photons 1 and 2 as g photons and expand each of the others into a K photon and a G photon. Each of the diagrams in such an expansion occurs $(n-2)!/(n_K!n_G!)$ equivalent ways, where $n_K+n_G=(n-2)$. Thus, the product of all combinatoric factors is

$$\frac{(n-2)!}{n_K!n_G!}\frac{n(n-1)}{n!}=\frac{1}{n_K!n_G!}\ ,$$

just as before. There is only one new feature: the integral over the momentum of a K photon cannot be factored out. This is because the momenta of the photons are not independent but are related by $\sum_{i=1}^{n}q_i=\Delta$. In the case of the square term $q_1{}^2/\Delta^2$, this complication is avoided as $q_1{}^2$ cancels the propagator for photon 1. Therefore, q_1 does not occur in the integrand explicitly. Thus, the integration over q_1 simply eliminates the δ function $\delta^{(4)}(\Delta-\sum_1^n q_i)$, and the integral over the momentum of a K photon can be factored out. In the present case, we may introduce the integral representation for the δ function:

$$(2\pi)^4\delta^{(4)}\left[\Delta-\sum_1^n q_i\right]=\int d^4x\,\exp\left[ix\cdot\left[\Delta-\sum_1^n q_i\right]\right]\ . \tag{2.21}$$

Then we may integrate over $q_1, q_2,\ldots,q_n$ without restrictions. Thus, summing the crossed terms in (2.18) over all n_K and n_G gives

$$\int d^4x e^{ix\cdot\Delta}M_c(x)e^{k(x)}\ , \tag{2.22a}$$

where

$$M_c(x)=\sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n+2}\frac{d^4q_i}{(2\pi)^4}\exp\left[-ix\cdot\sum_{1}^{n+2}q_i\right]\frac{q\cdot q'}{\Delta^2}$$

$$\times(\text{ integrand from all Feynman diagrams with two } g \text{ photons of momenta } q \text{ and } q' \text{ and } n\ G \text{ photons}) \tag{2.22b}$$

and

$$k(x)=\int\frac{d^4q}{(2\pi)^4}e^{-ix\cdot q}\times[\text{the integrand of (2.14)}]\ .$$

We note here that

$$k(0)=k\ .$$

Adding (2.20) and (2.22a), we obtain the sum of multiphoton exchange amplitudes as

$$e^kJ\ , \tag{2.23}$$

where

$$J=\mathcal{M}_s+\int d^4xe^{ix\cdot\Delta}M_c(x)\exp[k(x)-k]\ . \tag{2.24}$$

We note that $k(x)-k(0)$ is finite at $m=\lambda=0$. This is because the integral corresponding to this difference has no soft divergence at $q=0$ or anywhere else. Therefore, J is finite at $m=\lambda=0$, or is equal to a function $J(\theta)$ in the limit of $s\to\infty$ with θ fixed. Therefore, the energy dependence of the sum of multiphoton exchange amplitudes is entirely contained in the factor e^k in the limit $s\to\infty$ with θ fixed. In this limit the asymptotic form of k can be derived in a straightforward way. This derivation is given in Appendix A. We get

$$k\simeq\frac{\alpha}{\pi}\ln s\left[\ln\frac{2}{1+\cos\theta}-i\pi\right]\ ,\quad s\to\infty,\ \theta \text{ fixed}\ . \tag{2.25}$$

In summary, the multiphoton exchange amplitude for electron-electron scattering is equal to a sum of two terms, one of which is asymptotically equal to

$$s^{a(\theta)}J(\theta),\ s\to\infty,\ \theta \text{ fixed}\ , \tag{2.26}$$

where

$$a(\theta)\equiv\frac{\alpha}{\pi}\ln\frac{2}{1+\cos\theta}-i\alpha\ . \tag{2.27}$$

The other term is equal to the negative of (2.26) with particles 1 and 2 interchanged.

We make the following remarks.

(i) The multiphoton exchange amplitude is related to the relativistic potential scattering amplitude. More precisely, the former reduces to the latter if the mass of one of the incident particles is infinite. It is therefore interesting to compare (2.26) with the asymptotic scattering amplitude for the Klein-Gordon equation.[12] We note that the latter is in the form of (2.26) also. There is, however, a major difference. The exponent of s in the case of the Klein-Gordon equation is a purely imaginary number and is related to the infinite phase shift in Coulomb scattering, while $a(\theta)$ given by (2.27) has a real part.

(ii) The real part of $a(\theta)$ is positive. Thus, for electron-electron scattering the sum of multiphoton exchange amplitudes is larger than the one-photon exchange amplitude in the limit $s\to\infty$ with θ fixed. However, for electron-positron scattering, the exponent of s is the negative of $a(\theta)$, and the sum of multiphoton exchange amplitudes is smaller than the one-photon exchange amplitude in the limit $s\to\infty$ with θ fixed.

(iii) It is interesting to compare (2.26) with the Regge-pole term $\beta(t)s^{\alpha(t)}$, which is valid for certain amplitudes in the limit $s\to\infty$ with t fixed.

Photon self-energy. In this subsection we make a slight extension of the result in the preceding subsection. We consider the multiphoton exchange diagrams with a photon self-energy part inserted in each of the photon lines. The expression for this scattering amplitude can be obtained from the scattering amplitude of multiphoton exchange by making the replacement

$$\frac{e^2}{q^2}\to\frac{e^2(q^2)}{q^2}$$

for each of the photon propagators, where $e(q^2)$ is the renormalized charge with the subtraction point taken at q^2. All the arguments in deriving (2.23) go through. Thus, k is modified into K given by (1.2), and e^2 in the perturbation series for J is replaced by $e^2(q^2)$ for the virtual photon of momentum q.

III. ARBITRARY DIAGRAMS

It is not difficult to extend the method given in the preceding section to treat an arbitrary diagram. There are three kinds of divergences[13] which give rise to logarithmic factors of s: (1) mass divergences, (2) soft divergences, (3) ultraviolet divergences. We shall treat them in succession.

Let us first note that a virtual-photon line must belong to one of the following four classes: (a) it connects a point on the path of electron 1 with a point on the path of electron 2; (b) it connects either two points on the path of electron 1 or those of electron 2; (c) it connects a point on the path of either electron 1 or electron 2 with a point on an electron loop; or (d) it connects two points on the same or different electron loops. As we have discussed in the preceding section, photons of class (a) give infrared divergences, but not mass divergences. A photon of class (b) may give both infrared divergences and mass divergences. As we shall show, a photon of class (c) or class (d) gives no divergences whatsoever.

To see this, consider for definiteness the diagrams in Fig. 3. We shall show that the photons in this diagram give no divergences as we set $m=\lambda=0$. To prove this, let us concentrate on the possible divergences caused by the virtual photon of momentum q in the diagrams of Fig. 3. At $q=0$, the amplitude corresponding to the diagrams in Fig. 4 is finite provided that the other three external momenta in Fig. 4 are all off-shell. Since the integral $d^4q q^{-3}$ is convergent, the diagrams in Fig. 3 have no infrared divergences at $q=0$. Also, if $q \| p_1$, we find that for the same reason as in the remark following (2.3), the polarization of the photon of momentum q is parallel to q. By gauge invariance, the photon-photon scattering amplitude of Fig. 4 vanishes. Thus, the sum of scattering amplitudes corresponding to the diagrams in Fig. 3 has no mass divergences. This argument can be used to prove that a photon attached to an internal loop does not cause any divergence. As a consequence of this result, the amplitude corresponding to the diagrams in Fig. 3 is finite

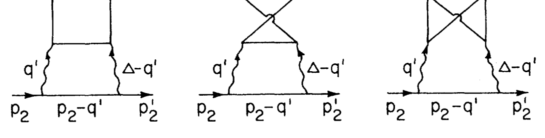

FIG. 3. Some diagrams for e-e scattering with an electron loop.

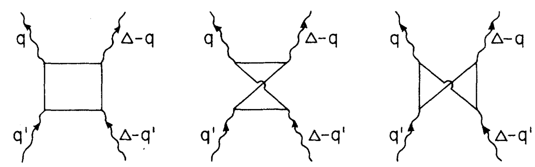

FIG. 4. The diagrams of γ-γ scattering contained in the diagrams in Fig. 3.

at $m=\lambda=0$. Hence, it is equal to a function of θ in the limit $s\to\infty$ with θ fixed. We remark that the logarithmic factors of the foward (or near forward) scattering amplitude come from electron loops only. It is interesting that the logarithmic factors of the nonforward scattering amplitude come from entirely different sources.

Mass divergences. From the above discussions only the photons of class (b) give mass divergences. We shall choose for such photons a gauge in which $N_{\mu\nu}$ is used in place of $g_{\mu\nu}$, where

$$N_{\mu\nu}=g_{\mu\nu}-\frac{n_\mu q_\nu+n_\nu q_\mu}{(n\cdot q)} . \tag{3.1}$$

In (3.1) n is a fixed vector of unit magnitude and q is the momentum of the photon in question. In this gauge the numerator has a zero at the surface of the phase space where mass divergences may occur. To see this, let us concentrate on photon 1 in Fig. 2. We shall denote q_1 by q for short. At the surface $q=xp_1$, photon 1 is of the polarization q_μ, as was deduced from (2.3). Now we have

$$N_{\mu\nu}q^\mu=-\frac{n_\nu q^2}{n\cdot q}=0 , \tag{3.2}$$

as $q^2=p_1{}^2=0$. Thus, the numerator has a zero at $q=xp_1$ if the gauge (3.1) is used. This argument obviously applies whenever the photon is attached to an external line, and most mass divergences are avoided this way. The only photons which will give mass divergences in the gauge (3.1) are the ones belonging to a self-energy part on the external electron line. In this case the zero from the numerator is not enough to make the integral for the wave-function renormalization constant to converge. Thus, the logarithmic factors of s from mass divergences are relegated to the wave functions. The wave-function renormalization constant for an electron of momentum p with the photons in the gauge (3.1) is a Dirac matrix which depends on both n and p. We shall show in Appendix B that in the limit of $|\vec{p}|\to\infty$, this matrix is a unity matrix or a scalar function. As we shall see, we shall be able to determine the form of this function.

Soft divergences. To factor out the logarithms from soft divergences, one needs to split the photons of class (a) and class (b) into a G photon plus a K photon. As we have mentioned, the K photon is defined only if a way to separate the p leg from the p' leg is given. This separation can be made at any vertex on the path of the external electron provided that the photon attached to this vertex of separation does not give soft divergences. If a diagram has one or more photons of class (c), then we may use any one of them for the separation. If a diagram has no photons of this class, then it must have photons of class (a). The treatment of such a diagram is exactly the same as that of the multiphoton exchange diagrams.

In all of these cases, the contribution of the K photons can be calculated exactly. Just as in the case of multiphoton exchange, these contributions are summed into an exponential function, with the exponent simply equal to a sum of terms in the form of (2.10). Each of these terms corresponds to a joining of two legs. For example, the four terms in (2.14) correspond to the joining of the p_2' leg and the p_1' leg, that of the p_2 leg with the p_1' leg, that of the p_2 leg with the p_1 leg, and that of the p_2' leg with the p_1 leg, respectively. The sign of a term is chosen to be plus if the two lines joined together are both incoming or both outgoing. Otherwise the sign is minus. For the electron-electron scattering amplitude with all diagrams included, there are four more terms in the exponent corresponding to the joining of the following four pairs of legs: the p_1 leg with the p_1 leg, the p_1 leg with the p_1' leg, the p_2 leg with the p_2 leg, and the p_2 leg with the p_2' leg. As we have mentioned, for such photons of class (b), we use the gauge defined in (3.1). Consequently, the definition of K and G corresponding to such a photon is no longer given by (2.4)—(2.7) but by

$$N_{\mu\nu}=G_{\mu\nu}^{(n)}+K_{\mu\nu}^{(n)}\ , \tag{3.3}$$

where

$$K_{\mu\nu}^{(n)}=\begin{cases} q_\mu q_\nu \dfrac{N_{\rho\sigma}(2p_i-q)^\rho(2p_f-q)^\sigma}{(2q\cdot p_f-q^2)(2q\cdot p_i-q^2)}, & p_i\neq p_f \\ \frac{1}{2}q_\mu q_\nu\left[\dfrac{N_{\rho\sigma}(2p-q)^\rho(2p-q)^\sigma}{(2q\cdot p-q^2)^2}+\text{preceding term with } q\to -q\right], & p_i=p_f=p\ . \end{cases} \tag{3.4}$$

The contribution of such K photons can be calculated easily. This calculation is exactly the same as the one given explicitly by Yennie and collaborators[11] and will not be repeated here. The result is that the K photons connecting two points on the path of electron 1 contribute to the exponent a term

$$\frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\frac{e^2N_{\mu\nu}}{q^2-\lambda^2}\left[\frac{(2p_1-q)^\mu}{2p_1\cdot q-q^2}-\frac{(2p_1'-q)^\mu}{2p_1'\cdot q-q^2}\right]\times\left[\frac{(2p_1-q)^\nu}{2p_1\cdot q-q^2}-\frac{(2p_1'-q)^\nu}{2p_1'\cdot q-q^2}\right]. \tag{3.5}$$

Substituting the explicit expression (3.1) for $N_{\mu\nu}$ into (3.5), we find that the last two terms in the right-hand side of (3.1) give zero. This is because either of the two sets of brackets in (3.5) vanishes if it is dotted with q. Thus, the contribution of the K photons to the electron-electron scattering amplitude gives an exponent function with the exponent $\mathscr{K}$ given by

$$\mathscr{K}=\mathscr{K}_1+\mathscr{K}_2+K\ , \tag{3.6a}$$

where

$$\mathscr{K}_1=\frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\frac{e^2(q^2)}{q^2-\lambda^2}\left[\frac{2p_1-q}{2p_1\cdot q-q^2}-\frac{2p_1'-q}{2p_1'\cdot q-q^2}\right]^2 \tag{3.6b}$$

and

$$\mathscr{K}_2=\mathscr{K}_1 \text{ with } p_1\to p_2 \text{ and } p_1'\to p_2'\ . \tag{3.6c}$$

In (3.6), we have included the contribution of photon self-energy parts.

Ultraviolet divergences. The diagrams for electron-electron scattering may have electron self-energy insertions and vertex insertions. When one or more of the external electron lines of such an insertion are off the mass shell, the corresponding (unrenormalized) propagator or vertex function has neither mass divergences nor soft divergences. However, these functions have ultraviolet divergences. To deal with such divergences, we must renormalize by making subtractions. If the subtractions made at the external electron lines of such insertions are on the mass shell, the subtracted functions have mass divergences and soft divergences. Consequently, the renormalized functions also do, and they give logarithmic functions of s.

In order to avoid the appearance of such logarithms, we shall make subtractions at the value t. By the Ward identity, the contributions of the vertex and the electron self-energy to the charge renormalization cancel each other. However, this cancellation is incomplete at the leftmost or the rightmost vertices on the path of an external electron. This is because the wave-function renormalization constant (ultraviolet divergent) for an external electron is evaluated at m^2, not at t. Thus, if we use the renormalized charge everywhere, we must compensate by multiplying the scattering amplitude by the product of ratios of the wave-function renormalization constant at m^2 with that at t. (The product is over the external electron lines.) Let us call this ratio $z_2{}^{1/2}$, which is ultraviolet finite. Then the e-e scattering amplitude is of the form

$$\mathscr{M}=\frac{1}{t}[z_2{}^{1/2}(p_1',n,t)z_2{}^{1/2}(p_2',n,t)]$$
$$\times\overline{\mathscr{M}}[z_2{}^{1/2}(p_1,n,t)z_2{}^{1/2}(p_2,n,t)]e^{K} . \quad (3.7)$$

We note that $\overline{\mathscr{M}}$ has neither soft divergences (which are contained in $e^{\mathscr{K}}$) nor mass divergences (which are contained in z_2). Thus, $\overline{\mathscr{M}}$ is a function of θ and $e(t)$ only. We also note that if we calculate, instead of the e-e scattering amplitude, the vertex functions of electron 1, say, we would get

$$\Gamma_\mu(p_1',p_1)=z_2{}^{1/2}(p_1,n,t)$$
$$\times\overline{\Gamma}_\mu z_2{}^{1/2}(p_1,n,t)e^{\mathscr{K}_1} , \quad (3.8)$$

where $\overline{\Gamma}_\mu$ is a function of θ and $e(t)$ only. Comparing (3.8) with (3.7), we obtain (1.1).

Finally, let us give a heuristic derivation of the asymptotic form of the vertex. A more rigorous derivation can be found in the literature.[1,2,3] Let us first consider all the diagrams with vertex insertions as well as the electron self-energy insertions but not photon self-energy insertions. In other words, all photons are bare photons. Let us go back to the Feynman gauge. We shall renormalize by making subtractions at off-shell values as before. By the Ward identity, the charge remains the bare charge e_0 as the contribution of the vertex function and the electron self-energy function cancel each other. However, this cancellation is complete only if each of the external electron lines contributes a factor $Z_2{}^{1/2}(t,e_0)$, where Z_2 is the wave-function renormalization constant in the Feynman gauge. Since the external electron lines are on the mass shell, each of them contributes a factor $Z_2{}^{1/2}(m^2,e_0)$ instead. Thus, the vertex function corresponding to all diagrams without photon self-energy insertions is factorized into $Z_2(m^2,e_0)/Z_2(t,e_0)$ times a function which does not have logarithmic factors of s due to ultraviolet divergences. To put it another way, these logarithms of the e-e scattering amplitude are contained in the overall factor $Z_2{}^{-1}(t,e_0)$. Next we recall that the soft divergences of the vertex function are factorized into $e^{\mathscr{K}_1}$, while the mass divergences of the scattering amplitudes are factorized into the overall renormalization constants in the gauge (3.1). (Such a renormalization constant is, by relativistic invariance, a function of $p\cdot n$, where p is the momentum of the electron.) It follows from these considerations that the vertex can be factorized into

$$\Gamma_\mu(p_1',p_1)=\frac{Z_2(m^2,e_0)}{Z_2(t,e_0)}e^{k_1}\exp[f(n\cdot p_1)+f^*(n\cdot p_1')]e^{G}\gamma_\mu , \quad (3.9)$$

where G is a function of θ and n only, independent of ω, and k_1 is equal to $\mathscr{K}_1$ defined in (3.6b) with $e(q^2)$ replaced by e_0. Since Γ_μ is independent of n, we conclude that

$$f(n\cdot p_1)+f^*(n\cdot p_1')+G$$

is independent of n. A little mathematics shows that f must have the form

$$f(n\cdot p_1)=2a\ln(n\cdot p_1)+b , \quad (3.10)$$

where a and b are constants. The proof is given in Appendix C. Thus, we get, ignoring photon self-energy parts,

$$\Gamma_\mu(p_1',p_1)=\frac{Z_2(m^2,e_0)}{Z_2(t,e_0)}e^{k_1}s^{a+a^*}H\gamma_\mu , \quad (3.11)$$

where H is a function of θ only in the limit $s\to\infty$ with θ fixed.

The constant $(a+a^*)$ can be determined as a perturbation series. For example, to obtain the lowest-order term in this perturbation series, we may calculate the $\ln^2 s$ terms and $\ln s$ terms in the lowest-order radiative corrections of the vertex function and compare them with (3.11). A somewhat easier way is to observe that if we use the definition (2.7) for b, it turns out that in the lowest-order the G photon does not give mass divergences. Thus, if we modify $\mathscr{K}_1$ by replacing q in the numerators in (3.6b) with $2q$, then the lowest-order term in $a+a^*$ is taken into account. We shall not write this out explicitly. Instead, let us discuss briefly what happens if we take into account the photon self-energy parts. In that case, we replace e_0 by the renormalized charge. Thus, we have

$$\Gamma_\mu(p_1',p_1)\simeq\gamma_\mu\frac{Z_2(m^2,e)}{Z_2(t,e)}H(\theta,e(t))\times\exp\left[2i\int\frac{d^4q}{(2\pi)^4}\frac{e^2(q^2)}{q^2-\lambda^2}\left[\frac{p_1-q}{2p_1\cdot q-q^2}-\frac{p_1'-q}{2p_1'\cdot q-q^2}\right]^2+O(e^4)\right]. \tag{3.12}$$

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APPENDIX A

Consider the integral

$$I\equiv\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-\lambda^2}\left[\frac{(p-q)\cdot(p'-q)}{[(p-q)^2-m^2][(p'-q)^2-m^2]}-\frac{1}{(p-q)^2-m^2}\right]. \tag{A1}$$

The second term in the integrand above serves to eliminate the ultraviolet divergence.

By introducing Feynman parameters and in the limit $\tau\gg m^2$, where $\tau\equiv-(p-p')^2$, we get

$$I=-\frac{i\tau}{32\pi^2}\int d^3\alpha\,\delta(1-\sum\alpha)\frac{\alpha_1}{(\alpha_2+\alpha_3)^2m^2+\alpha_1\lambda^2+\alpha_2\alpha_3\tau}+\frac{i}{16\pi^2}\ln\tau+O(1). \tag{A2}$$

After some algebra one obtains from (A2)

$$I=-\frac{i}{64\pi^2}\ln^2\left[\frac{\tau}{\lambda^2}\right]+O(1). \tag{A3}$$

To obtain (2.25) we identify in the first integral of (2.14) p_1' as p' and $-p_2'$ as p. Then τ is identified as $-s$. We interpret $-s$ as $e^{-i\pi}(s+i\epsilon)$ so that the argument of the logarithm is positive when s is negative. Thus, we have

$$k=\frac{e^2}{8\pi^2}\left[\ln^2\left[\frac{e^{-i\pi}s}{\lambda^2}\right]-\ln^2\left[\frac{|u|}{\lambda^2}\right]+O(1)\right], \tag{A4}$$

which leads readily to (2.25).

APPENDIX B

In this appendix, we prove two results valid at $m=0$: (a) in the gauge (3.1), the self-energy diagrams for an external electron give a factor proportional to the identity matrix in the Dirac spinor space; (b) the vertex function Γ_μ is proportional to γ_μ.[14]

If $m=0$, then the Lagrangian for QED is invariant under $\psi\to\gamma_5\psi$ and $\bar\psi\to\bar\psi\gamma_5$, where

$$\gamma_5=-i\begin{bmatrix}0 & I\\ I & 0\end{bmatrix},\quad \gamma_5{}^2=-1.$$

It follows from this chiral invariance that

$$\gamma_5\Sigma(p)\gamma_5=\Sigma(p) \tag{B1}$$

and

$$\gamma_5\Gamma_\mu(p',p)\gamma_5=\Gamma_\mu(p',p), \tag{B2}$$

where $\Sigma(p)$ is the mass operator for the electron.

In the gauge (3.1), $\Sigma(p)$ is a function of $\not p$ and $\not N$, where $N_\mu\equiv\omega n_\mu$. (N_μ is so defined that it has the same dimension as p_μ.) As a consequence of (B1), we have

$$\Sigma(p)=c_1\not p+c_2\not N. \tag{B3}$$

Terms such as $\not N\not p$ or $\not p\not N$, for example, do not satisfy (B1). The coefficients c_1 and c_2 are functions of p^2, n^2, and $(n\cdot p)$. However, c_2 must vanish if p is on the mass shell. This is because $\Sigma(p)$ is gauge invariant at such a point and is hence independent of N. Therefore, at the mass-shell point, $\Sigma(p)$ is proportional to $\not p$. Hence, result (a) is true.

Next, we have

$$\Gamma_\mu(p',p)=F_1\gamma_\mu+F_2(\not\Delta\gamma_\mu-\gamma_\mu\not\Delta),$$

where F_1 and F_2 are scalar functions. As a consequence of (B2), we have $F_2=0$. Thus, result (b) is true.

APPENDIX C

Let n be a unit vector with no time components. Let us also choose the vector $\vec{n}$ to lie in the scattering plane. Then we have

$$n\cdot p_1=-\omega\cos\theta\ ,\qquad n\cdot p_1'=-\omega\cos(\theta-\phi)\ , \tag{C1}$$

where θ is the angle between the vectors $\vec{n}$ and $\vec{p}_1$. Then

$$f(-\omega\cos\phi)+f^*(-\omega\cos(\theta-\phi))+G(\theta,\phi)$$

is independent of ϕ. Differentiating this expression with respect to ϕ and setting $\theta=\phi$, we get

$$\omega\sin\theta f'(-\omega\cos\theta)=G_\theta(\theta,\theta)\ , \tag{C2}$$

where G_θ denotes the partial derivative of G with respect to θ. From (C2) we get

$$f'(-\omega\cos\theta)=\frac{G_\theta(\theta,\theta)\cot\theta}{\omega\cos\theta}\ . \tag{C3}$$

Since the left-hand side of (C2) is a function of $\omega\cos\theta$ only, we conclude that

$$-G_\theta(\theta,\theta)\cot\theta=2a,\ \ a \text{ constant}\ .$$

Thus,

$$f'(-\omega\cos\theta)=\frac{2a}{-\omega\cos\theta}\ ,$$

or

$$f(-\omega\cos\theta)=2a\ln(-\omega\cos\theta)+b\ . \tag{C4}$$

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